

Jordan triple systems and K -spherical functions on Heisenberg groups

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Let $(K; H)$ be a pair of a Heisenberg Lie group H and compact group K acting on H as automorphisms. There are many studies about K -spherical functions on H if $(K \ltimes H, K)$ is a Gelfand pair (for example [BJR1] [BJR2] [D]). In this lecture, we treat the case that $H = V \times \mathbb{R}$, where V has PDHJTS structure, and that K is the connected component of automorphism group $\text{Aut}(V)$ of V . We see that each K -spherical function ϕ on H is described with the terms of Jordan triple systems and of Jordan algebras.

1. Jordan triple systems and Jordan algebras

A *Jordan triple system* (abbreviated as *JTS*) over a field \mathbb{K} is a vector space V over \mathbb{K} with trilinear map $\{\cdot, \cdot, \cdot\} : V \times V \times V \rightarrow V$ such that

$$\begin{aligned}\{x, y, z\} &= \{z, y, x\}, \\ \{a, b, \{x, y, z\}\} &= \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\},\end{aligned}$$

where $a, b, x, y, z \in V$. For $x, y \in V$ we define a linear map $x \square y : V \rightarrow V$ by $(x \square y)z = \{x, y, z\}$ for $z \in V$. A real JTS V is called *hermitian*, if V is a complex vector space and the trilinear map $(x, y, z) \mapsto \{x, y, z\}$ is \mathbb{C} -linear in x, z and \mathbb{C} -antilinear in y . We call a hermitian JTS V *positive definite* if the sesquilinear form $\text{tr}(x \square y)$ on V is positive definite. We often abbreviate positive definite hermitian JTS as *PDHJTS*.

The most important examples of JTS are Jordan algebras. A *Jordan algebra* over \mathbb{K} is a vector space V over \mathbb{K} with product satisfying

$$xy = yx, \quad x^2(xy) = x(x^2y),$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-T}\mathcal{E}\mathcal{X}$

for any $x, y \in V$. We write $xy = L(x)y$. Then $L(x)$ is a linear map on V . A real Jordan algebra V_0 is *Euclidean* if the symmetric form $(x, y) \mapsto \text{tr}L(xy)$ on V_0 is positive definite, and a complex Jordan algebra V is semisimple if the symmetric bilinear form $(x, y) \mapsto \text{tr}(xy)$ is non-degenerate. It is known that any semisimple complex Jordan algebra V has a Euclidean real form V_0 ; $V = V_0 \otimes \mathbb{C}$. Each semisimple complex Jordan algebra $V = V_0 \otimes \mathbb{C}$ has hermitian JTS structure with product

$$\{x, y, z\} = (x\bar{y})z + x(\bar{y}z) - \bar{y}(xz),$$

where $x, y, z \in V$ and \bar{y} is the conjugate with respect to V_0 . In particular, V is positive definite.

Any PDHJTS V naturally corresponds to a bounded symmetric domain (for example see [S]). The detail is as follows. We define a bounded domain \mathcal{D} by

$$\mathcal{D} = \{x \in V ; I_V - x \square x \text{ is positive definite}\},$$

where I_V is the identity map on V . Then \mathcal{D} is symmetric, the connected component G of isometry group $I(\mathcal{D})$ of \mathcal{D} is semisimple, the stabilizer K of $0 \in V$ in G is maximal compact and the pair (G, K) is a hermitian symmetric pair. We denote by $\mathfrak{g}, \mathfrak{k}$ the Lie algebras of G, K and $\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}$ their complexification respectively. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with Cartan involution θ and a decomposition of the complexification $\mathfrak{g}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_- + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+$, where \mathfrak{p}_+ (resp. \mathfrak{p}_-) is the sum of non-compact positive (resp. negative) root spaces of Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. We know that $\theta(\mathfrak{p}_{\pm}) = \mathfrak{p}_{\mp}$, $[\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}_{\mathbb{C}}$, $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\pm}] \subset \mathfrak{p}_{\pm}$ and define a trilinear product on \mathfrak{p}_+ by

$$\{x, y, z\} = -\frac{1}{2}[[x, \theta(y)], z], \quad x, y, z \in \mathfrak{p}_+.$$

Then \mathfrak{p}_+ is PDHJTS isomorphic to V .

