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ON THE EXPRESSION OF THE HARISH-CHANDRA C-FUNCTION OF $SU(n,1)$

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1. INTRODUCTION

The Harish-Chandra C-function was defined by Harish-Chandra as the constant terms of the asymptotic expansions of the Eisenstein integrals. The Harish-Chandra C-function plays a basic tool in studying harmonic analysis on semisimple Lie groups, because it closely relates to the Plancherel measure and the reducibility of the principal series representations. Moreover, the location of the singularities of the Harish-Chandra C-function is crucial for the proof of the Paley-Wiener type theorem or the various Schwartz type theorem. In the cases of $Spin(n,1)$ and $SU(n,1)$, there are no multiple irreducible unitary representations of M occurring in the irreducible unitary representation of K. In these cases, it is known that the restriction of the Harish-Chandra C-function to the M-irreducible component becomes a scalar-valued meromorphic function.

2. NOTATION AND PRELIMINARIES

Let G be a connected semisimple Lie group of real rank one with finite center, K a maximal compact subgroup of G and θ the corresponding Cartan involution. As usual, we shall use lower case German letters to denote the corresponding Lie algebras and upper case German letters their universal enveloping algebras. Let $\langle\cdot,\cdot\rangle$ denote the Killing form on $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$, define the inner product $\langle X,Y\rangle_\theta$ in $\mathfrak{g}$ by $\langle X,Y\rangle_\theta = -\langle X,\theta Y\rangle$.

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to θ. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra containing $\mathfrak{a}$ and $\mathfrak{h}_{\mathfrak{k}}=\mathfrak{h}\cap\mathfrak{k}$. Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}_{\mathfrak{k}}$. Fix an orthonormal basis of $\sqrt{-1}\mathfrak{h}_{\mathfrak{k}}$ and choose the orthonormal bases of $\sqrt{-1}\mathfrak{h}_{\mathfrak{k}}+\mathfrak{a}$ and $\sqrt{-1}\mathfrak{t}$ so that their restrictions to $\sqrt{-1}\mathfrak{h}_{\mathfrak{k}}$ coincide with the orthonormal basis of $\sqrt{-1}\mathfrak{h}_{\mathfrak{k}}$. As usual, we shall equip $\sqrt{-1}\mathfrak{h}_{\mathfrak{k}}+\mathfrak{a}$, $\sqrt{-1}\mathfrak{t}$ and $\sqrt{-1}\mathfrak{h}_{\mathfrak{k}}$ with the lexicographic ordering with respect to the orthonormal basis given above.

Let Σ denote the set of all nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and Σ^+ the subset of Σ consisting of all positive roots. With $\alpha\in\Sigma$, $\mathfrak{g}_\alpha$ denotes the corresponding root subspace of $\mathfrak{g}$ and $m_\alpha=\dim\mathfrak{g}_\alpha$. Let $\mathfrak{n}$ be the sum of all positive root subspaces. A and N denote the analytic subgroups of G coresponding to $\mathfrak{a}$ and $\mathfrak{n}$, respectively and $\bar{N}=\theta N$. Then $G=KAN$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ are the Iwasawa decompositions of G and $\mathfrak{g}$, respectively. For $g\in G$, g decomposes under $G=KAN$ as $g=\kappa(g)\exp H(g)n(g)$, where $\kappa(g)\in K$, $H(g)\in\mathfrak{a}$ and $n(g)\in N$.

We denote by M and M' the centralizer and the normalizer of $\mathfrak{a}$ in K, respectively. Then $W(\mathfrak{a})=M'/M$ is the Weyl group of G. Let Δ_K be the set of roots of $\mathfrak{k}_c$ with respect to $\mathfrak{t}_c$ and Δ_K^+ the subset of Δ_K consisting of all positive roots. Put $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma^+}m_\alpha\alpha$ and $\delta_K=\frac{1}{2}\sum_{\beta\in\Delta_K^+}\beta$.

We fix an orthonormal basis $\{X_{\alpha,1},\dots,X_{\alpha,m_\alpha}\}$ of $\mathfrak{g}_\alpha$ and $\{U_1,\dots,U_m\}$ ($m=\dim\mathfrak{m}$) of $\mathfrak{m}$ with respect to the inner product $\langle\cdot,\cdot\rangle_\theta$. Put $Y_{\alpha,i}=2^{-1/2}(X_{\alpha,i}+\theta X_{\alpha,i})$ and $Z_{\alpha,i}=$

$2^{-1/2}(X_{\alpha,i}-\theta X_{\alpha,i})$ for each i $(1 \leq i \leq m_\alpha)$. Choose $H \in \mathfrak{a}$ so that $\alpha(H)=1$, where $\alpha \in \Sigma^+$ denotes the simple root. Set $\omega_{\mathfrak{t}} = -\sum_{i=1}^{m} U_i^2 - \sum_{j=1}^{2}\sum_{i=1}^{m_{j\alpha}} Y_{j\alpha,i}^2$ and $\omega_{j\alpha} = -\sum_{i=1}^{m_{j\alpha}} Y_{j\alpha,i}^2$.

Let D_K and D_M denote the sets of dominant, analytically integral forms on $\mathfrak{t}_{\mathrm{c}}$ and $\mathfrak{h}_{\mathfrak{t}_d}$ respectively, with respect to the orderings given above. If $\lambda \in D_K$ and $\mu \in D_M$, we write $(\tau_\lambda, V_\lambda)$ and (σ_μ, H_μ) for the irreducible unitary representations of K and M whose highest weights are λ and μ, respectively. With $\tau \in \hat{K}$ and $\sigma \in \hat{M}$, $[\tau : \sigma]$ denotes the multiplicity of σ occurring in $\tau|_M$. Let $\hat{K}(\sigma)$ and $\hat{M}(\tau)$ be the subsets of $\hat{K}$ and $\hat{M}$ consisting of the elements satisfying $[\tau:\sigma] \neq 0$, respectively. Similarly $D_K(\mu)$ and $D_M(\lambda)$ denote the subsets of D_K and D_M consisting of the elements satisfying $[\tau_\lambda : \sigma_\mu] \neq 0$, respectively.

Finally we write the Haar measures dk and $d\bar{n}$ on K and $\bar{N}$, respectively, normalized as $\int_K dk = 1$ and $\int_{\bar{N}} \exp\{-2\rho(H(\bar{n}))\}d\bar{n} = 1$.

3. INFINITESIMAL OPERATOR OF THE PRINCIPAL SERIES

In this section, we shall introduce the formula of the inifinitesimal operator of the principal series representation that was shown by Thieleker [10]. We shall reform Thieleker's formula for our convenience so that we can get the recursion formula of the Harish-Chandra C-function.

We shall first review the compact picture of the principal series to explain the notation and the parametrization. Let $(\sigma, H_\sigma) \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathrm{c}}^*$. We set

$$C_\sigma^\infty(K) = \left\{\varphi \in C^\infty(K; H_\sigma) : \varphi(km) = \sigma(m)^{-1}\varphi(k)\right\}. \tag{3.1}$$

Let $\mathcal{H}^{\sigma,\nu}$ denote the completion of $C_\sigma^\infty(K)$ with respect to the inner product $(f,g) = \int_K (f(k), g(k))_{H_\sigma} dk$. Define the action $\pi_{\sigma,\nu}$ of G on $\mathcal{H}^{\sigma,\nu}$ by

$$(\pi_{\sigma,\nu}(g)\varphi)(k) = e^{-(\nu+\rho)(H(g^{-1}k))}\varphi(\kappa(g^{-1}k)). \tag{3.2}$$

For $\varphi \in \mathcal{H}^{\sigma,\nu}$, define the function φ_ν on G by $\varphi_\nu(g) = e^{-(\nu+\rho)(H(g))}\varphi(\kappa(g))$. With $(\tau, V_\tau) \in \hat{K}(\sigma)$, let $\mathcal{H}^{\sigma,\nu}(\tau)$ be the τ-isotypic component of $\mathcal{H}^{\sigma,\nu}$. Then Frobenius reciprocity implies the following lemma.

Lemma 3.1. *The correspondence* $T \otimes v \to f_{T\otimes v}(k) = T(\tau(k)^{-1}v)$ *is a* K*-module isomorphism of* $\mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau$ *onto* $\mathcal{H}^{\sigma,\nu}(\tau)$.

We set $\phi_Z(k) = \langle \mathrm{Ad}(k)^{-1}Z, H\rangle / \langle H, H\rangle$. The objective in this section is to prove the following theorem.

