表現論シンポジウム報告集, 1997
pp.1-8

A survey on "Matsushima-Murakami's formula"

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1. Dimension formula

Let G be a connected semisimple Lie group, Γ a uniform discrete subgroup, K a maximal compact subgroup of G, τ a representation of K on a finite-dimensional complex vector space V and λ a complex number. By an automorphic form of type (Γ, τ, λ), we mean a V-valued C^∞-function f on G satisfying the conditions:

1) $f(st) = \tau(t)^{-1}f(s)$ $(s \in G, \quad t \in K)$
2) $f(\gamma s) = f(s)$ $(\gamma \in \Gamma, s \in G)$
3) $Cf = \lambda f$

where C is the Casimir operator, a left invariant diffferential operator on G defined in terms of the Lie algebra of G. The space of all automorphic forms of type (Γ, τ, λ), which we denote by $A(\Gamma, \tau, \lambda)$, is of finite dimension. We know that the right regular representation of the group G on the Hilbert space $L^2(G)$ is a unitary representation of G and decomposes into sum of countable number of irreducible representations, among which each irreducible representaion π enters with a finite number of multiplicity $m(\pi)$. Now the so-called Matsushima-Murakami's formula is the following dimension formula:

(1.1) $\dim A(\Gamma, \tau, \lambda) = \Sigma\, m(\pi)(\pi|K : \tau^*)$

where Σ indicates that π runs over the set of all irreducible components of the regular representation such that $\pi(C)\phi = \lambda\phi$ for all ϕ in the domain of the operator $\pi(C)$ representing C under π; moreover $(\pi|K : \tau^*)$ denotes the intertwinnning number between the restriction $\pi|K$ of π to K and the representation τ^* of K contragradient to τ.

2. Cohomology groups

The dimension formula (1.1) is applied to prove vanishing theorems of certain cohomology groups. To explain this, we now assume that the symmetric space $X = G/K$ is Hermitian symmetric and the subgroup Γ of G acts freely on X. Put $M = \Gamma\backslash X$. Then M is a compact Kaehler manifold admittiing X as universal covering manifold. Let V be a finite dimensional complex vector space and $j : G \times X \to GL(V)$ be an <u>automorphic factor</u>, namely, a C^∞ map such that

1) $j(st, x) = j(s, tx)j(t, x)$ $(s, t \in G, \quad x \in X)$,

2) $j(s, x)$ is holomorphic in $x \in X$ for each $s \in G$.

Such a factor j defines an action of Γ on $X \times V$ by the rule

$$\gamma(x, v) = (\gamma x, j(\gamma, x)v) \qquad (\gamma \in \Gamma)$$

and the quotient space $\Gamma \backslash (X \times V)$ is a holomorphic vector bundle over M with typical fibre V , which we denote by E(j). The cohomology groups with which we are concerned are those of M with coefficients in the sheaf $\underline{E}(j)$ of germs of holomorphic sections of E(j). The q-th cohomology group will be denoted by $H^q(M, \underline{E}(j))$. Then we have the Dolbealt isomorphism:

(2.1) $\qquad H^q(M, \underline{E}(j)) = H^{0,q}(E(j))$

where the right hand side is the q-th cohomology group of the complex $A(E(j)) = \Sigma_q A^{0,q}(E(j))$ equipped with the differential operator d'' , $A^{0,q}(E(j))$ being the space of E(j)-valued forms of type (0,q) on M. In the following, we consider exclusively the cohomology for the case where j is the so-called <u>canonical automorphic factor</u>. Observe that any automorphic factor j defines a representation τ of K by

$$\tau(t) = j(t, x_0) \qquad (t \in K),$$

where x_0 is the point of X representing the coset K. We know that any representation τ of K is defined by a canonical automorphic factor J_τ and, in case that τ is irreducible. any j defining τ is equivalent to J_τ in a certain sense.

Important examples of the cohomology groups are:

1) τ is the trivial representation and $J_\tau(s,x) = 1$. In this case, $E(J_\tau)$ is a trivial bundle $M \times C$ over M, and $\dim H^{0,q}(M, E(J_\tau))$ is the (0,q)-Betti number of M.

