

## ON CERTAIN SMALL UNITARY REPRESENTATIONS OF INDEFINITE ORTHOGONAL GROUPS

CHEN-BO ZHU AND JING-SONG HUANG

**ABSTRACT.** For any  $n \in \mathbb{N}$  such that  $2n \leq \min(p, q)$ , we construct a representation  $\pi_n$  of  $O(p, q)$  with  $p + q$  even as the kernel of a commuting set of  $\frac{n(n+1)}{2}$  number of  $O(p, q)$ -invariant differential operators in the space of  $C^\infty$  functions on an isotropic cone with a distinguished  $GL_n^+(\mathbb{R})$ -homogeneity degree. We show (except when  $p = q = 2n$ ) that the  $K$ -finite vectors of  $\pi_n$  is the  $(\mathfrak{g}, K)$  module of an irreducible unitarizable representation of  $O(p, q)$  with the Gelfand-Kirillov dimension  $n(p + q - 2n - 1)$ .

### 1. CONSTRUCTION OF THE REPRESENTATION $\pi_n$

Let  $V = \mathbb{R}^p \oplus \mathbb{R}^q$  be the real vector space equipped with the quadratic form

$$(v_1, v_2) = v_1^t I_{p,q} v_2, \quad v_1, v_2 \in V,$$

where  $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ . Let  $G = O(p, q)$  be the isometry group.

Fix a natural number  $n$ . We always assume  $p + q$  even and  $2n \leq \min(p, q)$ . Let  $V^n$  be the direct sum of  $n$  copies of  $V$ , which can be identified with the space  $M_{p+q,n}(\mathbb{R})$  of  $(p + q) \times n$  real matrices. Let

$$X = (x_{ij})_{1 \leq i \leq p+q, 1 \leq j \leq n} \in M_{p+q,n}(\mathbb{R}),$$

and for  $1 \leq i, j \leq n$ , let  $r_{ij}$  be quadratic forms defined by

$$(1.1) \quad r_{ij}(X) = \sum_{t=1}^p x_{ti} x_{tj} - \sum_{t=p+1}^{p+q} x_{ti} x_{tj}.$$

Define the isotropic cone  $\mathcal{X}$ :

$$\mathcal{X} = \{X \in M_{p+q,n}(\mathbb{R}) \mid r_{ij}(X) = 0, 1 \leq i, j \leq n\},$$

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The details of this work will appear elsewhere.

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and the following dense subset of  $\mathcal{X}$ :

$$\mathcal{X}^{00} = \{X \in \mathcal{X} \mid \text{rank } X = n\}.$$

The group  $G = O(p, q)$  acts on  $V^n \simeq M_{p+q, n}(\mathbb{R})$  and the action is identified with left multiplication:

$$X \mapsto gX, \quad g \in O(p, q), \quad X \in M_{p+q, n}(\mathbb{R}).$$

The group  $H = GL_n(\mathbb{R})$  acts on  $M_{p+q, n}(\mathbb{R})$  on the right by

$$X \mapsto Xh^{-1}, \quad h \in GL_n(\mathbb{R}), \quad X \in M_{p+q, n}(\mathbb{R}).$$

Clearly the actions of  $G$  and  $H$  on  $M_{p+q, n}(\mathbb{R})$  commute, and both actions preserve the isotropic cone  $\mathcal{X}$ , and  $\mathcal{X}^{00}$ .

Define the following set of differential operators

$$(1.2) \quad \Delta_{ij} = \sum_{t=1}^p \frac{\partial^2}{\partial x_{ti} \partial x_{tj}} - \sum_{t=p+1}^{p+q} \frac{\partial^2}{\partial x_{ti} \partial x_{tj}},$$

$$(1.3) \quad E_{ij} = \sum_{t=1}^{p+q} x_{ti} \frac{\partial}{\partial x_{tj}},$$

for  $1 \leq i, j \leq n$ . Then  $\{r_{ij}, \Delta_{ij}, E_{ij}\}_{1 \leq i, j \leq n}$  generate the algebra of  $O(p, q)$ -invariant polynomial differential operators on  $M_{p+q, n}(\mathbb{R})$ .

