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Whittaker - 新谷関数

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村瀬，菅野両氏との共同研究で得られた 代数群上の特殊関数（球関数），
Whittaker - 新谷関数に関する結果を報告する.

SL_2 の正則保型形式の Fourier 係数を用いて L 関数が構成されることはよく知られている．表現論的には

$$\text{Fourier係数} \Leftarrow SL_2 \text{ の極大巾単部分群の 1 次表現}$$

であり，一般の場合でも種々の L 関数の構成に Whittaker 関数（$\Leftarrow$ reductive 群の極大巾単部分群の非退化 1 次表現）が役に立つ.

新谷は $U(2,1)$ の正則保型形式の

$$\text{Fourier-Jacobi係数} \Leftarrow \begin{array}{l} U(2,1) \text{ の極大巾単部分群（＝Heisenberg群）の} \\ \text{Schrödinger表現} \end{array}$$

を用いて L 関数を構成した [Sh2]. さらに [Sh3] では Sp_{2n} に対して Fourier-Jacobi 係数に対応する特殊関数を導入して，Siegel 保型形式の L 関数への応用を論じた.

この特殊関数が Whittaker - 新谷関数である．Whittaker - 新谷関数は，その後，村瀬，菅野によって研究 [MS1]，他の群への拡張（新谷関数；[MS2] ＝直交群, [MS3] ＝ GL_n）がなされ，保型 L 関数への応用が与えられている.

ここでは，上に現れた p 進体上の Whittaker - 新谷関数（の拡張）の重複度 1 定理，明示公式等を解説する [KMS]．これらは p 進体上の Whittaker 関数の場合（[Sh1], [CS], [K1]）の類似になっている（証明も）．なお，球関数論としての一般化については [K2]，保型 L 関数への応用については [M] も参照されたい.

以下，この稿では奇数次直交群を取り扱う．偶数次の場合も同様．なお，準備の都合で OHP 原稿（の一部分）を使わせて頂いたことをお許し願いたい.

$\boxed{\text{Groups}}$

- k : p-adic field $\supset \mathcal{O}$. integers
  π : prime ele. $\#(\mathcal{O}/\pi\mathcal{O}) = q$

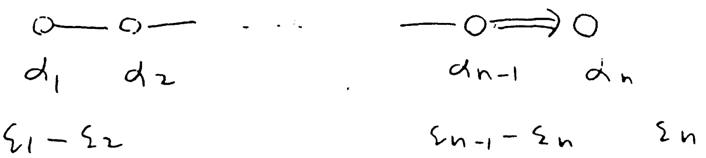
- $G = G(k) = O(2n+1)$ split orthog.
  $\subset GL_{2n+1}(k) \quad \updownarrow \quad J_{2n+1} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$
  symmetric

< subgroups >

$K = G(\mathcal{O})$: max. compact

$P = TN$: Borel (upper triangular)
T : max. torus (diagonal)
N. : max. unip.

- Φ : roots of G



- Q : max. parabolic $\longleftrightarrow \alpha_r$
  $(n = m + r)$

$$Q = L \cdot U \qquad \text{Levi decomp}$$

$$L = \underset{\substack{\parallel \\ L_1}}{GL(r)} \times \underset{\substack{\parallel \\ L_2}}{O(2m+1)}$$

$$Q = \left\{ \begin{pmatrix} \boxed{a} & & * \\ & \overset{L_2}{\boxed{O(2m+1)}} & \\ 0 & & \boxed{a'} \end{pmatrix} \ (a \in GL(r)) \right\}$$

$$(a' = J_r \, {}^t a^{-1} J_r)$$

- $G_0 = O(2m) \subset O(2m+1) = L_2$

  $\Phi_0 = \{ \pm \varepsilon_i \pm \varepsilon_j \ (r+1 \leq i < j \leq n) \} \subset \Phi$
  : roots of G_0

- $H \overset{\text{def}}{=} (N_1 \times G_0) \cdot U$

  N_1 = max. unipotent (unitriangular) of $GL(r) = L_1$

$$H = \left\{ \begin{pmatrix} \diagdown & & * \\ & \boxed{O(2m)} & \\ 0 & & \diagdown \end{pmatrix} \right\} \qquad \text{(cf. [GP])}$$

$$R_u(H) = N_1 \cdot U$$
$$H / R_u(H) \simeq G_0$$

$\left(\begin{array}{lll} r = 0. & H = G_0. & \text{reductive} \\ m = 0 & H = N & \text{max. unipotent} \end{array} \right)$