Theorem 3.2 (cf. [10]). *Let* $Z \in \mathfrak{p}_{\mathrm{c}}$ *and* $\varphi \in C_\sigma^\infty(K)$. *Then we have*

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k) = \frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle}(\phi_Z\varphi)(k) &+ \frac{1}{2\langle\alpha,\alpha\rangle}\left[(\phi_Z\varphi)(k;\omega_{\mathfrak{t}}) - \phi_Z(k)\varphi(k;\omega_{\mathfrak{t}})\right] \\ &- \frac{1}{4\langle\alpha,\alpha\rangle}\left[(\phi_Z\varphi)(k;\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_{2\alpha})\right].\end{aligned}$$

Proof. We first note

$$\mathrm{Ad}(k)^{-1}Z = \frac{\langle \mathrm{Ad}(k)^{-1}Z, H\rangle H}{\langle H, H\rangle} + \sum_{j=1}^{2}\sum_{i=1}^{m_{j\alpha}} \langle \mathrm{Ad}(k)^{-1}Z, Z_{j\alpha,i}\rangle Z_{j\alpha,i}. \tag{3.3}$$

It follows from the definition of $\varphi_\nu(g)$ that

$$\varphi_\nu(k;H) = -(\nu+\rho)(H)\varphi_\nu(k) \text{ for } H \in \mathfrak{a},\ k \in K, \tag{3.4}$$

$$\varphi_\nu(k;X) = 0 \text{ for } X \in \mathfrak{n},\ k \in K. \tag{3.5}$$

Noting $Z_{j\alpha,i} = -Y_{j\alpha,i} + \sqrt{2}X_{j\alpha,i}$, we obtain

$$\varphi_\nu(k;Z_{j\alpha,i}) = -\varphi_\nu(k;Y_{j\alpha,i}) = -\varphi(k;Y_{j\alpha,i}). \tag{3.6}$$

Taking into account (3.3) and (3.6), we have

(3.7)

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k) &= \varphi_\nu(-Z;k) = -\varphi_\nu(k;\operatorname{Ad}(k)^{-1}Z)\\ &= \langle \nu+\rho,\alpha\rangle\langle \operatorname{Ad}(k)^{-1}Z, H\rangle\varphi(k) + \sum_{j=1}^{2}\sum_{i=1}^{m_{j\alpha}}\langle \operatorname{Ad}(k)^{-1}Z, Z_{j\alpha,i}\rangle\varphi(k;Y_{j\alpha,i}).\end{aligned}$$

A simple calculation yields that

$$[H, Y_{j\alpha,i}] = jZ_{j\alpha,i},\ [Y_{j\alpha,i}, Z_{j\alpha,i}] = j\langle\alpha,\alpha\rangle H. \tag{3.8}$$

From (3.8), we have

$$\begin{aligned}\phi_Z(k;Y_{j\alpha,i}) &= \frac{\langle \operatorname{ad}(-Y_{j\alpha,i})\operatorname{Ad}(k)^{-1}Z, H\rangle}{\langle H,H\rangle}\\ &= \frac{-j\langle \operatorname{Ad}(k)^{-1}Z, Z_{j\alpha,i}\rangle}{\langle H,H\rangle}.\end{aligned} \tag{3.9}$$

Therefore, substituting (3.9) into (3.7), we obtain

(3.10)

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k) = \frac{\langle\nu+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) &- \frac{1}{\langle\alpha,\alpha\rangle}\sum_{i=1}^{m_\alpha}\phi_Z(k;Y_{\alpha,i})\varphi(k;Y_{\alpha,i})\\ &- \frac{1}{2\langle\alpha,\alpha\rangle}\sum_{i=1}^{m_{2\alpha}}\phi_Z(k;Y_{2\alpha,i})\varphi(k;Y_{2\alpha,i}).\end{aligned}$$

A simple calculation using (3.8) gives that

$$\phi_Z(k;U_i) = 0,\ \phi_Z(k;Y_{j\alpha,i}^2) = -j^2\langle\alpha,\alpha\rangle\phi_Z(k), \tag{3.11}$$

and hence

$$\begin{aligned}\phi_Z(k;\omega_{j\alpha}) &= j^2 m_{j\alpha}\langle\alpha,\alpha\rangle\phi_Z(k),\\ \phi_Z(k;\omega_t) &= (m_\alpha + 4m_{2\alpha})\langle\alpha,\alpha\rangle\phi_Z(k).\end{aligned} \tag{3.12}$$

By using Leibniz's formula, we have

(3.13)

$$\begin{aligned}(\phi_Z\varphi)(k;\omega_{j\alpha}) &= \phi_Z(k)\varphi(k;\omega_{j\alpha}) + \phi_Z(k;\omega_{j\alpha})\varphi(k) - 2\sum_{i=1}^{m_{j\alpha}}\phi_Z(k;Y_{j\alpha,i})\varphi(k;Y_{j\alpha,i})\\ &= \phi_Z(k)\varphi(k;\omega_{j\alpha}) + j^2 m_{j\alpha}\langle\alpha,\alpha\rangle\phi_Z(k)\varphi(k) - 2\sum_{i=1}^{m_{j\alpha}}\phi_Z(k;Y_{j\alpha,i})\varphi(k;Y_{j\alpha,i}).\end{aligned}$$

Therefore

(3.14)

$$-\sum_{i=1}^{m_{j\alpha}} \phi_Z(k; Y_{j\alpha,i})\varphi(k; Y_{j\alpha,i}) = \frac{1}{2}\left[(\phi_Z\varphi)(k;\omega_{j\alpha}) - \phi_Z(k)\varphi(k;\omega_{j\alpha}) - j^2 m_{j\alpha}\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)\right].$$

Substituting (3.14) into (3.10), we obtain

(3.15)

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\phi)(k) &= \frac{\langle\nu+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k)\\ &+ \frac{1}{2\langle\alpha,\alpha\rangle}\left[(\phi_Z\varphi)(k;\omega_\alpha) - \phi_Z(k)\varphi(k;\omega_\alpha) - m_\alpha\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)\right]\\ &+ \frac{1}{4\langle\alpha,\alpha\rangle}\left[(\phi_Z\varphi)(k;\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_{2\alpha}) - 4m_{2\alpha}\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)\right]\\ &= \frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) + \frac{1}{2\langle\alpha,\alpha\rangle}\left[(\phi_Z\varphi)(k;\omega_\alpha+\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_\alpha+\omega_{2\alpha})\right]\\ &\qquad - \frac{1}{4\langle\alpha,\alpha\rangle}\left[(\phi_Z\varphi)(k;\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_{2\alpha})\right].\end{aligned}$$

Noting (3.11), and using Leibniz's formula, we immediately obtain

$$(\phi_Z\varphi)(k;\omega_\alpha+\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_\alpha+\omega_{2\alpha}) = (\phi_Z\varphi)(k;\omega_{\mathfrak{t}}) - \phi_Z(k)\varphi(k;\omega_{\mathfrak{t}}). \tag{3.16}$$

Substituting (3.16) into the last expression in (3.15), we get the assertion. □

4. CONNECTION WITH C-FUNCTION AND INTERTWINING OPERATOR

In this section, we will summarize some known results on the relationship between the standard intertwining operator and the Harish-Chandra C-function. The results below are due to Knapp–Stein [6] and Wallach [13].

