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Quantization of Poisson structures on complex Grassmannians and some multidimensional q-Selberg integrals

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1. PROLOGUE

This paper is an informal account of some recent results concerning the quantization of complex Grassmannians and certain families of multivariable orthogonal polynomials of q-hypergeometric type. Part of the results discussed here is joint work with Masatoshi Noumi and Tetsuya Sugitani, another part was done in cooperation with Jasper Stokman. This paper does not contain any proofs. A more detailed account (including complete proofs) of the results presented here will be given elsewhere.

Over the last few years, important advances have been booked in the study of quantum analogues of compact symmetric spaces and their connection with q-hypergeometric orthogonal polynomials. Among many papers, I would like to cite Koornwinder [K2], Noumi [N], and Noumi and Sugitani [NS]. In the present paper, a generalization of certain results of Noumi and Sugitani [NS] in the case of complex Grassmannians is presented. In this case, there is a one-parameter family of quantum analogues, on which various (quite different) q-analogues of Selberg's multidimensional beta integral [Sb] and related families of orthogonal polynomials can be interpreted. A summary of these results is the content of sections 3 and 4.

In section 2 (which is incomplete and of a rather tentative nature), I sketch an approach that might lead to a better understanding of some of the phenomena alluded to above. According to Drinfel'd [Dr], a quantum group should be viewed as a "quantization" of a Poisson-Lie group rather than of a Lie group. Put in a different way, in the semi-classical limit, quantum group structures induce multiplicative Poisson structures on the underlying Lie groups. The study of these Poisson structures often provides important information about the quantum groups. Most of the papers on quantum groups and q-special functions, however, do not contain any reference to Poisson structures in the semi-classical limit (an exception should be made for some papers by Soibel'man and Vaksman [VS1], [SV]). The thrust of section 2 is that by applying Drinfel'd's and Soibel'man's ideas to homogeneous spaces one may reach, at the very least, a much better intuitive understanding of quantum symmetric spaces.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. Covariant Poisson structures on coadjoint orbits

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank r, and denote the underlying real Lie algebra by $\mathfrak{g}^{\mathbf{R}}$. Fix a compact real form $\mathfrak{u} \subset \mathfrak{g}$, a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{u}$ (hence a Cartan subalgebra $\mathfrak{h} := \mathfrak{t} \oplus i\mathfrak{t} \subset \mathfrak{g}$), and a choice of positive roots $R^+ \subset R$ for the root system $R = R(\mathfrak{g}, \mathfrak{h})$. There is the corresponding Iwasawa decomposition

$$\mathfrak{g}^{\mathbf{R}} = \mathfrak{u} \oplus \mathfrak{a} \oplus \mathfrak{n}_+, \quad \mathfrak{a} := i\mathfrak{t}. \tag{3.1}$$

The pair $(\mathfrak{u}, \mathfrak{a} \oplus \mathfrak{n}_+)$ together with the imaginary part $\Im\kappa$ of the Killing form κ on $\mathfrak{g}$ is a Manin triple (cf. Lu and Weinstein [LW1], Majid [Ma]), which essentially means that $\mathfrak{u}$ and $\mathfrak{a} \oplus \mathfrak{n}_+$ are isotropic w.r.t. the non-degenerate symmetric form $\Im\kappa$ on $\mathfrak{g}$ (and hence dual to each other). As a consequence, there are natural Lie bialgebra structures on $\mathfrak{u}$ and $\mathfrak{a} \oplus \mathfrak{n}_+$ (cf. loc. cit.). Let G be a connected (complex) Lie group with Lie algebra $\mathfrak{g}$, and let U, A, N be the connected (closed) subgroups of $G^{\mathbf{R}}$ corresponding to $\mathfrak{u}$, $\mathfrak{a}$ and $\mathfrak{n}_+$ respectively. The Lie bialgebra structures on $\mathfrak{u}$ and $\mathfrak{a} \oplus \mathfrak{n}_+$ can be integrated to multiplicative Poisson structures on U and AN (cf. loc. cit.). Thus, U and AN are dual Poisson-Lie groups.

The Poisson bracket on U is quasi-triangular (cf. Drinfel'd [Dr], see also [CP]), which essentially means that it can be described in terms of a solution of the modified Classical Yang-Baxter equation (CYBE). To describe a skew-symmetric solution $r_{\mathfrak{u}} \in \mathfrak{u} \otimes \mathfrak{u}$ of the modified CYBE, choose a Chevalley basis of $\mathfrak{g}$ consisting of $X_\alpha \in \mathfrak{g}$ ($\alpha \in R$) and $H_k \in \mathfrak{h}$ ($1 \leq k \leq r$) such that $\mathfrak{u} \subset \mathfrak{g}$ is the fixed point set of the conjugate-linear Cartan involution $\omega: \mathfrak{g} \to \mathfrak{g}$ given by $\omega(X_\alpha) = -X_{-\alpha}$, $\omega(H_k) = -H_k$. We may take $r_{\mathfrak{u}}$ to be

$$r_{\mathfrak{u}} := \frac{i}{2} \sum_{\alpha \in R^+} (X_\alpha \otimes X_{-\alpha} - X_{-\alpha} \otimes X_\alpha). \tag{3.2}$$

The Poisson bracket on U induced by $r_{\mathfrak{u}}$ can be written explicitly in terms of $r_{\mathfrak{u}}$ (cf. [CP, pp. 60–61], and is sometimes called the Sklyanin-Drinfel'd or Bruhat-Poisson bracket.