$\boxed{\text{Whittaker – Shintani function}}$

- $X_{nr}(T) \cong (\mathbb{C}^\times)^n$ unramified character
  $\quad \| $
  $\quad \mathrm{Hom}(T/T \cap K, \mathbb{C}^\times)$

$\mathcal{H} = C_c^\infty(G, K)$ Hecke alg. (convolution product)

$\lambda \in X_{nr}(T) \Longrightarrow \omega_\lambda : \mathcal{H} \longrightarrow \mathbb{C}$. alg. hom (Satake)

Similarly,

$X_{nr}(T_0) \ni \mu \Rightarrow \omega_\mu : \mathcal{H}_0 \longrightarrow \mathbb{C}$.
$\quad \quad \| $
$\quad C_c^\infty(G_0, K_0)$

- $\psi_0 : R \longrightarrow \mathbb{C}^\times$ non-triv additive character (conductor $\mathcal{O}$) fix

$\psi : R_u(H) = N_1 \cdot U \longrightarrow \mathbb{C}^\times$ (hom)

defined by

$$\begin{cases} \psi|_{X_{\varepsilon_i - \varepsilon_{i+1}}} = \psi_0 \\ \psi|_{X_{\varepsilon_r}} = \psi_0 \\ \psi|_{X_\alpha} \equiv 1 \quad \text{(otherwise)} \end{cases} \qquad \left(\begin{array}{l} \alpha \in \Phi \\ X_\alpha \overset{x_\alpha}{\cong} R \\ \text{root subgp} \end{array} \right)$$

. ψ : H - invariant. (cf. [GP])

<u>Def.</u> $(\lambda, \mu) \in X_{nr}(T) \times X_{nr}(T_0)$.

$WS(\lambda,\mu) \overset{\text{def}}{=} \{ F : G \to \mathbb{C}$ s.t.

(i) $L(\varphi)F = \omega_\lambda(\varphi)F \quad (\varphi \in \mathcal{H})$

(ii) $R(\varphi')F = \omega_\mu(\varphi')F \quad (\varphi' \in \mathcal{H}_0)$

(iii) $F(gu) = \psi(u)F(g) \quad (u \in R_u(H))\}$

<u>Whittaker - Shintani functions</u>

($G \underset{L}{\curvearrowright} G \underset{R}{\curvearrowleft} G_0$ regular representation)

$$\left(\begin{array}{lll} m=0. & \Rightarrow & \overset{\text{(unramified)}}{\text{class-1 Whittaker fn}} \\ r=0 & \Rightarrow & \text{Shintani fn [MS2]} \end{array} \right.$$

$\boxed{\text{Double coset decomposition}}$

$$g_0 \overset{\text{def}}{=} w_\ell \prod_{k=r+1}^{n} x_{\varepsilon_k}(1)$$

longest ele. in W_G (Weyl gp)

Th. 1. $G = \bigsqcup K\, d(\ell)\, g_0\, d_0(\ell')^{-1} K_0 R_u(H)$

$\ell = (\ell_1, \cdots \ell_n)$
$\ell_{r+1} \geqq \cdots \geqq \ell_n \geqq 0$
$\ell' = (\ell_1', \cdots, \ell_m')$
$\ell_1' \geqq \cdots \geqq \ell_m' \geqq 0$ ┘

$$\left(\begin{array}{l} d(\ell) = \begin{pmatrix} \pi^{\ell_1} & & & & & \\ & \ddots & & & & \\ & & \pi^{\ell_n} & & & \\ & & & 1 & & \\ & & & & \pi^{-\ell_n} & \\ & & & & & \ddots \\ & & & & & & \pi^{-\ell_1} \end{pmatrix} \in T. \\ d_0(\ell') = \begin{pmatrix} 1_r & & & \\ & \pi^{\ell_1'} & & \\ & & \ddots & \\ & & & \pi^{-\ell_1'} \\ & & & & 1_r \end{pmatrix} \in T_0 = T \cap G_0 \end{array} \right.$$