We note that a PDHJTS V is coming from a Jordan algebra if and only if the bounded symmetric domain \mathcal{D} corresponding to V is holomorphically diffeomorphic to a Siegel domain of tube type. In particular there exists a PDHJTS V which is not coming from some Jordan algebra (for example the space $\text{Mat}(m, n; \mathbb{C})$ of all $m \times n$ matrices with entries in \mathbb{C} ($m \neq n$). We discuss about it at section 4). However, any PDHJTS V has a 'maximal' subPDHJTS V_1 which is coming from a Jordan algebra and corresponding bounded symmetric domains have the same real rank (cf. [KW] [S]). In fact, we identify V as \mathfrak{p}_+ above. Then we can take a maximal family $\{\alpha_1, \dots, \alpha_r\}$ of strongly orthogonal positive non-compact roots of $\mathfrak{g}_{\mathbb{C}}$, and take a family $\{x_1, \dots, x_r\}$ of nonzero root vectors such that x_i belongs to

α_i -root space and is a tripotent: $\{x_i, x_i, x_i\} = x_i$ for any $1 \leq i \leq r$. We call the number r the *rank* of V also.

We see that the sum $x_0 = \sum_{i=1}^r x_i$ is also a tripotent and the 1-eigenspace of $x_0 \square x_0$ in \mathfrak{p}_+ is what we want.

We call a PDHJTS V is *simple* if corresponding bounded symmetric domain \mathcal{D} is irreducible. After this we assume that any PDHJTS we treat is simple.

2. Multiplicity-free decompositions of polynomial rings

An automorphism g of a PDHJTS V is the linear map on V such that

$$g \cdot \{x, y, z\} = \{g \cdot x, g \cdot y, g \cdot z\}, \quad x, y, z \in V.$$

We see that the identity component $\text{Aut}_0(V)$ of the group $\text{Aut}(V)$ of automorphisms of V is locally isomorphic to a maximal compact subgroup of identity component of the group of isometries on the bounded symmetric domain \mathcal{D} corresponding to V above. Moreover, $\text{Aut}_0(V)$ preserves some inner product (\cdot, \cdot) on V . Put $K = \text{Aut}_0(V)$ for simplicity. Then K acts on the ring $P(V)$ of polynomial functions on V by $(k \cdot p)(x) = p(k \cdot x)$ for $k \in K, p \in P(V)$ and $x \in V$.

We identify V and \mathfrak{p}_+ again, and recall the family $\{x_1, \dots, x_r\}$ of tripotents. For $1 \leq j \leq r$ we write $x^{(j)} = \sum_{i=1}^j x_i$. Then for each j the 1-eigenspace $V_1^{(j)}$ of $x^{(j)} \square x^{(j)}$ in \mathfrak{p}_+ is a subPDHJTS coming from a Jordan algebra.

Now we define the *determinant* of an element x in a Jordan algebra V over \mathbb{C} (cf. [FK2]). Let $\mathbb{C}[X]$ be the ring of polynomials of an indeterminate X . For each $x \in V$ there exists a monic polynomial $f(X) \in \mathbb{C}[X]$ having the minimal degree in the ideal $\{g(X); g(x) = 0\}$ of $\mathbb{C}[X]$. We denote by $r = r(V)$ the maximum of degrees of the minimal polynomial of $x \in V$. Then we see that for any x in some dense open set O of V the degree of the minimal polynomials equals to r . The number r is called the *rank* of V . For each $x \in O$ we define $\det x$ by the constant term of the minimal polynomial of x . We can extend \det to V .