Let  $GL_n^+(\mathbb{R})$  be the subgroup of  $GL_n(\mathbb{R})$  consisting of elements with positive determinant. For  $\alpha \in \mathbb{C}$ , let  $C^\infty(\mathcal{X}^{00}, \alpha)$  be the space of  $C^\infty$  functions  $f$  on  $\mathcal{X}^{00}$  such that

$$f(Xh) = (\det h)^\alpha f(X), \quad h \in GL_n^+(\mathbb{R}), \quad X \in \mathcal{X}^{00}.$$

Let  $U$  be an open conical neighborhood of  $\mathcal{X}^{00}$ , namely any open neighborhood of  $\mathcal{X}^{00}$  which is stable under  $GL_n^+(\mathbb{R})$ . Define  $C^\infty(U, \alpha)$  likewise. Let  $e_{ij}$  be the  $n \times n$  matrix with one at the  $(i, j)$  entry and zero's elsewhere. Together they form a basis for the Lie algebra  $\mathfrak{gl}_n(\mathbb{R})$  of  $GL_n(\mathbb{R})$ . We observe that the derived action of  $\mathfrak{gl}_n(\mathbb{R})$  on  $C^\infty(U)$  is given by:

$$e_{ij} \mapsto E_{ij}.$$

Thus if  $F \in C^\infty(U, \alpha)$ , then

$$E_{ij}F = \alpha \delta_{ij}F, \quad 1 \leq i, j \leq n,$$

where  $\delta_{ii} = 1$ , and  $\delta_{ij} = 0$  for  $i \neq j$ .

Let  $f \in C^\infty(\mathcal{X}^{00}, \alpha)$ . We would like to apply the differential operator  $\Delta_{kl}$  to it. To this end, we first extend  $f$  to a conical neighborhood  $U$  of  $\mathcal{X}^{00}$  in a conical way, namely we require the extension  $\tilde{f}$  to belong to  $C^\infty(U, \alpha)$ . This is always possible. We can apply  $\Delta_{kl}$  to the extension  $\tilde{f}$  and then restrict  $\Delta_{kl}(\tilde{f})$  back to  $\mathcal{X}^{00}$ . The problem is that the restriction may depend on the choice of the extension. We shall show for  $\alpha = n + 1 - \frac{p+q}{2}$ ,  $\Delta_{kl}(\tilde{f})|_{\mathcal{X}^{00}}$  is independent of the extension  $\tilde{f}$ . It clearly suffices to demonstrate the following

**Proposition 1.1.** *Let  $U$  be a conical neighborhood of  $\mathcal{X}^{00}$  in  $M_{p+q, n}(\mathbb{R})$ . If  $F \in C^\infty(U, n + 1 - \frac{p+q}{2})$  and  $F|_{\mathcal{X}^{00}} = 0$ , then  $\Delta_{kl}(F)|_{\mathcal{X}^{00}} = 0$  for  $1 \leq k, l \leq n$ .*

*Proof.* Since  $2n \leq \min(p, q)$ , we see that the  $\frac{n(n+1)}{2}$  polynomials  $r_{ij}$  ( $1 \leq i \leq j \leq n$ ) are algebraically independent. In fact the Jacobian matrix  $J(r_{11}, r_{12}, \dots, r_{nn})$  at any point of  $\mathcal{X}^{00}$  has full rank  $\frac{n(n+1)}{2}$  ([T], Lemma 2.9). Thus they are functionally independent. Using this fact and the fact  $F|_{\mathcal{X}^{00}} = 0$ , we can write in a unique way

$$F = \sum_{1 \leq i, j \leq n} r_{ij} F_{ij},$$

where  $F_{ij} = F_{ji} \in C^\infty(U')$ , for a small conical neighbourhood  $U' \subset U$  of  $\mathcal{X}^{00}$ .