In [6], they have constructed the integral expression of the intertwining operator between the principal series representations, which is called the standard intertwining operator. Let $(\sigma, H_\sigma) \in \hat{M}$ and $\nu \in \mathfrak{a}_c^*$. Fix $(\tau, V_\tau) \in \hat{K}(\sigma)$. We first note that if $T \in \mathrm{Hom}_M(V_\tau, H_\sigma)$ then $T\tau(w)^{-1} \in \mathrm{Hom}_M(V_\tau, H_{w\sigma})$, that is, $\dim\mathcal{H}^{\sigma,\nu}(\tau) = \dim\mathcal{H}^{w\sigma,w\nu}(\tau)$. Suppose $\mathrm{Re}\langle\nu,\alpha\rangle > 0$. Then the standard intertwining operator is defined as follows:

$$(A(w,\sigma,\nu)\varphi)(k) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))}\varphi(kw\kappa(\bar{n}))d\bar{n}, \quad (\varphi(k) \in C_\sigma^\infty(K)). \tag{4.1}$$

Then it is known that the integral defined above is well-defined. Moreover $A(w,\sigma,\nu)\varphi \in C_{w\sigma}^\infty(K)$ and it satisfies

$$A(w,\sigma,\nu)\pi_{\sigma,\nu}(g)\varphi(k) = \pi_{w\sigma,w\nu}(g)A(w,\sigma,\nu)\varphi(k). \tag{4.2}$$

In [6], they also showed that $A(w,\sigma,\nu)\varphi$, as a function of ν, can be extended to a meromorphic function on $\mathfrak{a}_c^*$. Let $T \otimes v \in \mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau$. Then it follows from Wallach (cf. [13]) that

$$(A(w,\sigma,\nu)f_{T\otimes v})(k) = T(C_\tau(\nu)\tau(w)^{-1}\tau(k)^{-1}v), \tag{4.3}$$

where

$$C_\tau(\nu) = \int_{\bar N} e^{-(\nu+\rho)(H(\bar n))}\tau(\kappa(\bar n))^{-1}d\bar n. \tag{4.4}$$

Here the function $C_\tau(\nu)$ is called the Harish-Chandra C-function associated with τ. Noting that $\tau(m)C_\tau(\nu) = C_\tau(\nu)\tau(m)$ for any $m \in M$, we see that $TC_\tau(\nu) \in \mathrm{Hom}_M(V_\tau, H_\sigma)$ and $TC_\tau(\nu)\tau(w)^{-1} \in \mathrm{Hom}_M(V_\tau, H_{w\sigma})$. Let $(R(w)\varphi)(k) = \varphi(kw)$ for $\varphi \in C_\sigma^\infty(K)$. Define the linear mapping

$$R_\tau(w)\colon \mathrm{Hom}_M(V_\tau, H_\sigma)\otimes V_\tau \to \mathrm{Hom}_M(V_\tau, H_{w\sigma})\otimes V_\tau \tag{4.5}$$

by $R_\tau(w)(T\otimes v) = T\tau(w)^{-1}\otimes v$. Then it is clear that

$$(R(w)f_{T\otimes v})(k) = f_{R_\tau(w)(T\otimes v)}(k). \tag{4.6}$$

Looking upon $C_\tau(\nu)$ as a linear mapping of $\mathrm{Hom}_M(V_\tau, H_\sigma)$, we set $C_\tau(\sigma:\nu) = \det C_\tau(\nu)$. We call $C_\tau(\sigma:\nu)$ the Harish-Chandra C-function associated with τ and σ. Our main concern in this paper is the case that $\dim \mathrm{Hom}_M(V_\tau, H_\sigma) = 1$. It is known that if $G = Spin(n,1)$ or $G = SU(n,1)$ then this assumption holds for all $\tau \in \hat K$ and $\sigma \in \hat M(\tau)$. Under this assumption, because $TC_\tau(\nu) = C_\tau(\sigma:\nu)T$, (4.3) can be written as follows.

Proposition 4.1. *Retain the above notation and assumption. We have*

$$(A(w,\sigma,\nu)f_{T\otimes v})(k) = C_\tau(\sigma:\nu)f_{R_\tau(w)(T\otimes v)}(k). \tag{4.7}$$

5. RECURSION FORMULA FOR C-FUNCTION

In this section, we shall show the recursion formula for the Harish-Chandra C-function under the assumption that $[\tau_\lambda : \sigma_\mu] = 0$ or 1. In addition, we assume that the unitary representation $(\mathrm{Ad}, \mathfrak{p}_c)$ of K has no multiple weights. This is the case that $G = Spin(2n+1,1)$ or semisimple G with $\mathrm{rank}\, G = \mathrm{rank}\, K$. Let $\Delta_\mathfrak{p}$ denote the set of all weights of $(\mathrm{Ad}, \mathfrak{p}_c)$ with respect to $\mathfrak{t}_c$. Under this assumption, the following lemma is valid.

Lemma 5.1. *Let* $\lambda \in D_K$. *Then*

$$\mathrm{Ad}\otimes\tau_\lambda = \sum_{\beta\in\Delta_\mathfrak{p}} m(\lambda+\beta)\tau_{\lambda+\beta},$$

where $m(\lambda+\beta) = 0$ *or* 1.

Let $E_{\lambda+\beta}$ denote the canonical projection of $\mathfrak{p}_c\otimes V_\lambda$ into $V_{\lambda+\beta}$ given by the decomposition in Lemma 5.1 satisfying $E_{\lambda+\beta}E_{\lambda+\beta}^* = I_{\lambda+\beta}$, where $E_{\lambda+\beta}^*$ and $I_{\lambda+\beta}$ denote the adjoint operator of $E_{\lambda+\beta}$ and the identity operator on $V_{\lambda+\beta}$, respectively.

Let $\lambda \in D_K$ and $\mu \in D_M(\lambda)$. For $T \in \mathrm{Hom}_M(V_\lambda, H_\mu)$, define $\tilde T \in \mathrm{Hom}_M(\mathfrak{p}_c\otimes V_\lambda, H_\mu)$ by

$$\tilde T(Z\otimes v) = \frac{\langle Z, H\rangle}{\langle H, H\rangle}T(v). \tag{5.1}$$

Define the linear mapping

$$\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)\colon \mathrm{Hom}_M(V_\lambda, H_\mu)\otimes V_\lambda \to \mathrm{Hom}_M(V_{\lambda+\beta}, H_\mu)\otimes V_{\lambda+\beta} \tag{5.2}$$

by

$$\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v) = \tilde T E_{\lambda+\beta}^*\otimes E_{\lambda+\beta}(Z\otimes v). \tag{5.3}$$

Here it is easy to check that $\tilde T E_{\lambda+\beta}^* \in \mathrm{Hom}_M(V_{\lambda+\beta}, H_\mu)$. Then we have the following proposition.

Proposition 5.2. *Keep the notation and assumption above. We drive*

$$(\phi_Z f_{T\otimes v})(k) = \sum_{\beta\in\Delta_p} m(\lambda+\beta) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k). \tag{5.4}$$

Proof. We compute

$$\begin{aligned}
(\phi_Z f_{T\otimes v})(k) &= \frac{\langle \mathrm{Ad}(k)^{-1}Z, H\rangle}{\langle H,H\rangle} T(\tau_\lambda(k)^{-1}v) \\
&= \tilde{T}((\mathrm{Ad}\otimes\tau_\lambda)(k)^{-1}(Z\otimes v)) \\
&= \tilde{T}\left((\mathrm{Ad}\otimes\tau_\lambda)(k)^{-1}\sum_{\beta\in\Delta_p} E^{*}_{\lambda+\beta}E_{\lambda+\beta}(Z\otimes v)\right) \\
&= \sum_{\beta\in\Delta_p} m(\lambda+\beta)\tilde{T}(E^{*}_{\lambda+\beta}\tau_{\lambda+\beta}(k)^{-1}E_{\lambda+\beta}(Z\otimes v)) \\
&= \sum_{\beta\in\Delta_p} m(\lambda+\beta) f_{\tilde{T}E^{*}_{\lambda+\beta}\otimes E_{\lambda+\beta}(Z\otimes v)}(k).
\end{aligned}$$

Therefore the assertion holds. □

For $\mu\in D_M$ and $w\in W(\mathfrak{a})$, define $w\mu\in D_M$ as $w\sigma_\mu=\sigma_{w\mu}$. In the following discussion, R_λ is an abbreviation of R_{τ_λ} and when there is no possibility of confusion, we shall use similar abbreviations. The next lemma is immediately obtained.

Lemma 5.3. *Retain the notation above. If $m(\lambda+\beta)\neq 0$ then it follows that*

$$R_{\lambda+\beta}(w)\mathcal{M}_\mu(Z;\lambda+\beta,\lambda) = -\mathcal{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w).$$

Proof. We compute

$$\begin{aligned}
(R(w)(\phi_Z f_{T\otimes v}))(k) &= \frac{\langle \mathrm{Ad}(kw)^{-1}Z, H\rangle}{\langle H,H\rangle} T(\tau_\lambda(kw)^{-1}v) \\
&= \frac{\langle \mathrm{Ad}(k)^{-1}Z, \mathrm{Ad}(w)H\rangle}{\langle H,H\rangle} T\tau_\lambda(w)^{-1}(\tau_\lambda(k)^{-1}v) \\
&= -(\phi_Z f_{R_\lambda(w)(T\otimes v)})(k).
\end{aligned} \tag{5.5}$$

Noting $f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}\in\mathcal{H}^{\sigma_\mu,\nu}(\tau_{\lambda+\beta})$ and $f_{R_\lambda(w)(T\otimes v)}\in\mathcal{H}^{w\sigma_\mu,w\nu}(\tau_\lambda)$, we see that

$$\begin{aligned}
(R(w)(\phi_Z f_{T\otimes v}))(k) &= \sum_{\beta\in\Delta_p} m(\lambda+\beta) f_{R_{\lambda+\beta}(w)\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k), \\
(\phi_Z f_{R_\lambda(w)(T\otimes v)})(k) &= \sum_{\beta\in\Delta_p} m(\lambda+\beta) f_{\mathcal{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w)(T\otimes v)}(k).
\end{aligned} \tag{5.6}$$

Substituting (5.6) into (5.5) and comparing side by side, we obtain the assertion. □

Combining Theorem 3.2 and Proposition 5.2, we have the following theorem.

