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On Dirichlet series and regular conjugacy classes in $GL(n, \mathbf{Z})$

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§1. Introduction.

Let $GL(n, \mathbf{C})$ be the group all $n \times n$ invertible matrices with entries in $\mathbf{C}$. An element A in $GL(n, \mathbf{C})$ is called *regular* if the centralizer T of A in $GL(n, \mathbf{C})$ forms a maximal split torus of the reductive group $GL(n, \mathbf{C})$. Let $GL(n, \mathbf{Z})$ be the modular group of degree n and ζ be a regular element in $GL(n, \mathbf{Z})$ with the characteristic polynomial $f(X)$. Let $G_{\mathbf{Z}}(f)$ be the subset of all roots of $f(X)$ in $GL(n, \mathbf{Z})$. The group $GL(n, \mathbf{Z})$ acts on $G_{\mathbf{Z}}(f)$ by the adjoint action. Let N_f be the number of orbits of $GL(n, \mathbf{Z})$ on $G_{\mathbf{Z}}(f)$. The purpose of this paper is to show that N_f is given by a special value of certain Dirichlet series $\zeta_R(s)$ of R at $s = 1$.

The finiteness of the space $G_{\mathbf{Z}}(f)/GL(n, \mathbf{Z})$ has been proved by [10], [14]. We remind that the zeta functions of various kinds have been introduced into the study of algebras. Especially, this paper has been strongly influenced by Solomon's idea in dealing with the group algebras [9] and its generalization by Bushnell-Reiner [2], [3].

§2. Preliminaries.

Let ζ be a regular element in the modular group $GL(n, \mathbf{Z})$ of degree n with the characteristic polynomial $f(X)$ and $R = \mathbf{Z}[\zeta]$ the ring generated by ζ over $\mathbf{Z}$. Since ζ is regular, the characteristic polynomial $f(X)$ of ζ has irreducible factors $f_1(X), f_2(X), \cdots, f_g(X)$ with multiplicity one. Let $\mathbf{Q}[X]$ be the polynomial ring in X with coefficients in $\mathbf{Q}$. The ring $\mathbf{Q}[\zeta]$, which is generated by ζ over $\mathbf{Q}$, is decomposed as follows: For $h_i(X) = f(X)/f_i(X)$ there exist $u_1(X), u_2(X), \cdots, u_g(X)$ in $\mathbf{Q}[X]$ such that $1 = \sum_{i=1}^{g} u_i(X)h_i(X)$. Put $e_i = u_i(\zeta)h_i(\zeta)$. Then we have

$$(2.1) \qquad 1 = \sum_{i=1}^{g} e_i, \; \textit{and} \; e_i e_j = \delta_{i,j} e_i$$

where $\delta_{i,j}$ is the Kronecker delta. Let ζ_i be the restriction of $\mathbf{Q}$-linear endomorphism ζ of $\mathbf{Q}[\zeta]$ to $\mathbf{Q}[\zeta]e_i$. Observe that ζ_i is a root of the irreducible polynomial $f_i(X)$, so that $k_i = \mathbf{Q}[\zeta_i]$ is an algebraic number field over $\mathbf{Q}$. The ring $\mathbf{Q}[\zeta]$ is decomposed into a direct sum of $k_i e_i$'s $(1 \leq i \leq g)$:

$$(2.2) \qquad \mathbf{Q}[\zeta] = k_1 e_1 \oplus k_2 e_2 \oplus \cdots \oplus k_g e_g.$$

Let O be the ring of algebraic integers in $\mathbf{Q}[\zeta]$. Since e_i is a root of the monic polynomial $X^2 - X$ in $\mathbf{Z}[X]$, e_i belongs to O. Let O_i be the ring of integers of k_i. Then we have

$$O = O_1 e_1 \oplus O_2 e_2 \oplus \cdots \oplus O_g e_g. \tag{2.3}$$

We shall define the norm and trace on $\mathbf{Q}[\zeta]$. Since all eigenvalues of ζ are mutually distinct, ζ is diagonalizable. Furthermore, there exist $\zeta, \zeta', \cdots, \zeta^{(n-1)}$ in T such that

$$\prod_{0 \leq i < j \leq n-1} (\zeta^{(i)} - \zeta^{(j)}) \in GL(n, \mathbf{C}), \ f(\zeta^{(j)}) = 0 \ (0 \leq j \leq n-1). \tag{2.4}$$

Put $\Omega = \{\zeta, \zeta', \cdots, \zeta^{(n-1)}\}$. Let $\mathbf{Q}[\Omega]$ be the commutative ring generated by Ω over $\mathbf{Q}$. Then

$$f(X) = (X - \zeta)(X - \zeta') \cdots (X - \zeta^{(n-1)}) \ in \ (\mathbf{Q}[\Omega])[X]. \tag{2.5}$$

The norm N and trace Tr in $\mathbf{Q}[\zeta]$ are defined, respectively, by

$$Np(\zeta) = p(\zeta)p(\zeta') \cdots p(\zeta^{(n-1)}), \tag{2.6}$$

$$Tr(p(\zeta)) = p(\zeta) + p(\zeta') + \cdots + p(\zeta^{(n-1)}) \tag{2.7}$$

where $p(X)$ is a polynomial in $\mathbf{Q}[X]$. Since $Np(\zeta)$ and $Tr(p(\zeta))$ are the symmetric polynomials in $\zeta, \zeta', \cdots, \zeta^{(n-1)}$, (2.5) implies that both $Np(\zeta)$ and $Tr(p(\zeta))$ are rational numbers. If α is an element in R, then $N\alpha$ (resp. $Tr(\alpha)$) is a rational integer. Let N_{k_i} be the norm of the algebraic number field k_i.