We compute:

$$\Delta_{kl} \left( \sum_{ij} r_{ij} F_{ij} \right) = \sum_{ij} r_{ij} \Delta_{kl}(F_{ij}) + \sum_{ij} [\Delta_{kl}, r_{ij}] F_{ij}.$$

We have

$$[\Delta_{kl}, r_{ij}] = \delta_{jk} H_{il} + \delta_{il} H_{jk} + \delta_{jl} H_{ik} + \delta_{ik} H_{jl},$$

where

$$(1.4) \quad H_{ij} = E_{ij} + \frac{p+q}{2} \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Now since  $F = \sum_{ij} r_{ij} F_{ij} \in C^\infty(U, \alpha)$ , we have (as functions in  $C^\infty(U')$ )

$$E_{kl} \left( \sum_{ij} r_{ij} F_{ij} \right) = \alpha \delta_{kl} \left( \sum_{ij} r_{ij} F_{ij} \right), \quad 1 \leq k, l \leq n.$$

Using  $E_{kl}(r_{ij}) = \delta_{li} r_{kj} + \delta_{lj} r_{ki}$ , we obtain from the above equation that

$$E_{kl}(F_{ij}) = \alpha \delta_{kl} F_{ij} - \delta_{ik} F_{lj} - \delta_{jk} F_{li},$$

and so

$$H_{kl}(F_{ij}) = (\alpha + \frac{p+q}{2})\delta_{kl}F_{ij} - \delta_{ik}F_{lj} - \delta_{jk}F_{li}.$$

We compute to get

$$\begin{aligned} \sum_{ij} [\Delta_{kl}, r_{ij}]F_{ij} &= \sum_{ij} (\delta_{jk}H_{il} + \delta_{il}H_{jk} + \delta_{jl}H_{ik} + \delta_{ik}H_{jl})F_{ij} \\ &= 4(\alpha + \frac{p+q}{2} - n - 1)F_{kl}. \end{aligned}$$

To summarize, we have

$$\Delta_{kl}(\sum_{ij} r_{ij}F_{ij}) = \sum_{ij} r_{ij}\Delta_{kl}(F_{ij}) + 4(\alpha + \frac{p+q}{2} - n - 1)F_{kl}.$$

If  $\alpha = n + 1 - \frac{p+q}{2}$ , then we clearly have

$$\Delta_{kl}(\sum_{ij} r_{ij}F_{ij})|_{\mathcal{X}^{00}} = 0. \quad \square$$

Thus for  $1 \leq k \leq l \leq n$ , we have the following differential operator  $\Delta'_{kl}$  defined on  $C^\infty(\mathcal{X}^{00}, n + 1 - \frac{p+q}{2})$ :

$$\Delta'_{kl}(f) = \Delta_{kl}(\tilde{f})|_{\mathcal{X}^{00}}, \quad f \in C^\infty(\mathcal{X}^{00}, n + 1 - \frac{p+q}{2}),$$

where  $\tilde{f}$  is a conical extension of  $f$  to a conical neighborhood  $U$  of  $\mathcal{X}^{00}$ .

Since the action of  $G$  commutes with  $\Delta_{kl}$ , the action of  $G$  also commutes with  $\Delta'_{kl}$ . Let

$$(1.5) \quad \mathcal{H}_n = \{f \in C^\infty(\mathcal{X}^{00}, n + 1 - \frac{p+q}{2}) \mid \Delta'_{kl}(f) = 0, \text{ for } 1 \leq k \leq l \leq n\}.$$

Then  $G$  acts on  $\mathcal{H}_n$ . We denote this representation of  $G$  by  $\pi_n$ , where  $2n \leq \min(p, q)$ .

**Theorem 1.2.** *Assume that  $2n \leq \min(p, q)$  (except for  $p = q = 2n$ ). Then*

*a) the  $K$ -finite vectors of  $\pi_n$  is the  $(\mathfrak{g}, K)$ -module of an irreducible unitarizable representation of  $O(p, q)$ , and*

*b)  $\pi_n$  has the Gelfand-Kirillov dimension  $n(p + q - 2n - 1)$ .*

*Remarks.* A) For  $n = 1$ , our construction reduces to that of Binegar and Zierau, which in turn is a generalization of the work of Kostant for  $p = q = 4$  ([K1], [K2]). It turns out that  $\pi_1$  is the minimal representation of  $O(p, q)$  ( $p+q$  even,  $\min(p, q) \geq 2$ ,

except for  $p = q = 2$ , cf. [BZ]). There the irreducibility and the unitarizability of  $\pi_1$  is proved by long calculations involving spherical functions and Gegenbauer polynomials. Our strategy is simple, and it is to identify the representation  $\pi_n$  with certain representation constructed via the formalism of theta correspondence. Then the statement on the irreducibility and the unitarizability will be derived from the powerful result of Li on theta lifting in the stable range ([L]).