Theorem 5.4. *Let $\mu \in D_M$ and $\lambda \in D_K(\mu)$. Then there exists $\eta_\lambda(\omega_{2\alpha}) \in \mathrm{C}$ such that*

$$\begin{aligned}
&(\pi_{\sigma_\mu,\nu}(Z)f_{T\otimes v})(k) \\
&\quad= \sum_{\beta\in\Delta_p}\left\{\frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle} + \frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle} - \frac{\eta_{\lambda+\beta}(\omega_{2\alpha})-\eta_\lambda(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\} \\
&\qquad\qquad \times m(\lambda+\beta)f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k),
\end{aligned}$$

$$\begin{aligned}
&(\pi_{w\sigma_\mu,w\nu}(Z)f_{R_\lambda(w)(T\otimes v)})(k) \\
&\quad= \sum_{\beta\in\Delta_p}\left\{-\frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle} + \frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle} - \frac{\eta_{\lambda+\beta}(\omega_{2\alpha})-\eta_\lambda(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\} \\
&\qquad\qquad \times m(\lambda+\beta)f_{\mathcal{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w)(T\otimes v)}(k).
\end{aligned}$$

Proof. By using Proposition 5.2, we deduce

$$\begin{aligned}
&(\phi_Z f_{T\otimes v})(k;\omega_t) - \phi_Z(k)f_{T\otimes v}(k;\omega_t) \\
&\qquad= \sum_{\beta\in\Delta_p}\langle 2\lambda+2\delta_K+\beta,\beta\rangle m(\lambda+\beta)f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k)
\end{aligned}$$

On the other side, under the assumption that $\dim \mathrm{Hom}_M(V_\lambda, H_\mu) = 1$, there exists $\eta_\lambda(\omega_{2\alpha}) \in \mathrm{C}$ such that

$$T\tau_\lambda(\omega_{2\alpha}) = \eta_\lambda(\omega_{2\alpha})T,$$

and hence

$$f_{T\otimes v}(k;\omega_{2\alpha}) = T\tau_\lambda(\omega_{2\alpha})(\tau_\lambda(k)^{-1}v) = \eta_\lambda(\omega_{2\alpha})f_{T\otimes v}(k).$$

Likewise we have

$$f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k;\omega_{2\alpha}) = \eta_{\lambda+\beta}(\omega_{2\alpha})f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).$$

Accordingly we get

$$\begin{aligned}
&(\phi_Z f_{T\otimes v})(k;\omega_{2\alpha}) - \phi_Z(k)f_{T\otimes v}(k;\omega_{2\alpha}) \\
&\qquad= \sum_{\beta\in\Delta_p}(\eta_{\lambda+\beta}(\omega_{2\alpha})-\eta_\lambda(\omega_{2\alpha}))m(\lambda+\beta)f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).
\end{aligned}$$

Noting

$$\begin{aligned}
f_{T\tau_\lambda(w)^{-1}\otimes v}(k;\omega_{2\alpha}) &= T\tau_\lambda(w)^{-1}\tau_\lambda(\omega_{2\alpha})(\tau_\lambda(k)^{-1}v) \\
&= T\tau_\lambda(\omega_{2\alpha})\tau_\lambda(w)^{-1}(\tau_\lambda(k)^{-1}v) \\
&= \eta_\lambda(\omega_{2\alpha})f_{T\tau_\lambda(w)^{-1}\otimes v}(k),
\end{aligned}$$

we can get immediately the second equation in Theorem 5.4. □

The objective in this section is to prove the next theorem.

Theorem 5.5. *Let $\mu \in D_M$ and $\lambda \in D_K(\mu)$. Suppose $\lambda + \beta \in D_K(\mu)$. If $m(\lambda+\beta) \neq 0$, then we have the following recursion formula.*

$$\begin{aligned}\left\{-\frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle} + \frac{\langle 2\lambda + 2\delta_K + \beta, \beta\rangle}{2\langle \alpha, \alpha\rangle} - \frac{\eta_{\lambda+\beta}(\omega_{2\alpha}) - \eta_\lambda(\omega_{2\alpha})}{4\langle \alpha, \alpha\rangle}\right\} C_{\tau_\lambda}(\sigma_\mu : \nu) \\ = -\left\{\frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle} + \frac{\langle 2\lambda + 2\delta_K + \beta, \beta\rangle}{2\langle \alpha, \alpha\rangle} - \frac{\eta_{\lambda+\beta}(\omega_{2\alpha}) - \eta_\lambda(\omega_{2\alpha})}{4\langle \alpha, \alpha\rangle}\right\} C_{\tau_{\lambda+\beta}}(\sigma_\mu : \nu).\end{aligned}$$

Proof. We first recall

$$A(w, \sigma_\mu, \nu)\pi_{\sigma_\mu,\nu}(Z) f_{T\otimes v} = \pi_{w\sigma_\mu, w\nu}(Z) A(w, \sigma_\mu, \nu) f_{T\otimes v}. \tag{5.7}$$

Applying Theorem 5.4 to (5.7), we have that

$$\begin{aligned}&\text{the right-hand side of (5.7)} = C_{\tau_\lambda}(\sigma_\mu : \nu)(\pi_{w\sigma_\mu, w\nu}(Z) f_{R_\lambda(w)(T\otimes v)})(k) \\ &= C_{\tau_\lambda}(\sigma_\mu : \nu) \sum_{\beta \in \Delta_{\mathfrak{p}}} \left\{-\frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle} + \frac{\langle 2\lambda + 2\delta_K + \beta, \beta\rangle}{2\langle \alpha, \alpha\rangle} - \frac{\eta_{\lambda+\beta}(\omega_{2\alpha}) - \eta_\lambda(\omega_{2\alpha})}{4\langle \alpha, \alpha\rangle}\right\} \\ &\qquad \times m(\lambda+\beta) f_{\mathcal{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w)(T\otimes v)}(k).\end{aligned}$$

Similarly, taking into account Lemma 5.3, we have that

$$\begin{aligned}&\text{the left-hand side of (5.7)} \\ &= A(w, \sigma_\mu, \nu)\Bigg[\sum_{\beta \in \Delta_{\mathfrak{p}}} \left\{\frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle} + \frac{\langle 2\lambda + 2\delta_K + \beta, \beta\rangle}{2\langle \alpha, \alpha\rangle} - \frac{\eta_{\lambda+\beta}(\omega_{2\alpha}) - \eta_\lambda(\omega_{2\alpha})}{4\langle \alpha, \alpha\rangle}\right\} \\ &\qquad \times m(\lambda+\beta) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k)\Bigg]\end{aligned}$$

$$\begin{aligned}&= \sum_{\beta \in \Delta_{\mathfrak{p}}} \left\{\frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle} + \frac{\langle 2\lambda + 2\delta_K + \beta, \beta\rangle}{2\langle \alpha, \alpha\rangle} - \frac{\eta_{\lambda+\beta}(\omega_{2\alpha}) - \eta_\lambda(\omega_{2\alpha})}{4\langle \alpha, \alpha\rangle}\right\} \\ &\qquad \times C_{\tau_{\lambda+\beta}}(\sigma_\mu : \nu) m(\lambda+\beta) f_{R_{\lambda+\beta}(w)\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k)\end{aligned}$$

$$\begin{aligned}&= \sum_{\beta \in \Delta_{\mathfrak{p}}} \left[-\left\{\frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle} + \frac{\langle 2\lambda + 2\delta_K + \beta, \beta\rangle}{2\langle \alpha, \alpha\rangle} - \frac{\eta_{\lambda+\beta}(\omega_{2\alpha}) - \eta_\lambda(\omega_{2\alpha})}{4\langle \alpha, \alpha\rangle}\right\}\right] \\ &\qquad \times C_{\tau_{\lambda+\beta}}(\sigma_\mu : \nu) m(\lambda+\beta) f_{\mathcal{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w)(T\otimes v)}(k).\end{aligned}$$

From the definition of $\mathcal{M}_\mu$, we see that if $[\tau_\lambda : \sigma_\mu] = [\tau_{\lambda+\beta} : \sigma_\mu] = 1$, then $\mathcal{M}_\mu(Z; \lambda+\beta, \lambda) \neq 0$ for some $Z \in \mathfrak{p}_c$. This completes the proof of theorem. □

6. EXPLICIT EXPRESSION OF THE RECURSION FORMULA

In the remainder of this report, we shall restrict our attention to the case of $SU(n, 1)$.