2) When τ is the holomorphic isotropic representation ad_+ defined later, the sheaf $\underline{E}(J_{ad_+})$ is the sheaf of germs of holomorphic vector fields on M, and the vanishing of the cohomology for this case was discussed by Calabi and Vesentini, By Kodaira's theory,the vanishing of the first cohomology implies the rigidity

of the complex structure of M .

3. Transition to $\Gamma \backslash G$

Recall that $E(j)$ is the holomorphic vector bundle over $M = \Gamma \backslash X$ obtained as the quotient $\Gamma \backslash (X \times V)$. Now we have a differentiable principal bundle over M with structure group K :

(#) $\quad \Gamma \backslash G \to M = \Gamma \backslash X = \Gamma \backslash G / K$.

Key lemma The vector bundle $E(j)$ is differentiably associated to the principal bundle (#) by the representation τ of K defined by the automorphic factor j . Namely, $E(j)$ is the quotient of $(\Gamma \backslash G) \times V$ by the right action of K given by

$$(y, v) k = (yk , \tau (k)^{-1} v) \qquad (k \in K)$$

with the projection $E(j) \to M$ such that the following is a commutative diagram:

$$\begin{array}{ccc} (\Gamma \backslash G) \times V & \to & E(j) \\ \downarrow & & \downarrow \\ \Gamma \backslash G & \to & M . \end{array} \tag{3.1}$$

Let $A^r(E(j))$ be the space of $E(j)$-valued r-forms on M and $A^r(\Gamma \backslash G, V)$ be the space of V-valued r-forms on $\Gamma \backslash G$. By the diagram (3.1), an $E(j)$-valued r-form on M induces a V-valued r-form on $\Gamma \backslash G$ so that we get an injection from $A^r(E(j))$ into $A^r(\Gamma \backslash G, V)$ whose image will be denoted by $A^r(\Gamma \backslash G, V, \tau)$:

$$A^r(E(j)) \cong A^r(\Gamma \backslash G, V, \tau) \subset A^r(\Gamma \backslash G, V) \tag{3.2}$$

Let g be the Lie algebra of the Lie group G and k be the subalgebra of g corresponding to the subgroup K . Since g consists of left invariant vector fields on G , g may be identified with a Lie algebra of vector fields on $\Gamma \backslash G$, The manifold $\Gamma \backslash G$ is then parallelizable by a basis of g . It follows that a V-valued r-form ω on $\Gamma \backslash G$ may be identified with an alternating r-form on the vector space g with values in $C^{\infty}(\Gamma \backslash G) \otimes V$, where $C^{\infty}(\Gamma \backslash G)$ is the space of complex-valued C^{∞}-functions on $\Gamma \backslash G$. Therefore we may put

$$A^r (\Gamma \backslash G, V) = C^{\infty}(\Gamma \backslash G) \otimes V \otimes \wedge^r (g^c)^* \tag{3.3}$$

where $\wedge^r (g^c)^*$ is the r-th exterior product of $(g^c)^*$, the dual space of the complexification g^c of g .

The subspace $A^r(\Gamma\backslash G, V, \tau)$ of $A^r(\Gamma\backslash G, V)$ consists of the forms ω with the properties:

1) $\omega \circ R_k = \tau(k)^{-1}\omega \quad (k \in K)$, and

2) ω vanishes on the fibres of the bundle $\Gamma\backslash G \to M$.

where R_k is the right translation of $\Gamma\backslash G$ by the element $k \in K$. Since the vector fields belonging to k span at each point the tangent space of the fibre through this point of the principal bundle $\Gamma\backslash G \to M$, the second condition is equivalent to:

2') $i(Y)\omega = 0 \quad (Y \in k)$

where $i(Y)$ denotes the interior product by the vector field Y. Let now $g = k + m$ be the Cartan decomposition of g. We will use the superscript C to designate complexification. The adjoint action of K leaves stable the subspace m and induces a representation ad_m of K on m^c. From the characteristic properties 1) and 2') of forms ω belonging to $A^r(\Gamma\backslash G, V, \tau)$, we see easily that the identification (3.3) gives rise to:

(3.4) $A^r(\Gamma\backslash G, V, \tau) = (C^\infty(\Gamma\backslash G) \otimes V \otimes \wedge^r(m^c)^*)^K$

where the left side denotes the subspace of K-invariant elements in the space $C^\infty(\Gamma\backslash G) \otimes V \otimes \wedge^r(m^c)^*$ on which K acts by

$(R \otimes \tau \otimes \wedge^r ad_m^*)(k) = R_k \otimes \tau(k) \otimes \wedge^r ad_m(k)^* \quad (k \in K)$.