Recall (cf. Semenov [Se]) that a Poisson homogeneous space of U is a Poisson manifold M with a (left) transitive U-action such that the action mapping $U \times M \to M$ is a Poisson mapping. The Poisson structure on M is then also called U-covariant. Let $K \subset U$ be a closed subgroup. The coset space U/K is called a Poisson coset space if there exists a (necessarily unique) Poisson structure on U/K such that the natural projection of U onto U/K is Poisson. This is the case if and only if K is a coisotropic submanifold of U, i.e. the subspace of smooth functions on U vanishing on K is closed under the Poisson bracket. The natural injection $C^\infty(U/K) \hookrightarrow C^\infty(U)$ then commutes with the Poisson brackets. A Poisson coset space is always Poisson homogeneous with the natural U-action. A Poisson homogeneous space M can be realized as a Poisson coset space if and only if M has zero-dimensional symplectic leaves (the stabilizer subgroup of a point where the Poisson tensor on M vanishes is a coisotropic submanifold of U).

One particular example of coisotropic subgroups is given by Poisson-Lie subgroups, i.e. closed subgroups K such that the Poisson tensor of U is tangent to K at all points of K. Recall the following result by Lu and Weinstein [LW1]:

Proposition 3.1 — *With the above notation, let $P \subset G$ be a standard parabolic subgroup (relative to the choice of Cartan subalgebra and positive roots). Then the subgroup $U \cap P \subset U$ is a Poisson-Lie subgroup of U endowed with the Bruhat-Poisson structure. The corresponding symplectic foliation of $U/U \cap P$ coincides with the Schubert cell decomposition of the generalized flag manifold G/P.*

The corresponding Poisson bracket on $U/U \cap P$ is also called Bruhat-Poisson.

Now consider a coadjoint orbit $\mathcal{O}_x \subset \mathfrak{u}^*$ such that the stabilizer of the point $x \in \mathcal{O}_x$ is of the form $U \cap P$ for some standard parabolic subgroup $P \subset G$. Besides the Bruhat-Poisson bracket on $\mathcal{O}_x$, there is another natural Poisson bracket on $\mathcal{O}_x$, viz. the symplectic Kirillov-Kostant bracket, which comes from the Lie algebra bracket on $\mathfrak{u}$. Whenever one has two Poisson brackets on the same space, the natural question arises as to whether the brackets are compatible, in the sense that their sum is again a Poisson bracket (i.e. satisfies the Jacobi identity). In this particular case, a complete answer to this question was given by Khoroshkin, Radul, and Rubtsov [KRR]:

Theorem 3.2 — *Let $\mathcal{O}_x \cong U/U \cap P$ (P standard parabolic) be a coadjoint orbit in $\mathfrak{u}^*$. Denote the Bruhat-Poisson and Kirillov-Kostant Poisson tensors by π_B and π_K respectively. Then the bivector field $\pi_\lambda := \pi_B + \lambda\pi_K$ ($\lambda \in \mathbb{R}$) is a Poisson tensor if and only if $\mathcal{O}_x$ is (Hermitian) symmetric. In this case, π_λ is U-covariant.*

This theorem raises the question for which values of $\lambda \in \mathbb{R}$ the Poisson homogeneous space $(\mathcal{O}_x, \pi_\lambda)$ can be realized as a Poisson coset space of U. We shall content ourselves with looking at a simple example, since this suffices to give an idea of the general picture. The general case will be considered in a later paper.

Consider the case in which $U = SU(2)$ and $P \subset SL(2, \mathbb{C})$ is the standard upper-triangular matrix group. Then $U/U \cap P$ is isomorphic to the 2-sphere S^2. The symplectic leaves of (S^2, π_λ) were determined by Lu and Weinstein [LW2]:

Proposition 3.3 — *Suppose π_K is suitably normalized relative to π_B. The $SU(2)$-covariant Poisson manifold (S^2, π_λ) ($\lambda \in \mathbb{R}$) has zero-dimensional leaves if and only if $\lambda \in [-2, 0]$.*

Note that the choice $\lambda = 0$ corresponds to the Bruhat-Poisson structure on S^2. In this case, the symplectic foliation (Schubert cell decomposition of the Riemann sphere) consists of a single point and an open two-dimensional disk. The case $\lambda = -2$ is similar, since it corresponds to the Bruhat-Poisson structure associated with the opposite choice of positive roots. In both cases, (S^2, π_λ) can be written as a quotient by a Poisson-Lie subgroup, viz. the maximal torus in $SU(2)$. In the case $\lambda \in (-2, 0)$ (cf. Lu and Weinstein [LW2]), the symplectic foliation consists of two open two-dimensional disks and a circle of points.

Recall that quantizing a Poisson algebra consists, roughly speaking, of replacing both the commutative algebra product and the Poisson bracket by a non-commutative product $\star_h$ depending on a formal parameter h such that (i) one reobtains the algebra product in the zero-order approximation (ii) the commutator bracket reduces to the Poisson bracket in the first-order approximation.

Our main interest in this paper is to do harmonic analysis on quantized complex Grassmannians $U(n)/U(n-l)\times U(l)$. In particular, our approach heavily relies on the existence of a suitable analogue of the stabilizer subgroup $K=U(n-l)\times U(l)$. The ideas sketched above suggest that the existence of such an analogue is most likely in the case of a Poisson structure with zero-dimensional leaves. There is a general mathematical principle (cf. Soibel'man and Vaksman [VS1], [So1], [So2], [So3], [SV]) for an application of this idea to the case of quantizations of a compact simple Lie group) that the irreducible $*$-representations of a quantization of the algebra of functions on a real Poisson manifold more or less correspond to the symplectic leaves. In particular, a quantized algebra of functions on a Poisson manifold with zero-dimensional symplectic leaves should have one-dimensional $*$-representations.