Let us set $G = SU(n,1)$ $(n \geq 2)$. The Iwasawa decomposition of G is given as follows:

(6.1)

$$K = \left\{ \begin{pmatrix} X & \\ & u \end{pmatrix} : X \in U(n), u \in U(1), u \det X = 1 \right\},$$

(6.2)

$$M = \left\{ \begin{pmatrix} X & & \\ & u & \\ & & u \end{pmatrix} : X \in U(n-1), u \in U(1), u^2 \det X = 1 \right\},$$

(6.3)

$$\mathfrak{a} = \mathbf{R}H, \text{ where } H = E_{n,n+1} + E_{n+1,n},$$

$$A = \left\{ a_t = \begin{pmatrix} I_{n-1} & & \\ & \cosh t & \sinh t \\ & \sinh t & \cosh t \end{pmatrix} : t \in \mathbf{R} \right\},$$

(6.4)

$$N = \left\{ n(z,u) = \begin{pmatrix} I_{n-1} & z & -z \\ -z^* & 1-\omega/2 & \omega/2 \\ -z^* & -\omega/2 & 1+\omega/2 \end{pmatrix} : z \in \mathbf{C}^{n-1}, u \in \sqrt{-1}\mathbf{R}, \omega = |z|^2 - 2u \right\},$$

(6.5)

$$\bar{N} = \left\{ \bar{n}(z,u) = \begin{pmatrix} I_{n-1} & -z & -z \\ z^* & 1-\omega/2 & -\omega/2 \\ -z^* & \omega/2 & 1+\omega/2 \end{pmatrix} : z \in \mathbf{C}^{n-1}, u \in \sqrt{-1}\mathbf{R}, \omega = |z|^2 - 2u \right\},$$

where I_{n-1} denotes the unit matrix of order $n-1$.

We shall regard $\nu \in \mathfrak{a}_c^*$ as a complex number under the identification $\nu \to \nu(H)$. The following lemma is easy to obtain and hence omit the proof.

Lemma 6.1. *Let $\bar{n}(z,u)$ be as above. Then we have*

$$H(\bar{n}(z,u)) = \log|1+\omega|H,$$

$$\kappa(\bar{n}(z,u)) = \begin{pmatrix} I_{n-1} - \frac{2zz^*}{1+\omega} & \frac{-2z}{|1+\omega|} & 0 \\ \frac{2z^*}{1+\omega} & \frac{1-\omega}{|1+\omega|} & 0 \\ 0 & 0 & \frac{1+\omega}{|1+\omega|} \end{pmatrix}.$$

We shall next write down Δ_K^+, $\Delta_\mathfrak{p}$, D_K and D_M.

(6.6)

$$\Delta_K^+ = \{\varepsilon_i - \varepsilon_j \ (1 \leq i < j \leq n)\},$$

(6.7)

$$\Delta_\mathfrak{p} = \{\beta_j = \varepsilon_j - \varepsilon_{n+1}, -\beta_j \ (1 \leq j \leq n)\},$$

(6.8)

$$\delta_K = \frac{1}{2}\sum_{p=1}^{n}(n-2p+1)\varepsilon_p,$$

(6.9)

$$D_K = \left\{ (\lambda_1, \ldots, \lambda_n) \in \left(\frac{1}{n+1}\mathbf{Z}\right)^n : \lambda_p - \lambda_{p+1} \in \mathbf{N} \ (1 \leq p \leq n-1) \right\},$$

(6.10)

$$D_M = \left\{ (\mu_1, \ldots, \mu_{n-1}) \in \left(\frac{1}{n+1}\mathbf{Z}\right)^{n-1} : \mu_p - \mu_{p+1} \in \mathbf{N} \ (1 \leq p \leq n-2) \right\}.$$

Let $\lambda \in D_K$ and $\mu \in D_M$ be given as above. Then $[\tau_\lambda : \sigma_\mu] = 1$ iff the betweenness condition below holds:

$$(6.11) \quad \lambda_p - \mu_p \in \mathbf{Z},\ \mu_p - \lambda_{p+1} \in \mathbf{Z}\ (1 \leq p \leq n-1) \text{ and } \lambda_1 \geq \mu_1 \geq \dots \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

It follows from $\langle X, Y\rangle = 2(n+1)\operatorname{tr} XY$ that for $\lambda = (\lambda_1, \dots, \lambda_n) \in D_K$,

$$(6.12) \qquad \frac{\langle 2\lambda + 2\delta_K + \beta_j, \beta_j\rangle}{2\langle \alpha, \alpha\rangle} = 2\lambda_j + 2|\tau_\lambda| + n - 2j + 3,\ (1 \leq j \leq n),$$

where $|\tau_\lambda| = \sum_{p=1}^{n} \lambda_p$. We shall now compute $\eta_\lambda(\omega_{2\alpha})$. To do this, we will utilize the Gel'fand–Tsetlin basis for $\mathfrak{u}(n)$. Let $\lambda \in D_K$ and $\mu \in D_M(\lambda)$. We set

(6.13)

$$V_\lambda = \operatorname{Span}_{\mathbf{C}} \{v(\mathrm{m}) : \text{Gel'fand–Tsetlin scheme } \mathrm{m} = (m_n, \dots, m_1) \text{ with } m_n = \lambda\},$$

(6.14)

$$V_\lambda(\mu) = \operatorname{Span}_{\mathbf{C}} \{v(\mathrm{m}) : \text{Gel'fand–Tsetlin scheme } \mathrm{m} \text{ with } m_n = \lambda, m_{n-1} = \mu\}.$$

Put $X_p = E_{p,p+1}$, $Y_p = E_{p+1,p}$, $H_p = E^i_{p,p} - E^i_{p+1,p+1}$ and $H_0 = \operatorname{diag}(-i, \dots, -i, ni)$, where $E^i_{p,q}$ denotes the matrix whose (k, l)-component is $i\delta_{pk}\delta_{ql}$. Then it is known (cf. [11]) that there exists an irreducible unitary representation $(\tau_\lambda, V_\lambda)$ of K satisfying the following condition.

$$(6.15) \qquad \tau_\lambda(X_p)v(\mathrm{m}) = \sum_{j=1}^{p} A^j_p(\mathrm{m})v(\mathrm{m}^{+j}_p),$$

$$(6.16) \qquad \tau_\lambda(Y_p)v(\mathrm{m}) = \sum_{j=1}^{p} B^j_p(\mathrm{m})v(\mathrm{m}^{-j}_p),$$

$$(6.17) \qquad \tau_\lambda(H_p)v(\mathrm{m}) = \left\{ 2\sum_{j=1}^{p} m_{j,p} - \sum_{j=1}^{p-1} m_{j,p-1} - \sum_{j=1}^{p+1} m_{j,p+1} \right\} \sqrt{-1}v(\mathrm{m}),$$

$$(6.18) \qquad \tau_\lambda(H_0)v(\mathrm{m}) = -(n+1)\sum_{j=1}^{n} m_{j,n}\sqrt{-1}v(\mathrm{m}),$$

where $\mathrm{m}^{\pm j}_p$ is the Gel'fand–Tsetlin scheme obtained by replacing $m_{j,p}$ with $m_{j,p} \pm 1$ in m_p of m. Moreover, as a representation of M, $(\tau_\lambda|_M, V_\lambda(\mu))$ is unitarily equivalent with (σ_μ, H_μ) and

$$(6.19) \qquad V_\lambda = \sum_{\mu \in D_M(\lambda)} V_\lambda(\mu).$$

We define the Gel'fand–Tsetlin scheme $\mathrm{m}_{\lambda,\mu} = (m_n, \dots, m_1)$ by

$$(6.20) \qquad m_n = \lambda = (\lambda_1, \dots, \lambda_n),\ m_p = (\mu_1, \dots, \mu_p),\ (1 \leq p \leq n-1).$$