Let $A^{0,q}(\Gamma\backslash G, V, \tau)$ be the subspace of $A^q(\Gamma\backslash G, V, \tau)$ corresponding to the subspace $A^{0,q}(E(j))$ by the isomorphism (3.2). To describe this space, note that the Cartan decoposition of g together with the complex stuructue of G/K gives rise to the decomposition of g^c:

(3.5) $g^c = k^c + n^+ + n^-$

where n^+ (n^-) consists of those elements of m^c which project to complex tangent vector of type (1,0) (resp. (0,1)). The adjoint action of K on g^c leaves stable the subspace n^+ so that we get a representation ad_+ of K on the vector space n^+. We denote by ad^q_+ the repres e ntation of K on the q-th exterior product $\wedge^q n^+$ of n^+ induced from ad_+. We may identify the dual space of n^- with n^+ by means of the Killing form of g^c. Then we get the following identification similar to (3.4).

(3.6) $A^{0,q}(\Gamma\backslash G, V, \tau) = (C^\infty(\Gamma\backslash G) \otimes V \otimes \wedge^q n^+)^K$

where K acts on $C^\infty(\Gamma\backslash G) \otimes V \otimes \wedge^q n^+$ by $R \otimes \tau \otimes ad^q_+$.

4. Application of the harmonic theory

Since M is a compact Kaehler manifold, we can apply the harmonic thoery to study the cohomology groups $H^{0,q}(E(j))$ after introducing Hermitian metrics in the fibres of the vector bundle E(j). By virtue of the Key Lemma, we can introduce Hemitian metrics in the fibers of E(j) by making use of a Hermitian metric in V invariant under $\tau(k)$ ($k \in K$).

In the following we assume that the automorphic factor j is a canonical automorphic factor J_τ . Then after a series of crucial calculus we get a simple formula for the Laplacian operator $\Box$ acting on $A^{0,q}(E(j))$. Since G/K is a Hermitian symmetric space, k contains a Cartan subalgebra h of g , and we choose a certain canonical linear ordering in the dual space h^* of h such that n^+ (n^-) is a sum of eigenspaces of some positive (resp. negative) roots of g^c. When an irreducible representation τ of K is given, denote by Λ the highet weight of τ relative to this ordering. Then the Laplacian operator $\Box$ acting on $A^{0,q}(E(j))$ is transformed via (3.5) into the action on the right hand side of (3.5):

(4.1) $\Box = \frac{1}{2}(- C + \langle\Lambda , \Lambda + 2\delta\rangle)$

where C is the Casimir operator acting on $C^\infty(\Gamma \backslash G)$, δ is the half-sum of positive roots and $\langle\ ,\ \rangle$ is the inner product defined by the Killing form of g^c .

Now the harmonic theory asserts that the cohomology groups $H^{0,q}(E(J_\tau))$ is canonically isomorphic to the space of forms ω of type (0,q) which are harmonic, i.e. such that $\Box\omega = 0$. It follows from (3.6) and (4.1) that the space of harmonic forms is isomorphic to the space of $V \otimes \Lambda^q n^+$-valued C^∞-functions f on $\Gamma \backslash G$ satisfying the following conditions.

1) $f(yk) = (\tau \otimes ad^q{}_+)(k)^{-1} f(y) \quad (y \in \Gamma \backslash G,\ k \in K)$

2) $Cf = \langle\Lambda , \Lambda + 2\delta\rangle f$.