In section 4 we will construct a family of quantized complex Grassmannians depending on (essentially) a real parameter $-\infty\leq\sigma\leq\infty$. They will be realized as certain coadjoint orbits in a quantized space of Hermitian matrices. As a criterion for the selection of the orbits we take the existence of a particular one-dimensional $*$-representation. In the case of quantum $SU(2)$, our quantized Grassmannians reduce exactly to those quantum spheres studied by Podleś [P] that have one-dimensional $*$-representations. In this case, by explicitly computing the "semi-classical limit" of the commutation relations of a suitable set of generators of the quantized algebra of functions, it can be shown that one recovers covariant Poisson structures of type π_λ on the 2-sphere, and that the parameter range $-\infty\leq\sigma\leq\infty$ corresponds exactly to the parameter range $\lambda\in[-2,0]$ ($\sigma=\pm\infty$ corresponding to $\lambda=0,-2$). A similar picture should hold in the case of general rank. For instance, in the case $\sigma=\pm\infty$, the quantum Grassmannian can be written as a quotient of the quantum unitary group by a quantum subgroup (in the strict sense). In the semi-classical limit, this means that the Grassmannian with its covariant Poisson structure can be realized as a quotient of $U(n)$ endowed with the Bruhat-Poisson structure by a Poisson-Lie subgroup, viz. $U(n-l)\times U(l)$ or $U(l)\times U(n-l)$ (which are the intersection of $U(n)$ and certain maximal parabolic subgroups in $GL(n,\mathbb{C})$). More details will be given in a later paper.

3. Some multidimensional q-Selberg integrals

Recall Selberg's multidimensional beta integral [Sb]:

$$\int_0^1\cdots\int_0^1\prod_{i<j}|t_i-t_j|^{2z}\prod_{j=1}^{l}t_j^{x-1}(1-t_j)^{y-1}dt_j = \prod_{j=1}^{n}\frac{\Gamma(x+(j-1)z)\Gamma(y+(j-1)z)\Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\Gamma(z+1)}, \tag{3.1}$$

where $l\geq 1$ is an integer, and $x,y,z\in\mathbb{C}$ are such that $\Re(x),\Re(y)>0$, $\Re(z)>-\max\{\frac{1}{l},\Re(x)/(l-1),\Re(y)/(l-1)\}$. In the case $l=1$, Selberg's integral reduces to Euler's classical beta integral.

In the remainder of this paper fix $0<q<1$. Recall the following standard notation:

$$(a;q)_k := \prod_{j=0}^{k-1}(1-aq^j),\ (a_1,\dots,a_s;q)_k := \prod_{j=1}^{s}(a_j;q)_k,\ (a;q)_\infty := \lim_{k\to\infty}(a;q)_k.$$

Recall Jackson's q-integral and the following q-analogue of the Γ-function:

$$\int_0^1 f(x)d_qx := (1-q)\sum_{k=0}^{\infty} f(q^k)q^k, \quad \Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{3.2}$$

Askey [A] considered the following q-analogue of Selberg's integral and conjectured its evaluation. A proof of this conjecture was independently given by Habsieger [Hb] and Kadell [Ka].

$$\begin{aligned}\int_0^1\cdots\int_0^1 \prod_{i<j} t_i^{2k}\left(\frac{t_jq^{1-k}}{t_i};q\right)_{2k} \prod_{j=1}^{l} t_j^{x-1}\frac{(t_jq;q)_\infty}{(t_jq^y;q)_\infty}d_qt_j \\ = q^{k\binom{l}{2}x+2k^2\binom{l}{3}}\prod_{j=1}^{l}\frac{\Gamma_q(x+(j-1)k)\Gamma_q(y+(j-1)k)\Gamma_q(jk+1)}{\Gamma_q(x+y+(n+j-2)k)\Gamma_q(k+1)},\end{aligned} \tag{3.3}$$

where $k \geq 0$ is an integer, and $\Re(x), \Re(y) > 0$. Note that, by definition of Jackson's q-integral, the measure defined on the left-hand side of (3.3) is supported on an infinite discrete set.

Macdonald [M1] (in a special case) and Koornwinder [K] considered a different q-analogue of Selberg's integral, which was first evaluated by Gustafson [Gu]:

$$\begin{aligned}\frac{1}{(2\pi i)^l}\int_{\mathbb{T}^l}\prod_{i<j}\frac{(t_it_j^{-1},t_i^{-1}t_j,t_it_j,t_i^{-1}t_j^{-1};q)_\infty}{(tt_it_j^{-1},tt_i^{-1}t_j,tt_it_j,tt_i^{-1}t_j^{-1};q)_\infty}\times \\ \prod_{j=1}^{l}\frac{(t_j^2;q)_\infty(t_j^{-2};q)_\infty}{(at_j,a/t_j,bt_j,b/t_j,ct_j,c/t_j,dt_j,d/t_j;q)_\infty}\frac{dt_j}{t_j} \\ = 2^l\, l!\prod_{j=1}^{l}\frac{(t;q)_\infty(t^{l+j-2}abcd;q)_\infty}{(t^j,q,abt^{j-1},act^{j-1},adt^{j-1},bct^{j-1},bdt^{j-1},cdt^{j-1};q)_\infty}.\end{aligned} \tag{3.4}$$

Here $a,b,c,d \in \mathbb{C}$ are such that $|a|,|b|,|c|,|d| < 1$, and $-1 < t < 1$. Under these conditions, the weight function on the left-hand side of (3.4) is positive and continuous on the compact torus $\mathbb{T}^l$. The symbol $\int_{\mathbb{T}^l} dt_1\cdots dt_l$ denotes n-fold complex integration along the unit circle in the positive direction. In the case $l=1$, the integral (3.4) reduces to the well-known Askey-Wilson q-beta integral [AW].