<u>Cor</u> (1) $F \in WS(\lambda, \mu)$

$$\operatorname{supp} F \subset \bigsqcup K\, d(\ell)\, g_0\, d_0(\ell')^{-1} K_0 R_u(H)$$

$$\left.\begin{array}{l} \ell, \ell' \text{ as before} \\ \ell_1 \geqq \cdots \geqq \ell_n \geqq 0 \end{array}\right\} \cdots (*)$$

(2) $\dim WS(\lambda, \mu) \leqq 1$

$\boxed{\text{Explicit formula}}$

$\lambda \in X_{nr}(T) \simeq (\mathbb{C}^\times)^n$

$\lambda(d(\ell)) = \lambda_1^{\ell_1} \cdots \lambda_n^{\ell_n} \qquad (\ell \in \mathbb{Z}^n)$

<u>Th. 2.</u> (1) $WS(\lambda, \mu) = \mathbb{C} \cdot {}^{\exists}F_{\lambda, \mu}$, $F_{\lambda,\mu}(1) = 1$

(2) $F_{\lambda, \mu}(d(\ell)\, g_0\, d_0(\ell')^{-1})$

$$= (\text{const}) \sum_{\substack{w \in W_G \\ w' \in W_{G_0}}} c(w\lambda, w'\mu)\, ((w\lambda)\, \delta^{1/2})(d(\ell)) \times$$

$$\times\, ((w'\mu)\, \delta_0^{1/2})(d_0(\ell'))$$

$$((\ell, \ell') \cdots (*))$$

$$= \quad 0 \qquad (\text{otherwise})$$

$c(\lambda, \mu)$: rational fn in λ, μ

"numerator" =

$$\prod_{\substack{1 \le i \le n \\ 1 \le j \le m}} (1 - q^{-1/2} (\lambda_i^{-1} \mu_j)^{\eta_{ij}})(1 - q^{-1/2} \lambda_i^{-1} \mu_j^{-1})$$

$$\eta_{ij} = \begin{cases} 1 & (i \leqq r+j) \\ -1 & (i > r+j) \end{cases}$$

"denominator" =

$$\prod_{1 \le i < j \le n} (1 - \lambda_i^{-1} \lambda_j)(1 - \lambda_i^{-1} \lambda_j^{-1}) \prod_{1 \le i \le n} (1 - \lambda_i^{-2})$$

$$\times \prod_{1 \le i < j \le m} (1 - \mu_i^{-1} \mu_j)(1 - \mu_i^{-1} \mu_j^{-1})$$

$\delta : P \longrightarrow \mathbb{R}_+^\times$ modulus character
$(\delta_0 : P_0 = G_0 \cap P \longrightarrow \mathbb{R}_+^\times)$

$$(\text{const}) = (1 - q^{-m}) \prod_{1 \le i \le m-1} (1 - q^{-2i})$$

<u>Remark</u>

$m = 0 \Rightarrow$ explicit formula for Whittaker fn [CS], [K1]

Local L-factors

$$\nu \in X_{nr}(T_{GL(r)}) \cong (\mathbb{C}^\times)^r$$

$$W_\nu : GL(r) \longrightarrow \mathbb{C}$$

class-1 Whittaker fn $(W_\nu(1)=1)$

explicit formula [Sh1]

Th. 3

$$\int_{T_{GL(r)}} W_\nu\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix}\right) F_{\lambda,\mu}\left(\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_r & & & \\ & & & 1_{2m+1} & & \\ & & & & t_r^{-1} & \\ & & & & & \ddots \\ & & & & & & t_1^{-1} \end{pmatrix}\right) \prod_{i=1}^{r} |t_i|^{s+2i-n-1-\frac{r}{2}}\, dt$$

$$= \frac{L(\nu \otimes \lambda, s)}{L(\nu \otimes \mu; s+\frac{1}{2})\, L(\nu, \mathrm{Alt}^2; 2s)}$$

$\lrcorner$

$$L(\nu \otimes \lambda, s) = \prod_{\substack{1 \le i \le n \\ 1 \le j \le r}} (1 - q^{-s}\lambda_i \nu_j)^{-1} (1 - q^{-s}\lambda_i^{-1}\nu_j)^{-1}$$

$$L(\nu, \mathrm{Alt}^2; s) = \prod_{1 \le i < j \le n} (1 - q^{-s}\nu_i \nu_j)^{-1}$$

etc. local L-factors

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