Because

$$(6.21) \qquad \begin{aligned} Y_{2\alpha} &= \frac{1}{2\sqrt{n+1}} \operatorname{diag}(\overbrace{0, \dots, 0}^{n-1}, i, -i) \\ &= \frac{-1}{2n\sqrt{n+1}}\{H_0 + H_1 + 2H_2 + \dots + (n-1)H_{n-1}\}, \end{aligned}$$

we have from (6.17) and (6.18) that

$$\tau_\lambda(Y_{2\alpha})v(\mathrm{m}_{\lambda,\mu}) = \frac{1}{2\sqrt{n+1}}(2|\tau_\lambda| - |\sigma_\mu|)\sqrt{-1}v(\mathrm{m}_{\lambda,\mu}), \tag{6.22}$$

where $|\tau_\lambda| = \sum_{p=1}^{n} \lambda_p$ and $|\sigma_\mu| = \sum_{p=1}^{n-1} \mu_p$. Since $\omega_{2\alpha} = -Y_{2\alpha}^2$, it follows

$$\tau_\lambda(\omega_{2\alpha})v(\mathrm{m}_{\lambda,\mu}) = \frac{1}{4(n+1)}(2|\tau_\lambda| - |\sigma_\mu|)^2 v(\mathrm{m}_{\lambda,\mu}). \tag{6.23}$$

Therefore, it follows from $Tv(\mathrm{m}_{\lambda,\mu}) \neq 0$ that

$$\eta_\lambda(\omega_{2\alpha}) = \frac{1}{4(n+1)}(2|\tau_\lambda| - |\sigma_\mu|)^2. \tag{6.24}$$

Taking into account $|\tau_{\lambda+\beta_j}| = |\tau_\lambda| + 1$, we have

$$\frac{\eta_{\lambda+\beta_j}(\omega_{2\alpha}) - \eta_\lambda(\omega_{2\alpha})}{4(\alpha,\alpha)} = 2|\tau_\lambda| - |\sigma_\mu| + 1. \tag{6.25}$$

By using these results, we will here write down the recursion formula of the Harish-Chandra C-function. Put $\hat{\mu} = (\mu_1, \ldots, \mu_{n-1}, \mu_{n-1}) \in D_K(\mu)$. Then from (6.12) and (6.25), Theorem 5.5 can be written as follows: if $[\tau_{\lambda+\beta_j} : \sigma_\mu] = 1$ then

$$(-\nu+2\lambda_j+|\sigma_\mu|+n-2j+2)C_{\tau_\lambda}(\sigma_\mu : \nu) = -(\nu+2\lambda_j+|\sigma_\mu|+n-2j+2)C_{\tau_{\lambda+\beta_j}}(\sigma_\mu : \nu). \tag{6.26}$$

Accordingly, using the preceding recursion formula and shifting the parameters as $\mu_p \to \lambda_p$ and $\mu_{n-1} \to \lambda_n$, we can find the following theorem.

Theorem 6.2. *Keep the notation above. We get the following.*

$$\begin{aligned} C_{\tau_\lambda}(\sigma_\mu : \nu) = \\ (-1)^{|\tau_\lambda|-|\tau_{\hat{\mu}}|} \prod_{j=1}^{n-1} & \frac{\left(\frac{-\nu+n+|\sigma_\mu|}{2} + \mu_j - j + 1\right)_{\lambda_j-\mu_j}}{\left(\frac{\nu+n+|\sigma_\mu|}{2} + \mu_j - j + 1\right)_{\lambda_j-\mu_j}} \frac{\left(\frac{-\nu+n-|\sigma_\mu|}{2} - \mu_{n-1}\right)_{\mu_{n-1}-\lambda_n}}{\left(\frac{\nu+n-|\sigma_\mu|}{2} - \mu_{n-1}\right)_{\mu_{n-1}-\lambda_n}} \\ & \times C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu). \end{aligned}$$

7. EXPRESSION OF $C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu)$

Theorem 6.2 says that for getting the expression of the Harish-Chandra C-function, it suffices to calculate $C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu)$. To do this, we shall get the integral expression of $C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu)$.

Let $\mu = (\mu_1, \ldots, \mu_{n-1}) \in D_M$. Then, as mentioned in §6, the Gel'fand–Tsetlin basis $v(\mathrm{m}_{\hat{\mu}}) = v(\mathrm{m}_{\hat{\mu},\mu})$ is a highest weight vector of $(\tau_{\hat{\mu}}, V_{\hat{\mu}})$. We choose $T \in \mathrm{Hom}_M(V_{\hat{\mu}}, V_{\hat{\mu}}(\mu))$ as a canonical projection. Noting $Tv(\mathrm{m}_{\hat{\mu}}) = v(\mathrm{m}_{\hat{\mu}}) \neq 0$ and remembering $TC_{\tau_{\hat{\mu}}}(\nu) = C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu)T$, we have

$$\begin{aligned} C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu) &= \langle TC_{\tau_{\hat{\mu}}}(\nu)v(\mathrm{m}_{\hat{\mu}}), v(\mathrm{m}_{\hat{\mu}})\rangle \\ &= \langle C_{\tau_{\hat{\mu}}}(\nu)v(\mathrm{m}_{\hat{\mu}}), v(\mathrm{m}_{\hat{\mu}})\rangle \\ &= \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))}\langle \tau_{\hat{\mu}}(\kappa(\bar{n})^{-1})v(\mathrm{m}_{\hat{\mu}}), v(\mathrm{m}_{\hat{\mu}})\rangle d\bar{n}. \end{aligned} \tag{7.1}$$

Putting $\phi_{\hat{\mu}}(k) = \langle \tau_{\hat{\mu}}(k) v(\mathrm{m}_{\hat{\mu}}), v(\mathrm{m}_{\hat{\mu}}) \rangle$, we obtain

$$C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))} \phi_{\hat{\mu}}(\kappa(\bar{n})^{-1}) d\bar{n}. \tag{7.2}$$

As a preparation for computing $\phi_{\hat{\mu}}(\kappa(\bar{n})^{-1})$, we shall write down the action of $\tau_\Lambda(\kappa(\bar{n})^{-1})$ in the case of the alternating tensor representation Λ. Let (Φ, C^n) be the usual representation of K, that is, for $k = \begin{pmatrix} X & \\ & u \end{pmatrix} \in K$ and $z \in \mathrm{C}^n$, $\Phi(k)z = u^{-1}Xz$. We denote by (Φ_0, C) the representation K defined by $\Phi_0(k)z = u^{-1}z$. Let $(\Phi_r, \wedge^r \mathrm{C}^n)$ be the alternating tensor representation of Φ. Then Φ_r $(1 \le r \le n-1)$ is irreducible with highest weight $\Lambda_r = \varepsilon_1 + \cdots + \varepsilon_r - r\varepsilon_{n+1}$ and $e_1 \wedge \cdots \wedge e_r$ is its highest weight vector. A simple computation implies

$$\langle \Phi(\kappa(\bar{n}(z,u))^{-1}) e_p, e_q \rangle = \frac{1+\omega}{|1+\omega|} \left(\delta_{pq} - \frac{2 z_q \bar{z}_p}{1+\bar{\omega}} \right). \tag{7.3}$$

Thus we have

$$\phi_{\Lambda_r}(\kappa(\bar{n}(z,u))^{-1}) = \left(\frac{1+\omega}{|1+\omega|} \right)^r \left(1 - \frac{2\sum_{p=1}^r |z_p|^2}{1+\bar{\omega}} \right). \tag{7.4}$$

In order to compute $\phi_{\hat{\mu}}(\kappa(\bar{n})^{-1})$ for any $\mu \in D_M$, we shall apply the Gauss decomposition of $K_c \cong GL(n, \mathrm{C})$ and prove that $\phi_{\hat{\mu}}(\kappa(\bar{n})^{-1})$ can be represented as the products of $\phi_{\Lambda_r}(\kappa(\bar{n})^{-1})$. We consider τ_λ as a holomorphic representation on K_c. We write $\mathfrak{k}_+$ (resp. $\mathfrak{k}_-$) for the sum of all positive root subspaces (resp. negative root subspaces) with respect to $(\mathfrak{k}_c, \mathfrak{t}_c)$. Let K_+ and K_- denote the analytic subgroups of K_c corresponding to $\mathfrak{k}_+$ and $\mathfrak{k}_-$, respectively. We note that $K_1 \exp \mathfrak{t}_c K_2$ is dense in K_c and ϕ_λ is holomorphic on K_c. For $\lambda \in D'_K$, it follows from the definition of ϕ_λ that

$$\phi_\lambda(k_1 \exp H k_2) = \phi_\lambda(\exp H) = e^{\lambda(H)}, \quad (k_1 \in K_+, k_2 \in K_-, H \in \mathfrak{t}_c). \tag{7.5}$$