Given these integrals, it is natural to ask what kind of orthogonal polynomials correspond to them. It is well-known (cf. [HS]) that Selberg's integral arises in connection with generalized Jacobi polynomials associated with the root system BC_l.

Let us first consider the integral (3.4). Macdonald [M1] (in a special case) and Koornwinder [K] defined a family of multivariable Askey-Wilson polynomials depending rationally on q and five additional parameters a, b, c, d, t. We recall the precise definition as given in [K1], [S3].

Write $P_\Sigma := \bigoplus_{1\le k\le l} \mathbb{Z}\varepsilon_k$ for the weight lattice of the root system $\Sigma = BC_l$. A dominant weight $\lambda = \sum_{k=1}^{l} \lambda_k\varepsilon_k \in P_\Sigma^+$ is characterized by the condition $\lambda_1 \ge \ldots \ge \lambda_l \ge 0$. Recall that $\lambda \le \mu$ (dominance ordering on weights) if and only if $\sum_{i=1}^{k} \lambda_i \le \sum_{i=1}^{k} \mu_i$ for all $1 \le k \le l$. Write x_i for the formal exponential e^{ε_i}. Let $\mathbb{C}[x^{\pm 1}] := \mathbb{C}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]$ denote the algebra of trigonometric polynomials associated with the root system BC_l. The Weyl group $W := \mathbb{Z}_2^l \rtimes \mathfrak{S}_l$ of BC_l acts on $\mathbb{C}[x^{\pm 1}]$ by permutations and sign changes of the x_k. Let $\mathbb{C}[x^{\pm 1}]^W \subset \mathbb{C}[x^{\pm 1}]$ denote the subalgebra of W-invariant elements.

Let $N \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$. Define the truncated Jackson q-integral by:

$$\int_\alpha^\beta f(x)d_{q,N}x := \int_0^\beta f(x)d_{q,N}x - \int_0^\alpha f(x)d_{q,N}x,$$

$$\int_0^\beta f(x)d_{q,N}x := \sum_{k=0}^{N} f(\beta q^k)(\beta q^k - \beta q^{k+1}) \text{ if } N \ge 0,$$

$$\int_\alpha^\beta f(x)d_{q,N}x := 0 \text{ if } N < 0.$$

Take V_{AW} to be the set of quadruples (a, b, c, d) such that

(1) a, b, c, d are real, or appear in complex conjugate pairs,
(2) $ab, ac, ad, bc, bd, cd \notin \mathbb{R}_{\ge 1}$.

Now fix $(a, b, c, d) \in V_{AW}$. Set $t := q^k$ $(k \in \mathbb{Z}_+)$. For $e \in \{a, b, c, d\}$ such that $|e| > 1$, set N_e equal to the largest integer such that $|eq^{N_e}| > 1$. Set $N_e := -1$ if $|e| \le 1$. For any $0 \le m \le l$ define the following inner product on the space $\mathbb{C}[x^{\pm 1}]^W$ (cf. Stokman [S3]):

$$\langle f, g\rangle_m := \frac{2^m \binom{l}{m}}{(2\pi i)^{l-m}} \sum_{e_1,\ldots,e_m} \int_{x_1=0}^{e_1} \cdots \int_{x_m=0}^{e_m} \int \cdots \int_{(x_{m+1},\ldots,x_l)\in T_{l-m}} f(x)\overline{g(x)}\Delta_{AW,m}(x) \frac{d_{q,N_{e_1}}x_1}{(1-q)x_1} \cdots \frac{d_{q,N_{e_m}}x_m}{(1-q)x_m} \frac{dx_{m+1}}{x_{m+1}} \cdots \frac{dx_l}{x_l}, \tag{3.5}$$

where the sum is taken over all $e_i \in \{a, b, c, d\}$. The following notation is used:

$$\Delta_{AW,m}(x) := \prod_{r=1}^{m} w_d(x_r; e_r; f_r, g_r, h_r; q) \times \prod_{s=m+1}^{l} w_c(x_s; a, b, c, d; q) \prod_{i<j} (x_ix_j, x_ix_j^{-1}, x_i^{-1}x_j, x_i^{-1}x_j^{-1}; q)_k, \tag{3.6}$$

$$w_c(x; a, b, c, d; q) := \frac{(x^2, x^{-2}; q)_\infty}{(ax, a/x, bx, b/x, cx, c/x, dx, d/x; q)_\infty}, \tag{3.7}$$

$$w_d(eq^i; e; f, g, h; q) := \frac{(e^{-2}; q)_\infty}{(q, ef, f/e, eg, g/e, eh, h/e; q)_\infty} \times \frac{(e^2, ef, eg, eh; q)_i}{(q, eq/f, eq/g, eq/h; q)_i} \frac{(1 - e^2 q^{2i})}{(1 - e^2)} \left(\frac{q}{efgh}\right)^i. \tag{3.8}$$

Here the f, g, h are by definition such that (e, f, g, h) is a permutation of (a, b, c, d).