Lemma 2.1. *Let $\alpha = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_g e_g$ where $\alpha_i \in k_i$ for $i = 1, 2, \cdots, g$. Then we have*

$$N\alpha = N_{k_1}\alpha_1 N_{k_2}\alpha_2 \cdots N_{k_g}\alpha_g \times 1_n.$$

Here 1_n is the identity matrix of degree n.

Define the unit group $\mathbf{E}_O$ of $\mathbf{Q}[\zeta]$ by

$$\mathbf{E}_O = \{\epsilon \in O : N\epsilon = \pm 1\}. \tag{2.8}$$

Lemma 2.2. *Let $\mathbf{E}_i$ be the unit group of the algebraic number field k_i. Then we have*

$$\mathbf{E}_O = \mathbf{E}_1 e_1 \oplus \mathbf{E}_2 e_2 \oplus \cdots \oplus \mathbf{E}_g e_g. \tag{2.9}$$

We now turn to the ideals of R. An ideal $\mathfrak{a} \subset R$ is said to be *nosingular* if the index $(R : \mathfrak{a})$ (as group) is finite, in which case the norm of $\mathfrak{a}$, $N\mathfrak{a}$, is defined to be this index. An R- submodule $\mathfrak{a}$ of $\mathbf{Q}[\zeta]$ is a *fractional ideal* of R if there exists an invertible element α in $\mathbf{Q}[\zeta]$ such that $\alpha\mathfrak{a}$ is nonsingular ideal of R. For a fractional ideal $\mathfrak{a}$ of R, we define the norm $N\mathfrak{a}$ of $\mathfrak{a}$ by $N\mathfrak{a} = (N(\alpha))^{-1}N(\alpha\mathfrak{a})$.

DEFINITION 2.1. *Let $\mathfrak{a}$ be a nonsingular ideal of R. The pseudo inverse ideal $\check{\mathfrak{a}}$ of $\mathfrak{a}$ is defined by*

$$\check{\mathfrak{a}} = \{\mu \in \mathbf{Q}[\zeta] : \mu\mathfrak{a} \subset R\}.$$

The psuedo inverse ideal $\check{\mathfrak{a}}$ of a given nonsingular ideal $\mathfrak{a}$ is fractional.

We remark that if a nonsingular ideal $\mathfrak{a}$ is invertible (ie. there exists a fractional ideal $\mathfrak{b}$ of R such that $\mathfrak{a}\mathfrak{b} = R$), then $\check{\mathfrak{a}}$ is actually an inverse ideal of $\mathfrak{a}$. In general, however, there is a nonsingular ideal which has no inverse ideal (see Example below).

Example. Take

$$\zeta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \in GL(4, \mathbf{Z}).$$

Then the chracteristic polynomial f of ζ is decomposed into $f(X) = F(X)F(-X)$, $F(X) = X^2 + X + 1$. $F(X)$ is irreducible modulo 2 and $f(X) - F(X)^2 = -2F(X)X$. Let $(2, F(\zeta))$ be the ideal of $R = \mathbf{Z}[\zeta]$ generated by 2 and $F(\zeta)$. Then $(2, F(\zeta))^2 = 2(2, F(\zeta))$. This implies that the ideal $(2, F(\zeta))$ is not invertible.

§3. Theorem of Latimer and MacDuffee.

Let $\Omega = \{\zeta, \zeta', \cdots, \zeta^{(n-1)}\}$ be the conjugate system of ζ (see (2.5)). Let $\alpha^{(j)}$ be the j-th conjugate of α in $\mathbf{Q}[\zeta]$ which is defined by $\alpha^{(j)} = p(\zeta^{(j)})$ where $\alpha = p(\zeta)$ and $p(X) \in \mathbf{Q}[X]$. Let $GL(n, \mathbf{Q})$ be the group of rational matrices in $GL(n, \mathbf{C})$.