We return to the argument of the PDHJTS V . Let Δ_i denote the determinant function on $V_1^{(i)}$. We regard Δ_i as a polynomial function on V . Let $\mathbf{k} = (k_1, k_2, \dots, k_r)$ a partition, that is, the sequence of nonnegative integers such that $k_1 \geq k_2 \geq \dots \geq k_r$. then we define $\Delta_{\mathbf{k}} = \Delta_1^{k_1 - k_2} \Delta_2^{k_2 - k_3} \dots \Delta_r^{k_r}$. We see that the subPDHJTS $V_1 = V_1^{(r)}$ contains all $V_1^{(j)}$ and we regard $\Delta_{\mathbf{k}}$ as a polynomial function on V_1 . Moreover the connected component $K_1 = \text{Aut}(V_1)$ of the automorphism group of V_1 is isomorphic to a closed subgroup of $\text{Aut}(V)$.

Theorem 2.1. ([FK1], Theorem 2.1) *We have decompositions*

$$P(V) = \sum_{\mathbf{k}} P_{\mathbf{k}}, \quad P(V_1) = \sum_{\mathbf{k}} P_{1,\mathbf{k}},$$

where \mathbf{k} is a partition and $P_{\mathbf{k}}$ (resp. $P_{1,\mathbf{k}}$) is the irreducible representation of K (resp. K_1) with highest weight vector $\Delta_{\mathbf{k}}$. Moreover we have that $P_{\mathbf{k}}$ (resp. $P_{1,\mathbf{k}}$) is not equivalent to P_1 (resp. $P_{1,1}$) unless $\mathbf{k} = 1$.

Obviously $P_{\mathbf{k}}$ consists of homogeneous polynomials on V . So we define $\deg \mathbf{k}$ by the degree of $p \in P_{\mathbf{k}}$.

3. K -spherical functions on Heisenberg groups

We recall that the connected component K of automorphism group of a PDHJTS V preserves some inner product (\cdot, \cdot) on V . We define a Heisenberg Lie group $H_V = V \times \mathbb{R}$ with product

$$(x, t)(x', t') := (x + x', t + t' - \text{Im}(x, x')),$$

where $x, x' \in V$ and $t, t' \in \mathbb{R}$. Then K acts on H_V as automorphisms by $k \cdot (x, t) = (k \cdot x, t)$ ($k \in K, x \in V, t \in \mathbb{R}$). By Theorem 2.1 and [BJR1], Theorem 4.6 we have that the pair $(K \ltimes H_V, K)$ is a Gelfand pair. The bounded K -spherical function ϕ on H_V is of the form

$$\phi(x) = \int_K \langle \rho(k \cdot x)v, v \rangle dk,$$

where ρ is an irreducible unitary representation of H_V , $\langle \cdot, \cdot \rangle$ the inner product on the representation space H_ρ of ρ and dk the normalized Haar measure on K . We treat only the infinite-dimensional irreducible unitary representation. For simplicity we assume that $\rho(0, t) = e^{\sqrt{-1}t} I_{H_\rho}$, where I_{H_ρ} is the identity operator on H_ρ . We realize the representation ρ on the Fock space $\mathcal{F}(V)$ of all entire holomorphic functions f on V which satisfy the condition

$$\|f\|^2 := \frac{1}{\pi^n} \int_V |f(x)|^2 e^{-|x|^2} dx < \infty,$$

where $n = \dim_{\mathbb{C}} V$ and dx the Lebesgue measure on $2n$ -dimensional Euclidean space V . The representation operator is the follows:

$$\rho(x, t)f(y) := e^{\sqrt{-1}t - (y, x) - |x|^2/2} f(y + x).$$

We know that the ring $P(V)$ of polynomial functions on V is included densely in $\mathcal{F}(V)$. Hence we have Hilbert direct sum decomposition $\mathcal{F}(V) = \bigoplus_{\mathbf{k}} P_{\mathbf{k}}$ as K