Define a new inner product by summing up the ones in (3.5):

$$\langle f, g\rangle_{AW} := \sum_{i=0}^{m} \langle f, g\rangle_m. \tag{3.9}$$

Under the conditions $|a|, |b|, |c|, |d| < 1$, $t = q^k$ $(k \in \mathbb{Z}_+)$, the inner product (3.9) reduces to the inner product associated with the weight function on the left-hand side of (3.4).

Now define the Askey-Wilson polynomials $P_\lambda \in \mathbb{C}[x^{\pm 1}]^W$ $(\lambda \in P_\Sigma^+)$ by the conditions

$$(i)\ P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu, \quad (ii)\ \langle P_\lambda, m_\mu\rangle_{AW} = 0 \ \ (\mu < \lambda). \tag{3.10}$$

Here $m_\lambda := \sum_{\mu \in W\lambda} x^\mu$ is the orbit sum. The following theorem was proved by Stokman [S3].

Theorem 3.1 — *Let $(a, b, c, d) \in V_{AW}$, $t := q^k$ $(k \in \mathbb{Z}_+)$. Then the multivariable Askey-Wilson polynomials $P_\lambda = P_\lambda(x; a, b, c, d; q, t)$ satisfy*

$$\langle P_\lambda, P_\mu\rangle_{AW} = 0, \quad \lambda \neq \mu.$$

A similar theorem was proved by Koornwinder [K] subject to the parameter conditions $|a|, |b|, |c|, |d| < 1$, $t \in (-1, 1)$. Note that these theorems are non-trivial, since the dominance ordering on weights is not total (if $l \geq 2$).

Next consider Askey's integral (3.3). Stokman [S1] defined a family of so-called multivariable little q-Jacobi polynomials, and proved that they are orthogonal with respect to the weight function on the left-hand side of (3.3). The precise details are as follows.

Fix an integer $k \geq 0$, and set $t := q^k$. Let V_L^q denote the set of real pairs (a, b) such that

$$0 < a < \frac{1}{q}, \quad -\infty < b < \frac{1}{q}. \tag{3.11}$$

On the algebra $\mathbb{C}[x]^{S_l} := \mathbb{C}[x_1, \ldots, x_l]^{S_l}$ of symmetric polynomials define the inner product

$$\langle f, g\rangle_L := \int_{x_1=0}^{1} \int_{x_2=0}^{1} \cdots \int_{x_l=0}^{1} f(x)\overline{g(x)}\Delta_L(x) d_q x_1 d_q x_2 \cdots d_q x_l \tag{3.12}$$

with the following notation:

$$\Delta_L(x) := \left(\prod_{i=1}^{l} w_L(x_i)\right) \Delta(x) \prod_{i<j} x_i^{2k-1} \left(q^{1-k}\frac{x_j}{x_i}; q\right)_{2k-1},$$

$$\Delta(x) := \prod_{i<j}(x_i - x_j), \quad w_L(x) := \frac{(qx;q)_\infty}{(qbx;q)_\infty} x^\alpha \ (a = q^\alpha).$$

Define the little q-Jacobi polynomials $P_\lambda \in \mathbb{C}[x]^{S_l}$ ($\lambda \in P_\Sigma^+$) by the conditions

$$(i)\ P_\lambda = \tilde{m}_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}\tilde{m}_\mu, \quad (ii)\ \langle P_\lambda, \tilde{m}_\mu\rangle_L = 0 \ \ (\mu < \lambda). \tag{3.13}$$

Here $\tilde{m}_\lambda := \sum_{\mu \in S_l\lambda} x^\mu$ is a monomial symmetric polynomial.

The following theorem was proved by Stokman [S1]:

Theorem 3.2 — *Let $(a,b) \in V_L$, $t := q^k$ ($k \in \mathbb{Z}_+$). Then the multivariable little q-Jacobi polynomials $P_\lambda = P_\lambda(x;a,b;q,t)$ satisfy*

$$\langle P_\lambda, P_\mu\rangle_L = 0, \quad \lambda \neq \mu.$$

In the case $l = 1$, the P_λ reduce to little q-Jacobi polynomials [AA].

4. Quantized complex Grassmannians

A complex Grassmannian of rank l can be realized as a (co-)adjoint orbit of the unitary group $U = U(n)$. The orbit is completely determined by specifying two distinct real eigenvalues with multiplicities $n - l$ and l. The stabilizer subgroup with respect to a suitably chosen base point is equal to $K = U(n-l) \times U(l)$. Below we discuss a quantum analogue of this picture.

The definition of the quantum unitary group is well-known. We adopt the notation of [N]. Fix $n \geq 2$ in the remainder of this paper. Let $\mathcal{A}_q = \mathcal{A}_q(U(n))$ denote the quantized algebra of functions on $U(n)$. The generators t_{ij} and $\det_q^{-1}$ satisfy the usual $RT_1T_2 = T_2T_1R$ commutation relations, where R is given by

$$R := \sum_{ij} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji}. \tag{4.1}$$

Let $P = \bigoplus_{1\leq i\leq n} \mathbb{Z}\epsilon_i$ denote the rational character lattice of $GL(n,\mathbb{C})$. The quantized universal enveloping algebra $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{gl}(n))$ is generated by q^h ($h \in P^*$) and e_i, f_i ($1 \leq i \leq n-1$) subject to the well-known quantized Weyl-Serre relations. The algebra $\mathcal{U}_q$ is also generated by the L-operators L_{ij}^+, $L_{ij}^- \in \mathcal{U}_q$ (cf. [RTF], [N]). Their commutation relations can be expressed by means of the R-matrix (4.1) (cf. loc.cit.).