This space is clearly isomorphic to the space of automorphic forms $A(\Gamma , \tau \otimes ad^q{}_+, \langle\Lambda , \Lambda + 2\delta\rangle)$ in the terminology of section 1 . Thus we get the isomorphism:

$$H^{0,q}(M, \underline{E}(J_\tau)) = A(\Gamma , \tau \otimes ad^q{}_+, \langle\Lambda , \Lambda + 2\delta\rangle)$$

Applying the dimension formula (1.1) and (2.1), we get finally:

(4.2) $\dim H^q(M, \underline{E}(J\ )) = \Sigma\ m(\pi)(\pi|K : (\tau \otimes ad^q{}_+)^\pi)$

where π runs over the irreducible components of the regular representation of G for which $\pi(C) = \langle \Lambda, \Lambda + 2\delta \rangle$.

5. Vanishing thoerems

Applying (4.2) we can derive a series of vanishing theorems for the cohomology groups such as vanishing theorems for the (0,q) Betti numbers and a theorem of Calabi and Vesentini. By a very refined argument depending on deep results of unitary representations, Parthasarathy and Williams show that when G/K is an irreducible Hermitian symmetric space, under widely applicable conditions on Λ, the q-th cohomology group $H^q(M, \underline{E}(J_\tau))$ vanishes except for q in the following Table. Examples of non-vanishing cohomology for almost all q in the Table are constructed by C.W. Anderson.

TABLE

Type	G	Set of numbers q
$\mathrm{I}_{n,m}$	$SU(n,m),\ n \geq m$	$\{nm - n'm' \mid 0 \leqslant n' \leqslant n,\ 0 \leqslant m' \leqslant m\}$
II_n	$SO^*(2n),\ n > 3$	$\left\{\frac{n(n-1)}{2} - \frac{j(j-1)}{2} \middle\| j = 3, \ldots, n\right\} \cup \left\{\frac{n(n-1)}{2} - i \middle\| i = 0, 1, \ldots, n-1\right\}$
III_n	$Sp(n,\mathrm{R})$	$\{0\} \cup \{n + (n-1) + \cdots + (n-j) \mid j = 0, 1, \ldots, n-1\}$
IV_n	$SO_0(n,2),\ n > 2$	$\{0\} \cup \left\{\left[\frac{n+1}{2}\right], \ldots, n\right\}$
V_6	real form of E_6	$\{0, 8, 11, 12, 13, 14, 15, 16\}$
VI_7	real from of E_7	$\{0, 17, 21, 22, 23, 24, 25, 26, 27\}$

4. Brief history

Vanishing thoerems of sheaf cohomology groups were one of the objects of research in the second half of 1950's, because they imply rigidity of many interesting stuctures such as riemannian structures, complex structures, as well as local rigidity of discrete subgroups in a Lie group. The theory developed by Matsushima and myself [1],[2] is concerned with the sheaf cohomology groups on symmetric Riemannian manifolds G/K defined by a representation of the group G or a canonical automorphic factor. We get general vanishing theorems which imply vanishing theorems obtained by Weil and by Calabi and Vesentini. We refer to my report [a] for a quick survey of this theory as well as its applications, and to my lecture notes [b] for an elementary presentation of the whole theory.

The dimension formula (2.1) was obtained first by Matsushima [3] for special but distinguished case and later generalized in the joint work [4]. The application of this formula to the cohomology was initiated by Hotta and Wallach [5], which was followed by Hotta and myself [6]. From this work we see that the dimension formula provides a new proof for the vanishing theorem of Calabi-Vesentini type (i.e.,of d''-cohomology), which was previously obtained through a vanishing thoerem of Weil type (i.e.,of d-cohomology). The study of vanishing of cohomology along this line is the object of the works by Parthasarthy [8] and Williams [9],[10],[11]. (I am not aware of further developement of the study about this subject after the middle of 1980's.)

The present report is based on my article [c] where the vanishing theorems of Williams are explained in detail.

Expository works by S. Murakami:

[a] Cohomology of vector-valued forms on compact, locally symmetric Riemannian manifolds, Proceedings Symposia in Pure Mathematics, vol 9, Algebraic groups and discontinuous subgroups, 387-399.

[b] Cohomology of vector-valued forms on symmetric spaces, Lecture Notes, Chicago University, 1966.

[c] Vanishing theorems on cohomology associated to Hermitian symmetric spaces, Ann.Inst.Fourier, 37(1987), 225-233.

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