Because $\hat{\mu} = \sum_{p=1}^{n-2} (\mu_p - \mu_{p+1}) \Lambda_p - (n+1)\mu_{n-1}\varepsilon_{n+1}$, we obtain

$$\phi_{\hat{\mu}}(k_1 \exp H k_2) = \prod_{p=1}^{n-2} \phi_{\Lambda_p}(k_1 \exp H k_2)^{\mu_p - \mu_{p+1}} \phi_{\Lambda_0}(k_1 \exp H k_2)^{(n+1)\mu_{n-1}}. \tag{7.6}$$

Therefore we have

$$\begin{aligned} &\phi_{\hat{\mu}}(\kappa(\bar{n}(z,u))^{-1}) \\ &= (1+\omega)^{(|\sigma_\mu| + 2\mu_{n-1})/2} (1+\bar{\omega})^{-(|\sigma_\mu| + 2\mu_1)/2} \prod_{p=1}^{n-2} \left(1 + \bar{\omega} - 2\sum_{j=1}^p |z_j|^2 \right)^{\mu_p - \mu_{p+1}} \end{aligned} \tag{7.7}$$

Using (7.7) and carrying out the integration, we can get the explicit expression of $C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu)$. In this case, $\rho = n$ and $\bar{N} = \mathrm{C}^{n-1} \times \mathrm{R}$ and

$$\int_{\mathrm{C}^{n-1} \times \mathrm{R}} |1+\omega|^{-2n} dz d\bar{z} du = \frac{(2\pi)^n}{(n-1)!} (= c_n, \text{ say}). \tag{7.8}$$

Applying Lemma 5.1 and (7.7), we have

$$\begin{aligned} c_n C_{\tau_{\hat{\mu}}}(\sigma_\mu : \nu) = \int_{\mathrm{C}^{n-1} \times \mathrm{R}} &(1+\omega)^{-(\nu+n-|\sigma_\mu| - 2\mu_{n-1})/2} (1+\bar{\omega})^{-(\nu+n+|\sigma_\mu|+2\mu_1)/2} \\ &\times \prod_{p=1}^{n-2} \left(1 + \bar{\omega} - 2\sum_{j=1}^p |z_j|^2 \right)^{\mu_p - \mu_{p+1}} \quad dz d\bar{z} du. \end{aligned} \tag{7.9}$$

Here we use the following lemma.

Lemma 7.1 (cf. [5], [9]). *Let* $\lambda \in \mathbf{C}$, $\ell \in \mathbf{Z}$, $q_j \in \mathbf{N}$ $(1 \leq q_j \leq n-1)$, *and* $F = 1 + \frac{1}{2}(|z_1|^2 + \cdots + |z_{n-1}|^2) + \sqrt{-1}u$. *Then*

$$\int_{\mathbf{C}^{n-1}\times\mathbf{R}} \bar{F}^{\frac{\lambda+\ell}{2}} F^{\frac{\lambda-\ell}{2}} \prod_{p=1}^{n-1} \left(\bar{F} - \sum_{j=1}^{p} |z_j|^2 \right)^{q_p} dz d\bar{z} du =$$

$$\frac{(2\pi)^n 2^{\lambda+n+q_1+\cdots+q_{n-1}} \Gamma(-\lambda - n - q_1 - \cdots - q_{n-1})}{\prod_{j=1}^{n-1} \left(-\frac{\lambda+\ell}{2} - q_1 - \cdots - q_{j-1} - j\right) \Gamma\left(-\frac{\lambda+\ell}{2} - q_1 - \cdots - q_{n-1} - n + 1\right) \Gamma\left(-\frac{\lambda-\ell}{2}\right)}.$$

Taking into account (7.8), we obtain the following proposition.

Proposition 7.2. *With the notation above, we have the following.*

$$C_{\tau_\mu}(\sigma_\mu : \nu) = \frac{(n-1)! 2^{-\nu+n} \Gamma(\nu)}{\prod_{j=1}^{n-1} \left(\frac{\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right) \Gamma\left(\frac{\nu+n-|\sigma_\mu|}{2} - \mu_{n-1}\right) \Gamma\left(\frac{\nu-n+|\sigma_\mu|}{2} + 1 + \mu_{n-1}\right)}.$$

8. EXPRESSION OF THE HARISH-CHANDRA C-FUNCTION

Combining Theorem 6.1 and Proposition 7.2, we can get the explicit expressions of the Harish-Chandra C-functions for $SU(n,1)$.

Theorem 8.1. *The Harish-Chandra C-functions $C_{\tau_\lambda}(\sigma_\mu : \nu)$ for $SU(n,1)$ associated with $\tau_\lambda \in \hat{K}$ and $\sigma_\mu \in \hat{M}(\tau_\lambda)$ are given as follows:*

$$C_{\tau_\lambda}(\sigma_\mu : \nu) = \frac{(n-1)! 2^{-\nu+n} \Gamma(\nu) \prod_{j=1}^{n-1} \Gamma\left(\frac{\nu-n-|\sigma_\mu|}{2} + j - \mu_j\right) \prod_{j=1}^{n-1} \Gamma\left(\frac{\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right)}{\prod_{j=1}^{n} \Gamma\left(\frac{\nu-n-|\sigma_\mu|}{2} + j - \lambda_j\right) \prod_{j=1}^{n} \Gamma\left(\frac{\nu+n+|\sigma_\mu|}{2} - j + 1 + \lambda_j\right)}.$$

As well known, because the Plancherel measure $\mu(\sigma_\mu, \nu)$ can be given by the Harish-Chandra C-function $C_{\tau_\lambda}(\sigma_\mu : \nu)$ by using the relation $\mu(\sigma_\mu, \nu) = |C_{\tau_\lambda}(\sigma_\mu : \nu)|^{-2}$, Theorem 8.1 implies immediately the following corollary.

Corollary 8.2. *The Plancherel measure for $SU(n,1)$ is following.*

If $n - |\sigma_\mu| - 2\mu_{n-1}$ is an odd integer,

$$\mu(\sigma_\mu, \nu) = \frac{-\pi\nu}{(n-1)!^2 2^{2n-1}} \prod_{j=1}^{n-1} \left(\frac{\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right) \left(\frac{-\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right) \tan\frac{\pi\nu}{2},$$

If $n - |\sigma_\mu| - 2\mu_{n-1}$ is an even integer,

$$\mu(\sigma_\mu, \nu) = \frac{\pi\nu}{(n-1)!^2 2^{2n-1}} \prod_{j=1}^{n-1} \left(\frac{\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right) \left(\frac{-\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right) \cot\frac{\pi\nu}{2}.$$

9. DETERMINATION OF THE COMPOSITION SERIES

In this section, we shall show that the expression of the Harish-Chandra C-function can be utilizable to write down the composition series of the nonunitary principal series representations.

Let $\mu = (\mu_1, \ldots, \mu_{n-1}) \in D_M$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in D_K(\mu)$. Suppose $\nu \in \mathbf{R}$ and $\nu > 0$. For $1 \leq j \leq n-1$, we set $h_j = (\nu - n - |\sigma_\mu|)/2 + j - \mu_j$ and $k_j = (\nu + n + |\sigma_\mu|)/2 - j + \mu_j$ and assume $h_j \in \mathbf{Z}$ and $k_j \in \mathbf{Z}$. We choose $0 \leq a, b \leq n-1$ satisfying the following conditions:

(9.1)
$$\begin{gathered} h_1 < \cdots < h_a \leq 0 < h_{a+1} < \cdots < h_{n-1}, \\ k_1 > \cdots > k_b > 0 \geq k_{b+1} > \cdots > k_{n-1}. \end{gathered}$$

When $h_a = 0$ and $k_{b+1} \neq 0$, we set

(9.2)
$$\begin{aligned} S^{\sigma_\mu}_{a,-}(b) &= \{\lambda = (\lambda_1, \ldots, \lambda_n) \in D_K(\mu) : \mu_{b+1} \leq \lambda_{b+1} \leq a + b - n - |\sigma_\mu| - \mu_a\}, \\ \mathcal{H}^{\sigma_\mu}_{a,-}(b) &= \sum_{\lambda \in S^{\sigma_\mu}_{a,-}(b)} V_\lambda. \end{aligned}$$