B) We remark also that for  $n = 1$ , the infinitesimal structure of  $C^\infty(\mathcal{X}^{00}, \alpha)$  as an  $O(p, q)$  module is the main object of study in Howe and Tan's work [HT], and is completely determined for any  $\alpha$ . In particular, they pointed out that when  $p + q$  is even, the space  $C^\infty(\mathcal{X}^{00}, 2 - \frac{p+q}{2})$  contains a unitary constituent whose  $K$ -types are distributed along a single line (a ladder representation), and they went on to say that this representation should be obtained from the theta lift of the trivial representation for the dual pair  $(O(p, q), SL_2(\mathbb{R}))$ . Our article is partly motivated by this important observation.

C) The representations  $\pi_n$  should be unipotent in the sense of Vogan, and can be thought of as the orthogonal analog of some unipotent representations of hermitian symmetric groups (e.g.  $Sp(2n, \mathbb{R})$ ) constructed by Sahi ([S]). All of them are theta lifts of unitary characters in the stable range. C.f. [LZ1], [LZ2].

D) Philosophically it is rather conceivable that the representation  $\pi_n$  has the GK-dimension  $n(p + q - 2n - 1)$ . The space of  $C^\infty$  functions on the isotropic cone with a fixed  $GL_n^+(\mathbb{R})$  homogeneity degree satisfies  $\frac{n(n+1)}{2} + n^2$  algebraic equations. As the kernel of  $\frac{n(n+1)}{2}$  differential operators on that space, the representation  $\pi_n$  "should" have GK-dimension  $n(p+q) - (\frac{n(n+1)}{2} + n^2) - \frac{n(n+1)}{2} = n(p+q-2n-1)$ . Our argument relies on a description of  $K$ -types of  $\pi_n$  and thus its asymptotic behavior is readily computed.

## 2. HOWE QUOTIENTS $\Omega(0)$ AND THEIR $K$ -TYPES

Let  $(G, G')$  be the following reductive dual pair:

$$(O(p, q), Sp(2n, \mathbb{R})) \subset Sp(2(p+q)n, \mathbb{R}).$$

Denote  $Sp = Sp(2(p+q)n, \mathbb{R})$ . Let  $\tilde{Sp}$  be the metaplectic cover of  $Sp$ . For a reductive subgroup  $E$  of  $Sp$ , let  $\tilde{E}$  be the inverse image of  $E$  inside  $\tilde{Sp}$ .

Let  $K \cong O(p) \times O(q)$  and  $K' \cong U(n)$  be maximal compact subgroups of  $G$  and  $G'$ , respectively. We may choose a maximal compact subgroup  $U$  of  $Sp$  in such a way that

$$K = U \cap G \text{ and } K' = U \cap G'.$$

Let  $\omega$  be the oscillator representation of  $\tilde{S}p$  which we fix by specifying a non-trivial unitary additive character of  $\mathbb{R}$ . We shall apply Howe Duality Theorem [H2] in the context of  $(\mathfrak{g}, \tilde{K})$ -modules. Let  $S$  be the space of  $\tilde{U}$ -finite vectors of  $\omega$ , which consists of polynomials in a Fock model of  $\omega$ .  $S$  is naturally a  $(\mathfrak{g}', \tilde{K}')$ -module. Let  $\Omega(0)$  be the maximal quotient of  $S$  on which  $(\mathfrak{g}', \tilde{K}')$  acts trivially. Howe Duality Theorem [H2] tells us that  $\Omega(0)$  is a quasi-simple  $(\mathfrak{g}, \tilde{K})$ -module of finite length and has a unique irreducible quotient, denoted by  $Q(0)$ . Thus  $Q(0)$  is the representation of  $(\mathfrak{g}, \tilde{K})$  which corresponds to the trivial  $(\mathfrak{g}', \tilde{K}')$  module under Howe's quotient correspondence.