$\mathcal{A}_q$ and $\mathcal{U}_q$ are in natural Hopf $*$-algebra duality, and $\mathcal{A}_q$ is an $\mathcal{U}_q$-bimodule with two-sided symmetry. Let $W(\lambda) \subset \mathcal{A}_q$ be the subspace spanned by the matrix coefficients of the irreducible representation $V(\lambda)$ of $\mathcal{U}_q$ with highest weight $\lambda \in P^+$. Then

$$\mathcal{A}_q = \bigoplus_{\lambda \in P^+} W(\lambda). \tag{4.2}$$

Let $h\colon \mathcal{A}_q \to \mathbb{C}$ denote the Haar functional on $\mathcal{A}_q$. Then $\langle a,b\rangle := h(b^*a)$ defines a positive definite inner product on $\mathcal{A}_q$ with respect to which the subspaces $W(\lambda) \subset \mathcal{A}_q$ are mutually orthogonal (Schur orthogonality).

We now discuss a quantum analogue of the (co-)adjoint orbit picture in the case of $U(n)$ (cf. [NDS]). The algebra $\mathcal{C}_q = \mathcal{C}_q(n)$ of polynomial functions on the quantum space of q-Hermitian matrices is generated by the symbols x_{ij} $(1 \leq i, j \leq n)$ subject to the relations given by the following reflection equation:

$$R_{12}X_1R_{12}^{-1}X_2 = X_2R_{21}^{-1}X_1R_{21}. \tag{4.3}$$

Here $X_1 := X \otimes \mathrm{id}$ and $X_2 := \mathrm{id} \otimes X$ are Kronecker matrix products, $R_{12} = R$, $R_{21} = PRP$ (P being the permutation operator). There is a unique $*$-operation on $\mathcal{C}_q$ such that $x_{ij}^* = x_{ji}$. In shorthand notation, we write $X^* = X$. There is a unique $*$-algebra homomorphism $\delta: \mathcal{C}_q \to \mathcal{A}_q \otimes \mathcal{C}_q$ such that

$$\delta(x_{ij}) = \sum_{r,s} t_{ir}t_{js}^* \otimes x_{rs} \quad \text{or} \quad \delta(X) = TXT^*. \tag{4.4}$$

The mapping δ is a comodule mapping, and serves as our quantum analogue of the adjoint action.

The next step is to look for a one-dimensional $*$-representation of $\mathcal{C}_q$ which allows us to select the right quantum coadjoint orbit. In the remainder of this paper fix an integer $1 \leq l \leq \left[\frac{n}{2}\right]$. Let $-\infty < \sigma \leq \infty$ be a parameter. Set

$$\begin{aligned} J^\sigma &:= \sum_{1\leq k\leq l} q^\sigma(q^{-\sigma} - q^\sigma)e_{kk} + \sum_{l<k<l'} e_{kk} - \sum_{k\leq l \text{ or } k\geq l'} q^\sigma e_{kk'} \quad (\sigma \text{ finite}), \\ J^\infty &:= \sum_{1\leq k\leq l'} e_{kk}, \end{aligned} \tag{4.5}$$

where $k' := n + 1 - k$ $(1 \leq k \leq n)$. Observe that J^σ has two distinct eigenvalues with multiplicities $n - l$ and l.

Theorem 4.1 — *The matrix J^σ satisfies the reflection equation* (4.3) *for any $-\infty < \sigma \leq \infty$.*

Define a $*$-algebra homomorphism $\varepsilon^\sigma: \mathcal{C}_q \to \mathbb{C}$ by $\varepsilon^\sigma(X) = J^\sigma$. The mapping $\Psi^\sigma := (\mathrm{id} \otimes \varepsilon^\sigma) \circ \delta$ of $\mathcal{C}_q$ into $\mathcal{A}_q$ is a $*$-algebra homomorphism intertwining the natural (left) coactions of $\mathcal{A}_q$ on $\mathcal{C}_q$ and itself. Write $\mathcal{B}_q^\sigma \subset \mathcal{A}_q$ for the image of Ψ^σ, which is a $*$-subalgebra and left coideal in $\mathcal{A}_q$.

We look for a stabilizer "subgroup" of $\mathcal{B}_q^\sigma$. Set

$$M^\sigma := L^+J^\sigma - J^\sigma L^- \in \mathrm{End}(V) \otimes \mathcal{U}_q \quad (\sigma \text{ finite}). \tag{4.6}$$

The subspace $\mathfrak{k}^\sigma \subset \mathcal{U}_q$ (σ finite) is by definition spanned by the coefficients of the matrix M^σ. We may take $\mathfrak{k}^\infty$ to be equal to the subspace spanned by

$$L_{ij}^\pm \ (1 \leq i \neq j \leq l' \text{ or } l' + 1 \leq i \neq j \leq n), \quad L_{ii}^+ - L_{ii}^- \ (1 \leq i \leq n). \tag{4.7}$$

The embedding $\mathfrak{k}^\infty \subset \mathcal{U}_q$ corresponds to the natural surjection $\mathcal{A}_q(U(n-l)) \otimes \mathcal{A}_q(U(l)) \to \mathcal{A}_q$. We might also define a coideal $\mathfrak{k}^{-\infty}$ in the obvious way, but this will not yield anything new.

The subspace $\mathfrak{k}^\sigma \subset \mathcal{U}_q$ $(-\infty < \sigma \leq \infty)$ is a two-sided coideal invariant under the involution $\tau := * \circ S$. An element $a \in \mathcal{A}_q$ is called left $\mathfrak{k}^\sigma$-invariant if $\mathfrak{k}^\sigma \cdot a = 0$. For finite σ, the following theorem was announced in [NDS].