When $h_a \neq 0$ and $k_{b+1} = 0$, we set

(9.3)
$$\begin{aligned} S^{\sigma_\mu}_{b,+}(a) &= \{\lambda = (\lambda_1, \ldots, \lambda_n) \in D_K(\mu) : a + b - n - |\sigma_\mu| - \mu_{b+1} + 2 \leq \lambda_{a+1} \leq \mu_a\}, \\ \mathcal{H}^{\sigma_\mu}_{b,+}(a) &= \sum_{\lambda \in S^{\sigma_\mu}_{b,+}(a)} V_\lambda. \end{aligned}$$

When $h_a \neq 0$ and $k_{b+1} \neq 0$, we set

(9.4)
$$\begin{aligned} S^{\sigma_\mu}_{\nu,+}(a) &= \left\{\lambda = (\lambda_1, \ldots, \lambda_n) \in D_K(\mu) : \frac{\nu - n - |\sigma_\mu|}{2} + a + 1 \leq \lambda_{a+1} \leq \mu_a\right\}, \\ S^{\sigma_\mu}_{\nu,-}(b) &= \left\{\lambda = (\lambda_1, \ldots, \lambda_n) \in D_K(\mu) : \mu_{b+1} \leq \lambda_{b+1} \leq -\frac{\nu + n + |\sigma_\mu|}{2} + b\right\}, \\ \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) &= \sum_{\lambda \in S^{\sigma_\mu}_{\nu,+}(a)} V_\lambda, \\ \mathcal{H}^{\sigma_\mu}_{\nu,-}(b) &= \sum_{\lambda \in S^{\sigma_\mu}_{\nu,-}(b)} V_\lambda. \end{aligned}$$

In addition, in case $a \neq b$, we set

(9.5)
$$\begin{aligned} S^{\sigma_\mu}_{\nu}(a,b) &= S^{\sigma_\mu}_{\nu,+}(a) \cap S^{\sigma_\mu}_{\nu,-}(b), \\ \mathcal{H}^{\sigma_\mu}_{\nu}(a,b) &= \sum_{\lambda \in S^{\sigma_\mu}_{\nu}(a,b)} V_\lambda. \end{aligned}$$

Theorem 9.1 (cf. [8]). *Retain the notation and assumption above.* $\pi_{\sigma_\mu,\nu}$ *is reducible iff* $(\nu - n - |\sigma_\mu|)/2 + j - \mu_j \in \mathbf{Z}\backslash\{0\}$ *or* $(\nu + n + |\sigma_\mu|)/2 - j + \mu_j \in \mathbf{Z}\backslash\{0\}$ *for* $1 \leq j \leq n-1$.

Theorem 9.2 (cf. [8]). *Assume $\pi_{\sigma_\mu,\nu}$ is reducible. Choose $0 \leq a \leq b \leq n-1$ satisfying the relations in (9.1). Then the composition series of $\pi_{\sigma_\mu,\nu}$ are given as follows:*

(1) *$h_a = 0$ and $k_{b+1} \neq 0$.*

$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{a,-}(b) \supset \{0\}.$$

(2) *$h_a \neq 0$ and $k_{b+1} = 0$.*

$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{b,+}(a) \supset \{0\}.$$

(3) *$h_a \neq 0$ and $k_{b+1} \neq 0$ and $a = b$.*

$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(a) \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) \supset \{0\},$$
$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(a) \supset \mathcal{H}^{\sigma_\mu}_{\nu,-}(a) \supset \{0\}.$$

(4) *$h_a \neq 0$ and $k_{b+1} \neq 0$ and $a < b$.*

$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(b) \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) \supset \mathcal{H}^{\sigma_\mu}_{\nu}(a,b) \supset \{0\},$$
$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(b) \supset \mathcal{H}^{\sigma_\mu}_{\nu,-}(b) \supset \mathcal{H}^{\sigma_\mu}_{\nu}(a,b) \supset \{0\}.$$

We shall next write down the composition series of $\pi_{\sigma_\mu,0}$. Let $\mu = (\mu_1, \ldots, \mu_{n-1}) \in D_M$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in D_K(\mu)$. Then Theorem 5.4 can be written as follows:

(9.6)

$$\begin{aligned}(\pi_{\sigma_\mu,0}(Z)f_{T\otimes v})(k) = &\sum_{j=1}^{n}(n - 2j + 2 + 2\lambda_j + |\sigma_\mu|)m(\lambda + \beta_j)f_{\mathcal{M}_\mu(Z;\lambda+\beta_j,\lambda)(T\otimes v)}(k) \\ &+ \sum_{j=1}^{n}(-n + 2j - 2\lambda_j - |\sigma_\mu|)m(\lambda - \beta_j)f_{\mathcal{M}_\mu(Z;\lambda-\beta_j,\lambda)(T\otimes v)}(k).\end{aligned}$$

Consequently, we have the following theorem.

Theorem 9.3 (cf. [8]). *Retain the notation above.*

(1) *$\pi_{\sigma_\mu,0}$ is reducible iff $(-n - |\sigma_\mu|)/2 + j - \mu_j \in \mathbf{Z}\backslash\{0\}$ $(1 \leq j \leq n-1)$.*

(2) *Assume $\pi_{\sigma_\mu,0}$ is reducible. Choose $0 \leq a \leq n-1$ satisfying the relation in (9.1). Then the composition series of $\pi_{\sigma_\mu,0}$ can be written as follows:*

$$\mathcal{H}^{\sigma_\mu,0} \supset \mathcal{H}^{\sigma_\mu}_{0,+}(a) \supset \{0\},$$
$$\mathcal{H}^{\sigma_\mu,0} \supset \mathcal{H}^{\sigma_\mu}_{0,-}(a) \supset \{0\}.$$

We shall determine which parts of the composition series are unitarizable. To do this, by using the expression of the Harish-Chandra C-function, we shall get the conditions for the subquotients of the composition series to admit the positive definite Hermitian sesquilinear forms.

Theorem 9.4 (cf. [5], [7]). *Retain the notation above. The following subquotients admit the positive definite Hermitian sesquilinear forms.*

(1) *$h_a = 0$ and $k_{b+1} \neq 0$.*
$\mathcal{H}^{\sigma_\mu}_{a,-}(b)$ with $\mu_{a+1} = \cdots = \mu_b$ and $h_{a+1} = 1$,
$\mathcal{H}^{\sigma_\mu,\nu}(K)/\mathcal{H}^{\sigma_\mu}_{a,-}(b)$ with $\mu_{a+1} = \cdots = \mu_b$ and $h_{a+1} = k_b = 1$ (that is, $\nu = b-a+1$).

(2) $h_a \neq 0$ *and* $k_{b+1} = 0$.
$\mathcal{H}^{\sigma_\mu}_{b,+}(a)$ *with* $\mu_{a+1} = \cdots = \mu_b$ *and* $k_b = 1$,
$\mathcal{H}^{\sigma_\mu,\nu}(K)/\mathcal{H}^{\sigma_\mu}_{b,+}(a)$ *with* $\mu_{a+1} = \cdots = \mu_b$ *and* $h_{a+1} = k_b = 1$ *(that is,* $\nu = b-a+1$*).*
(3) $h_a \neq 0$, $k_{b+1} \neq 0$, *and* $a = b$.
$\mathcal{H}^{\sigma_\mu}_{\nu,+}(a)$ *and* $\mathcal{H}^{\sigma_\mu}_{\nu,-}(a)$.
$\mathcal{H}^{\sigma_\mu,\nu}(K)/(\mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(a))$ *with* $\nu = 1$.
(4) $h_a \neq 0$, $k_{b+1} \neq 0$, *and* $a < b$.
$\mathcal{H}^{\sigma_\mu}_{\nu}(a,b)$ *with* $\mu_{a+1} = \cdots = \mu_b$,
$\mathcal{H}^{\sigma_\mu}_{\nu,+}(a)/\mathcal{H}^{\sigma_\mu}_{\nu}(a,b)$ *with* $\mu_{a+1} = \cdots = \mu_b$ *and* $k_b = 1$,
$\mathcal{H}^{\sigma_\mu}_{\nu,-}(b)/\mathcal{H}^{\sigma_\mu}_{\nu}(a,b)$ *with* $\mu_{a+1} = \cdots = \mu_b$ *and* $h_{a+1} = 1$,
$\mathcal{H}^{\sigma_\mu,\nu}(K)/(\mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(b))$ *with* $\mu_{a+1} = \cdots = \mu_b$ *and* $h_{a+1} = k_b = 1$ *(that is,* $\nu = b - a + 1$*).*

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