Since  $2n \leq \min(p, q)$ , the dual pair in question is stable with  $G'$  the small member. Further since  $p + q$  is even, the coverings  $\tilde{G} \rightarrow G$ ,  $\tilde{G}' \rightarrow G'$  are both trivial. From Li's result [L] we see Howe's correspondence will then give rise to an injection  $\hat{G}' \hookrightarrow \hat{G}$  (except for  $p = q = 2n$ ). This implies in particular that  $Q(0)$  is unitary. Moreover by using Li's construction of Howe's correspondence, we can show that  $\Omega(0) = Q(0)$  in this case. Thus we have

**Proposition 2.1.** *If  $2n$  is less than or equal to  $\min(p, q)$  (except for  $p = q = 2n$ ), then  $\Omega(0)$  is irreducible and unitary.*

Next we describe the  $K$ -types of  $\Omega(0)$ . We shall adopt Weyl's convention ([W], [A]) to parameterize representations of  $O(p)$ , namely by restriction from  $U(p)$ .

**Proposition 2.2.** *Let  $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$  be the reductive dual pair in  $Sp(2(p + q)n, \mathbb{R})$ ,  $\omega$  be the oscillator representation of  $\tilde{S}p(2(p + q)n, \mathbb{R})$ . Assume  $2n \leq \min(p, q)$  as before. Then*

(a) *the  $K$ -types in  $\Omega(0)$  are of the form*

$$(a_1, a_2, \dots, a_n, 0, \dots, 0; 1) \otimes (b_1, b_2, \dots, b_n, 0, \dots, 0; 1),$$

where  $a_1 \geq \dots \geq a_n \geq 0$ ,  $b_1 \geq \dots \geq b_n \geq 0$ , and  $a_i + \frac{p}{2} = b_i + \frac{q}{2}$  for  $1 \leq i \leq n$ . Moreover, each such  $K$ -type occurs with multiplicity one.

(b) *The Gelfand-Kirillov dimension of  $\Omega(0)$  is equal to  $n(p + q - 2n - 1)$ .*

The proof of part a) is quite similar to that of Proposition 2.1 of [KR1], and Proposition 2.1 of [LZ1]. Also Part a) allows us to estimate the growth of the dimension of  $K$ -types of  $\Omega(0)$ . Part b) then follows from this estimate.

3. IDENTIFICATION OF  $\pi_n$  WITH THE HOWE QUOTIENT

Let  $S^{p,n}$  be the following Stiefel manifold:

$$S^{p,n} = \{V \in M_{p,n}(\mathbb{R}) \mid V^t V = I_n\}.$$

Similarly we define the Stiefel manifold  $S^{q,n}$ . Let  $P_n$  be the set of  $n \times n$  positive definite symmetric matrices.

Let  $X = \begin{pmatrix} Y \\ Z \end{pmatrix} \in \mathcal{X}^{00}$ , where  $Y \in M_{p,n}(\mathbb{R})$ ,  $Z \in M_{q,n}(\mathbb{R})$ . Then we have the following ‘‘polar coordinates’’ decomposition:

$$X = \begin{pmatrix} V \\ W \end{pmatrix} R^{\frac{1}{2}}, \quad V \in S^{p,n}, \quad W \in S^{q,n}, \quad R \in P_n.$$

Fix  $R \in P_n$ . Given any neighborhood  $U$  of  $\mathcal{X}^{00}$  and  $f \in C^\infty(U)$ , define

$$J_R(f) = \int_{S^{p,n} \times S^{q,n}} f\left(\begin{pmatrix} V \\ W \end{pmatrix} R^{\frac{1}{2}}\right) dV dW,$$

where  $dV, dW$  are the rotation-invariant probability measures on  $S^{p,n}$  and  $S^{q,n}$  respectively. It is easy to see that

- a)  $J_R(f)$  depends only on  $f|_{\mathcal{X}^{00}}$ , and
- b)  $J_R(f_1(\Delta_{kl}f_2)) = J_R((\Delta_{kl}f_1)f_2)$ , for  $f_1, f_2 \in C^\infty(U)$ .

Recall we have the oscillator representation  $\omega$  of  $\tilde{Sp}(2(p+q)n, \mathbb{R})$  associated to the dual pair  $(G, G') = (O(p, q), Sp(2n, \mathbb{R})) \subset Sp(2(p+q)n, \mathbb{R})$ . It is realised on  $L^2(V^n)$  via the Schrodinger model. Let  $S = \mathcal{S}(V^n)$  be the space of Schwartz functions on  $V^n$ . Let  $R(0) = S/\omega(\mathfrak{g}')S$ , where  $\mathfrak{g}'$  is the Lie algebra of  $G' = Sp(2n, \mathbb{R})$  as before. We have