Theorem 4.2 — *Let $-\infty < \sigma \leq \infty$. The subalgebra $\mathcal{B}_q^\sigma \subset \mathcal{A}_q$ consists precisely of all left $\mathfrak{k}^\sigma$-invariant elements in $\mathcal{A}_q$.*

From now on, we write $\mathcal{A}_q(\mathfrak{k}^\sigma \backslash U)$ for the subalgebra $\mathcal{B}_q^\sigma$, and call it a quantized algebra of functions on the complex Grassmannian of rank l. This algebra can be written as the subalgebra of all functions in $\mathcal{A}_q$ invariant with respect to a quotient Hopf $*$-algebra of $\mathcal{A}_q$ (quantum subgroup) if and only if $\sigma = \infty$. In the limit $q \to 1$, $\mathcal{A}_q(\mathfrak{k}^\sigma \backslash U)$ is $U(n)$-isomorphic to the classical complex Grassmannian of rank l ($-\infty < \sigma \leq \infty$).

Let us now turn to the study of $\mathfrak{k}^\sigma$-biinvariant functions. More generally, let $-\infty < \tau \leq \infty$ be a second parameter. Let $\mathcal{H}^{(\sigma,\tau)} \subset \mathcal{A}_q$ denote the $*$-subalgebra of left $\mathfrak{k}^\sigma$-invariant and right $\mathfrak{k}^\tau$-invariant functions. Set $\mathcal{H}^{(\sigma,\tau)}(\lambda) := \mathcal{H}^{(\sigma,\tau)} \cap W(\lambda)$ ($\lambda \in P^+$).

We identify elements $\lambda \in P^+$ with sequences $(\lambda_1, \ldots, \lambda_n)$ of integers such that $\lambda_1 \geq \ldots \geq \lambda_n$. Let $P_\mathfrak{k}^+ \subset P^+$ be the subset of (spherical) dominant weights of the form

$$\lambda = (\mu_1, \ldots, \mu_l, 0, \ldots, 0, -\mu_l, \ldots, -\mu_1), \quad \mu_1 \geq \ldots \mu_l \geq 0. \tag{4.8}$$

Note that this defines a bijection from $P_\mathfrak{k}^+$ onto the set P_Σ^+ of dominant weights associated with $\Sigma = BC_l$.

The following theorem was announced in [NDS] for finite values of σ, τ. For the infinite case a proof is given in [DS].

Theorem 4.3 — *Let σ, τ be either both finite or both infinite. The subspace $\mathcal{H}^{(\sigma,\tau)}(\lambda) \subset \mathcal{H}^{(\sigma,\tau)}$ is non-zero if and only if $\lambda \in P_\mathfrak{k}^+$. One has the decomposition*

$$\mathcal{H}^{(\sigma,\tau)} = \bigoplus_{\lambda \in P_\mathfrak{k}^+} \mathcal{H}^{(\sigma,\tau)}(\lambda). \tag{4.9}$$

Each of the subspaces $\mathcal{H}^{(\sigma,\tau)}(\lambda)$ ($\lambda \in P_\mathfrak{k}^+$) is one-dimensional.

Any non-zero element $\varphi_\lambda \in \mathcal{H}^{(\sigma,\tau)}(\lambda)$ ($\lambda \in P_\mathfrak{k}^+$) is called a zonal spherical function corresponding to the spherical weight λ.

Let us write $m_i \in \mathbb{C}[x^{\pm 1}]^W$ for the orbit sum corresponding to the i-th fundamental weight $\varepsilon_1 + \cdots + \varepsilon_i \in P_\Sigma^+$. Similarly, write $\tilde{m}_i$ for the i-th elementary symmetric polynomial in $\mathbb{C}[x]^{\mathfrak{S}_l}$.

The following theorem was announced in [NDS].

Theorem 4.4 — *Let σ, τ be finite. There are $e_1, \ldots, e_l \in \mathcal{H}^{(\sigma,\tau)}$ such that:*

1. *the e_i ($1 \leq i \leq l$) commute, are algebraically independent, and generate the algebra $\mathcal{H}^{(\sigma,\tau)}$,*
2. *the assignment $e_i \mapsto m_i$ defines an algebra isomorphism*

$$\mathcal{H}^{(\sigma,\tau)} \xrightarrow{\simeq} \mathbb{C}[x^{\pm 1}]^W, \tag{4.10}$$

3. *any zonal spherical function φ_λ ($\lambda \in P_\mathfrak{k}^+$) is mapped under this isomorphism onto a scalar multiple of the multivariable Askey-Wilson polynomial*

$$P_\mu(x; -q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+2(n-2l)+1}; q^2, q^2),$$

where $\mu \in P_\Sigma^+$ is defined as in (4.8).

Denote the measure defined in (3.9) by $d\alpha_q(x) = d\alpha_{a,b,c,d,q,t}(x)$.

Theorem 4.5 — *Let σ, τ be finite. The restriction of the Haar functional $h : \mathcal{A}_q \to \mathbb{C}$ to $\mathcal{H}^{(\sigma,\tau)}$ is given by*

$$h(P(e_1, \dots, e_l)) = c(\sigma,\tau) \int P(m_1(x), \dots, m_l(x)) d\alpha_{q^2}(x) \quad (P \in \mathbb{C}[x_1, \dots, x_l])$$

with parameters

$$a = -q^{\sigma+\tau+1}, \; b = -q^{-\sigma-\tau+1}, \; c = q^{\sigma-\tau+1}, \; d = q^{-\sigma+\tau+2(n-2l)+1}, \; t = q^2.$$

The constant $c(\sigma, \tau)$ is a normalization constant defined by $c(\sigma,\tau) := 1/\int d\alpha_{q^2}(x)$. In case σ, τ are such that the parameters a, b, c, d defined above are less than one in absolute value, the normalization constant $c(\sigma, \tau)$ is explicitly given by the left-hand side of (3.4) with q replaced by q^2, $t = q^2$, and a, b, c, d as above.