module. We denote by $K_{\mathbf{k}}(\cdot, \cdot)$ the reproducing kernel of $P_{\mathbf{k}}$ and we define a function $p_{\mathbf{k}}$ by $p_{\mathbf{k}}(x) = \frac{1}{d_{\mathbf{k}}} K_{\mathbf{k}}(x, x)$ ($x \in V$). We see that $p_{\mathbf{k}}$ is K -invariant. By [BJR2], Proposition 4.2 we have for any K -spherical function ϕ associated to ρ is of the form

$$\phi(x, t) = \phi_{\mathbf{k}}(x, t) = e^{\sqrt{-1}t} e^{-|x|^2} q_{\mathbf{k}}(x),$$

where $q_{\mathbf{k}}$ is a K -invariant polynomial determined as follows. Let L denote the stabilizer of x_0 in K , where x_0 is defined in Section 1. By Theorem 2.1 of [FK1] each $P_{\mathbf{k}}$ has a unique L -invariant polynomial $\Phi_{\mathbf{k}}$ such that $\Phi_{\mathbf{k}}(x_0) = 1$. For two partitions \mathbf{k}, \mathbf{l} , we define the *generalized binomial coefficient* $\binom{\mathbf{k}}{\mathbf{l}}$ by

$$\Phi_{\mathbf{k}}(x_0 + x) = \sum_{\mathbf{l}} \binom{\mathbf{k}}{\mathbf{l}} \Phi_{\mathbf{l}}(x).$$

Now we define $q_{\mathbf{k}}$ by

$$q_{\mathbf{k}} = \sum_{\mathbf{l}} (-1)^{\deg \mathbf{l}} \binom{\mathbf{k}}{\mathbf{l}} p_{\mathbf{l}}.$$

We call $q_{\mathbf{k}}$ a *generalized Laguerre polynomial*. Similarly, for V_1 we define $\rho_1, \mathcal{F}(V_1), K_{1,\mathbf{k}}, p_{1,\mathbf{k}}, \phi_{1,\mathbf{k}}$ and $\Phi_{1,\mathbf{k}}$. Then the restriction $\Phi_{\mathbf{k}}$ on V_1 equals to $\Phi_{1,\mathbf{k}}$ by Theorem 2.1 of [FK1], hence the generalized binomial coefficient $\binom{\mathbf{k}}{\mathbf{l}}$ coincides with that defined by V . (See also [D], [Y]).

It is known that the restriction of $\Phi_{\mathbf{k}}$ on $W = \sum_{i=1}^r \mathbb{C}x_i$ is the scalar multiple of Jack polynomial $J_{\mathbf{k}}^{(2/d)}$, where d is the dimension of common $\frac{1}{2}$ -eigenspace of $x_i \square x_i$ and $x_j \square x_j$ for some i, j ($i \neq j$) (cf. [M1] [M2]).

4. Example 1: $U(m) \times U(n)$ -spherical functions

Let $V = \text{Mat}(m, n; \mathbb{C})$ ($m \leq n$) be the vector space of all $m \times n$ -matrices with entries in \mathbb{C} . We define a PDHJTS structure by

$$\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x),$$

where $x, y, z \in V$ and y^* is the adjoint matrix of y . The hermitian symmetric spaces corresponding to V is $M = \text{SU}(m, n)/\text{S}(U(m) \times U(n))$. The action of $\text{S}(U(m) \times U(n))$ on V is the following

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot x := axd^{-1},$$

where $a \in U(m), d \in U(n)$ and $x \in V$. So we extend the action to that of $U(m) \times U(n)$.