$$\omega(\mathfrak{g}') = \text{span}\{r, \Delta, H\},$$

where  $r = \text{span}\{r_{ij}\}_{1 \leq i, j \leq n}$ ,  $\Delta = \text{span}\{\Delta_{ij}\}_{1 \leq i, j \leq n}$ ,  $H = \text{span}\{H_{ij}\}_{1 \leq i, j \leq n}$ . See Equations (1.1), (1.2), (1.3), (1.4) in §1 for the definitions of  $r_{ij}$ ,  $\Delta_{ij}$  and  $H_{ij}$ . Note that the underlying Harish-Chandra module of  $R(0)$  is the Howe quotient  $\Omega(0)$  (cf. §2).

We define an  $O(p, q)$ -equivariant map

$$T: C^\infty(\mathcal{X}^{00}, n+1 - \frac{p+q}{2}) \longrightarrow R(0)$$

as follows. For  $f \in C^\infty(\mathcal{X}^{00}, n+1 - \frac{p+q}{2})$ , extend  $f$  conically to  $\tilde{f} \in C^\infty(U, n+1 - \frac{p+q}{2})$  where  $U$  is an open conical neighborhood of  $\mathcal{X}^{00}$ . Define  $Tf \in R(0)$  to be the

coset of  $\tilde{f}$  in  $R(0)$ . The map  $T$  is well-defined since any two extensions  $\tilde{f}_1$  and  $\tilde{f}_2$  of  $f$  differ by an element of  $\text{span}\{r_{ij}\mathcal{S}\}_{1 \leq i, j \leq n} \subseteq \omega(\mathfrak{g}')\mathcal{S}$ . Since  $O(p, q)$  commutes with the actions of  $H = GL_n(\mathbb{R})$  and  $\omega(\mathfrak{g}')$ , we see that  $T$  is  $O(p, q)$ -equivariant.

Recall the space  $\mathcal{H}_n$ :

$$H_n = \{f \in C^\infty(\mathcal{X}^{00}, n+1 - \frac{p+q}{2}) : \Delta_{kl}\tilde{f}|_{\mathcal{X}^{00}} = 0, 1 \leq k, l \leq n\},$$

on which  $O(p, q)$  acts by the representation  $\pi_n$ .

**Proposition 3.2.**  $T|_{\mathcal{H}_n}$  is an  $O(p, q)$ -equivariant embedding with dense image.

*Proof.* Let  $f \in \mathcal{H}_n$ . Suppose  $Tf = 0$ , namely  $\tilde{f} \in \omega(\mathfrak{g}')\mathcal{S}$ . Since  $H$  is the commutator algebra of  $r$  and  $\Delta$ , we may assume  $\tilde{f} \in \text{span}\{r\mathcal{S}, \Delta\mathcal{S}\}$ . Thus we can write

$$\tilde{f} = \sum_{ij} \Delta_{ij}(h_{ij}) + \sum_{ij} r_{ij}f_{ij},$$

where  $f_{ij}, h_{ij} \in \mathcal{S}$ . Set  $h = \sum_{ij} \Delta_{ij}(h_{ij})$ . Clearly we have  $\tilde{f}|_{\mathcal{X}^{00}} = h|_{\mathcal{X}^{00}}$ .

For any  $R \in P_n$ , we have  $J_R(\tilde{f}\Delta_{kl}(h_{kl})) = J_R((\Delta_{kl}\tilde{f})h_{kl}) = 0$ , since  $\Delta_{kl}\tilde{f}|_{\mathcal{X}^{00}} = 0$ . This implies that  $J_R(h\Delta_{kl}(h_{kl})) = 0$ , and so  $J_R(h^2) = 0$ . We then conclude that  $h|_{S^{p,n} \times S^{q,n} \times \{R\}} = 0$ . Since  $R \in P_n$  is arbitrary, we see that  $h|_{\mathcal{X}^{00}} = 0$ , and so  $f = \tilde{f}|_{\mathcal{X}^{00}} = 0$ . This proves that  $T$  is injective.

Since the Harish-Chandra module of  $R(0)$ , namely  $\Omega(0)$ , is irreducible, we see that the image of  $T|_{\mathcal{H}_n}$  must be dense in  $R(0)$ .  $\square$

Proposition 3.2 and Propositions 2.1, 2.2 will imply Theorem 1.2.

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