In the case $\sigma, \tau = \infty$ we have (cf. [DS]):

Theorem 4.6 — *Let $\sigma, \tau = \infty$. There are $\tilde{e}_1, \dots, \tilde{e}_l \in \mathcal{H}^{(\infty,\infty)}$ such that:*

(1) *the $\tilde{e}_i$ $(1 \le i \le l)$ commute, are algebraically independent, and generate the algebra $\mathcal{H}^{(\infty,\infty)}$,*
(2) *the assignment $\tilde{e}_i \mapsto \tilde{m}_i$ defines an algebra isomorphism*

$$\mathcal{H}^{(\infty,\infty)} \xrightarrow{\simeq} \mathbb{C}[x]^{\mathfrak{S}_l}, \tag{4.11}$$

(3) *any zonal spherical function φ_λ $(\lambda \in P_{\mathfrak{k}}^+)$ is mapped under this isomorphism onto a scalar multiple of the multivariable little q-Jacobi polynomial*

$$P_\mu(x; q^{2(n-2l)}, 1; q^2, q^2),$$

where $\mu \in P_\Sigma^+$ is defined as in (4.8).

Denote the measure defined in (3.12) by $d\lambda_q(x) = d\lambda_{a,b,q,t}(x)$.

Theorem 4.7 — *Let $\sigma, \tau = \infty$. The restriction of the Haar functional $h : \mathcal{A}_q \to \mathbb{C}$ to $\mathcal{H}^{(\infty,\infty)}$ is given by*

$$h(P(\tilde{e}_1, \dots, \tilde{e}_l)) = c(\infty,\infty) \int P(\tilde{m}_1(x), \dots, \tilde{m}_l(x)) d\lambda_{q^2}(x) \quad (P \in \mathbb{C}[x_1, \dots, x_l])$$

with parameters $a = q^{2(n-2l)}$, $b = 1$, $t = q^2$. The normalization constant $c(\infty, \infty)$ can be directly computed from (3.3).

Remarks 4.8 — Some comments on the above results.

(1) The isomorphism (4.10) can be realized as restriction to the standard maximal torus in the quantum unitary group. This is not possible for the

isomorphism (4.11). In fact, it can be shown that the restriction to the standard maximal torus of $\mathfrak{k}^{\infty}$-biinvariant functions is always zero.

(2) The proof of the above theorems is rather long and consists of several steps. The proof of Theorem 4.4 (cf. [NDS]) is based on the fact that the radial part of a certain quantum Casimir operator essentially coincides with Koornwinder's partial q-difference operator diagonalized by the multivariable Askey-Wilson polynomials [K] (for the indicated values of the parameters). Theorem 4.5 can be derived from this using the results by Stokman [S3] and Gustafson [Gu]. The proof of Theorem 4.6 is based on Theorem 4.4 and the fact that the limit transition from multivariable Askey-Wilson polynomials to multivariable little q-Jacobi polynomials as described in [SK], [S2] is compatible with the limit $\sigma, \tau \to \infty$ on quantum Grassmannians. For the details see [DS]. Theorem 4.7 follows from Theorem 4.6 and the results by Stokman [S1], Habsieger [Hb], and Kadell [Ka].

(3) As far as I know, an explicit formula for the constant term $c(\sigma, \tau)$ in Theorem 4.5 (arbitrary values of σ, τ) has not been published. However, Cherednik [C] has proved similar formulas (conjectured by Macdonald) for reduced root systems by using shift operators. Macdonald [M2] has claimed that Cherednik's methods extend to the case of multivariable Askey-Wilson polynomials. The details have not yet been disclosed.

(4) There are similar results for the case in which σ is finite and τ infinite (or vice versa). The resulting zonal spherical functions can be expressed as multivariable big q-Jacobi polynomials [S1]. The corresponding constant term identity is another q-analogue of Selberg's integral, which was conjectured by Askey [A] and proved by Evans [E]. For details see [DS].

(5) In the case $l = 1$, all the results stated in this section are contained in [DN]. See the introduction of that paper for a discussion of other contributions to the rank-one case.

5. Epilogue

The results discussed in this paper raise a number of obvious questions. For instance, the result by Khoroshki, Radul, and Rubtsov (Theorem 3.2) suggests that much of the content of this paper might generalize to arbitrary (classical) Hermitian symmetric spaces. As a matter of fact, I have found a one-parameter family of solutions J^{σ} to the reflection equation in the case of the Hermitian symmetric space $SO(2n)/U(n)$. In the special case $\sigma = 0$, the zonal spherical functions were shown to be expressible in terms of multivariable Askey-Wilson polynomials by Noumi and Sugitani [NS]. I hope to pursue this direction of research in more detail in the near future.

Second, it would be desirable to obtain a more direct proof of Theorems 4.5 and 4.7 that does not depend so much on known results about multivariable orthogonal polynomials. This might be done by taking the primitive ideal space of the various quantized algebras of functions into account, which is in its turn related to the symplectic foliation of the Poisson structure induced in the semi-classical limit. In the rank-one situtation, there is some interesting work in this direction by Vaksman and Soibel'man [VS2] and Koelink and Verding [KV].

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