We can take a family of tripotents $\{e_{11}, e_{22}, \dots, e_{mm}\}$, where e_{ij} is the matrix whose (k, l) -component is $\delta_{ik}\delta_{jl}$ for $1 \leq k \leq m, 1 \leq l \leq n$. We put $e = \sum_{i=1}^m e_{ii}$. Then the maximal PDHJTS V_1 is identified with $\text{Mat}(m, m; \mathbb{C})$. We see that Δ_i is the principal minor of degree i . We have irreducible decomposition: $P(V) = \sum_{\mathbf{k}} P_{\mathbf{k}}$. The irreducible component $P_{\mathbf{k}}$ is equivalent to $M_{\mathbf{k}} \otimes M_{0, -\mathbf{k}'}$, where $M_{\mathbf{k}}$ is the irreducible $U(m)$ -module of highest weight \mathbf{k} and $M_{0, -\mathbf{k}'}$ the irreducible $U(n)$ -module of highest weight $(0, -\mathbf{k}') = (0, \dots, 0, -k_m, -k_{m-1}, \dots, -k_1)$. Easy calculation shows that $K_{\mathbf{k}}(x, x) = K_{1, \mathbf{k}}(x, x) = K_{1, \mathbf{k}}(xx^*, e)$. The stabilizer L_1 of e in $K_1 = U(m) \times U(m)$ is isomorphic to $U(m)$ and the L_1 -invariant polynomial $\Phi_{1, \mathbf{k}}$ is of the form

$$\Phi_{1, \mathbf{k}}(x) = \frac{S_{\mathbf{k}}(x)}{\dim M_{\mathbf{k}}} = \frac{\det(e_{k'_i - i + j})}{\dim M_{\mathbf{k}}},$$

where $S_{\mathbf{k}}$ is the Schur function corresponding to \mathbf{k} , $\mathbf{k}' = (k'_1, k'_2, \dots, k'_s)$ the conjugate of \mathbf{k} and e_i the L_1 -invariant function determined by $\sum_{i=0}^m (-1)^i t^{m-i} e_i(x) = \det(te - x)$. Moreover we have

$$K_{1, \mathbf{k}}(x, e) = \frac{d_{\mathbf{k}}}{(m)_{\mathbf{k}}} \Phi_{1, \mathbf{k}}(x) = d_{\mathbf{k}} \prod_{i=1}^m \frac{1}{k_i!} \binom{k_i + m - i}{m - i}^{-1} \Phi_{1, \mathbf{k}}(x),$$

where $d_{\mathbf{k}} = \dim P_{1, \mathbf{k}} = (\dim M_{\mathbf{k}})^2$ (cf. [FK2], Ch.XI). We know that $S_{\mathbf{k}}(e + x) = \sum_1 \det\left(\binom{k_i + m - i}{l_j + m - j}\right) S_1(x)$ (see [M2], I. 3. Example 10). Therefore we have the generalized Laguerre polynomial $q_{\mathbf{k}}$ is of the form

$$q_{\mathbf{k}}(x) = \prod_{s=1}^m \binom{n-s}{n-m} \binom{k_s + n - s}{n-m}^{-1} \prod_{r=1}^m \frac{1}{k_r!} \binom{k_r + m - r}{m-r}^{-1} \\ \times \prod_{1 \leq p < q \leq m} \frac{q-p}{k_p - k_q + q - p} \det(e_{k'_i - i + j}(xx^*)).$$

5. Example 2. $\mathbb{T} \times \text{SO}(n)$ -spherical functions

Let $V = \mathbb{C}^n$ ($n \geq 3$) be the vector space with Jordan algebra structure

$$(x_1, x_2, \dots, x_n)(x'_1, x'_2, \dots, x'_n) := (x_1 x'_1 - \sum_{i=2}^n x_i x'_i, x_1 x'_2 + x_2 x'_1, \dots, x_1 x'_n + x_n x'_1).$$

The compact group $\mathbb{T} \times \text{SO}(n)$ acts on V by

$$(e^{\sqrt{-1}t}, g) \cdot x = e^{\sqrt{-1}t} x g^{-1},$$

where $t \in \mathbb{R}$ and $g \in \text{SO}(n)$. We see that V is PDHJTS of rank 2 and the elements $e_1 = \frac{1}{2}(1, \sqrt{-1}, 0, \dots, 0)$, $e_2 = \frac{1}{2}(1, -\sqrt{-1}, 0, \dots, 0)$ are tripotents. Moreover the connected component of $\text{Aut}(V)$ of PDHJTS V is locally isomorphic to $\mathbb{T} \times \text{SO}(n)$.

Thanks to the theory of spherical harmonics the action of $\mathbb{T} \times \text{SO}(n)$ on the ring $P(V)$ of polynomial functions is multiplicity-free:

$$P(V) = \sum_{\mathbf{k}} P_{\mathbf{k}},$$

where $P_{(k_1, k_2)} = P_{(k_1 - k_2, 0)} \otimes \mathbb{C}r^{2k_2}$, $r^2 = \sum_{i=1}^n x_i^2$ and $P_{(k, 0)}$ is the space of harmonic polynomials of degree k .

We get $\Phi_{\mathbf{k}}$ by the same way to get ultraspherical polynomials. We have that $\Phi_{(k_1, k_2)}(x) = \Phi_{(k_1 - k_2, 0)}(x)r^{2k_2}$ and that

$$\Phi_{(k, 0)} = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{k!}{(k-2i)!i!} \frac{(k+n-2)_{k-2i}}{2^k \binom{n-1}{k-i}} x_1^{k-2i} r^{2i},$$

where $\lfloor a \rfloor$ is the largest integer which is not more than a and $(a)_i = a(a+1)\cdots(a+i-1)$ for positive integer i . Put $d = n - 2$. Then the restriction of $\Phi_{\mathbf{k}}$ on the space $\mathbb{C}e_1 + \mathbb{C}e_2$ is the scalar multiple of two-variable Jack polynomial $J_{\mathbf{k}}^{(2/d)}$. The generalized binomial coefficient $\binom{\mathbf{k}}{\mathbf{l}}$ is the following:

$$\begin{aligned} \binom{(k_1, k_2)}{(l_1, l_2)} &= \frac{(k_1 - l_2 + 1 + d/2)_{l_2}}{(l_1 - l_2 + 1 + d/2)_{l_2}} \binom{k_2}{l_2} \\ &\times \sum_{i=0}^{\lfloor \frac{l_1 - l_2}{2} \rfloor} (-1)^i \frac{(k_1 - l_2 - i + 1 + d/2)_i}{(l_1 - l_2 - i + d/2)_i} \binom{k_1 + k_2 - 2l_2 - 2i}{l_1 - l_2 - 2i} \binom{k_2 - l_2}{i}. \end{aligned}$$

We define $\Psi_{\mathbf{k}}(|x|^2, |r^2|^2)$ by replacing x_1, r^2 with $|x|^2, |r^2|^2$ respectively. We have

$$\begin{aligned} p_{\mathbf{k}}(x) &= \frac{1}{d_{\mathbf{k}}} K_{\mathbf{k}}(x, x) = \frac{1}{\binom{n}{2}_{\mathbf{k}}} \Psi_{\mathbf{k}}(|x|^2, |r^2|^2) \\ &= \frac{n}{(2k_1 + n)(k_2 + 1)} \Psi_{\mathbf{k}}(|x|^2, |r^2|^2). \end{aligned}$$

Therefore the generalized Laguerre polynomial $q_{\mathbf{k}}$ is the following:

$$q_{\mathbf{k}}(x) = \sum_{\mathbf{l}} (-1)^{\text{deg } \mathbf{l}} \binom{\mathbf{k}}{\mathbf{l}} \frac{n}{(2k_1 + n)(k_2 + 1)} \Psi_{\mathbf{k}}(|x|^2, |r^2|^2).$$

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