LEMMA 3.1. *Any two matrices in $G_Z(f)$ are $GL(n, \mathbf{Q})$-conjugate.*

Proof. Let γ be an element in $G_Z(f)$. Since all roots of $f(X)$ are simple, there exists h in $GL(n, \mathbf{Q})$ such that

$$h\gamma h^{-1} = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \gamma_g \end{pmatrix}$$

where γ_i is an integral matrix of degree n_i with characteristic polynomial $f_i(X)$ (cf. Theorem III.12 and p.55, Exercise 7, [8]). Consequently it is sufficient to prove this lemma when $f(X)$ is irreducible over $\mathbf{Q}$. Define an integral matrix γ_0 by $\gamma_0\mathbf{x}_0 = \zeta\mathbf{x}_0$ where $\mathbf{x}_0 = (1, \zeta, \cdots, \zeta^{n-1})^T$. Since $f(\gamma_0)\mathbf{x}_0^{(j)} = 0\,(0 \leq j \leq n-1)$ and the matrix $(\mathbf{x}_0, \mathbf{x}_0', \cdots, \mathbf{x}_0^{(n-1)})$ is invertible, we have $f(\gamma) = 0$. Moreover, $\gamma_0 \in G_Z(f)$ since $\zeta^{-1} \in R$ and $\gamma_0^{-1}\mathbf{x}_0 = \zeta^{-1}\mathbf{x}_0$. Hence the proof is reduced to the following: Each element γ in $G_Z(f)$ is conjugate to γ_0 under the adjoint action of $GL(n, \mathbf{Q})$. Put $B(X) = X1_n - \gamma$, and let $C(X)$ be the algebraic complement of $B(X)$. For the first column vector $(c_1(X), c_2(X), \cdots, c_n(X))^T$ of $C(X)$, we put $\mathbf{x} = (c_1(\zeta), c_2(\zeta), \cdots, c_n(\zeta))^T$. Since $B(X)C(X) = f(X)1_n$ and $deg\, c_i \leq n-1\,(1 \leq$

$i \leq n$), it is easy see that $\gamma \mathbf{x} = \zeta \mathbf{x}$ and $\mathbf{x} \neq 0$. From the irreducibility of the chracteristic polynomial $f(X)$ of γ it follows that the matrix $(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$ is invertible. Hence $\{c_1(\zeta), c_2(\zeta), \cdots, c_n(\zeta)\}$ is a Q-basis of $\mathbf{Q}(\zeta)^n$. Define an element h in $GL(n, \mathbf{Q})$ by letting $h\mathbf{x} = \mathbf{x}_0$. Then we have $h\gamma h^{-1} = \gamma_0$ as claimed.

Lemma 3.2. *Let γ be an element in $G_Z(f)$. Then there exists a vector $\mathbf{x}$ in R^n such that $\gamma \mathbf{x} = \zeta \mathbf{x}$ and $(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$ is invertible. Furthermore, $\mathbf{x}$ is uniquely determined up to scalar multiplication by an invertible element in $\mathbf{Q}[\zeta]$.*

Proof. For $\mathbf{x}_0 = (1, \zeta, \cdots, \zeta^{n-1})^T$, we define a matrix γ_0 in $G_Z(f)$ by $\gamma_0 \mathbf{x}_0 = \zeta \mathbf{x}_0$. Let γ be each element in $G_Z(f)$. By Lemma 3.1 there exists h in $GL(n, \mathbf{Q})$ such that $\gamma = h\gamma_0 h^{-1}$. Put $\mathbf{x} = h\mathbf{x}_0$. Then $\gamma \mathbf{x} = \zeta \mathbf{x}$ and $(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$ is invertible. It remains to show the uniqueness of $\mathbf{x}$ up to a scalar multiplication by element in $\mathbf{Q}[\zeta]$. Suppose there exists $\mathbf{y} \in \mathbf{Q}[\zeta]^n$ such that $\gamma \mathbf{y} = \zeta \mathbf{y}$. Put $C = (\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)}), C^{-1} = (d_{ij})$. Observe that

$$(det(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)}))^2 d_{j1} \; (1 \leq j \leq n)$$

is a symmetric polynomial in $\zeta', \zeta^{(2)}, \cdots, \zeta^{(n-1)}$. Then by (2.5), $d_{j1} \in \mathbf{Q}[\zeta]$ for all $j = 1, 2, \cdots, n$. Since $CC^{-1} = 1_n$, we have

$$\mathbf{e}_i = \sum_{j=1}^{n} d_{ji} \mathbf{x}^{(j-1)} \; for \; i = 1, 2, \cdots, n$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is the canonical basis of $\mathbf{Q}[\zeta]^n$. Therefore $\mathbf{y}$ is written as $\mathbf{y} = \sum_{j=1}^{n} b_j \mathbf{x}^{(j-1)}$ with $b_j \in \mathbf{Q}[\Omega]$. In particular $b_1 \in R$. Since $\gamma \mathbf{x}^{(j)} = \zeta^{(j)} \mathbf{x}^{(j)}$ $(0 \leq j \leq n-1)$, it follows from the invertibility of $\zeta - \zeta^{(j)}$ that $\mathbf{y} = b_1 \mathbf{x}$, $b_j = 0 \; for \; all \; j \neq 1$. This implies that

$$det(\mathbf{y}, \mathbf{y}', \cdots, \mathbf{y}^{(n-1)}) = N b_1 det(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)}).$$

Hence we have $Nb_1 \neq 0$. Consequently b_1 is invertible in $\mathbf{Q}[\zeta]$. This gives the uniqueness of $\mathbf{x}$.

Let us define the ideal class semigroup of R. We denote by $\mathbf{A}$ (resp. $\mathbf{A}_O$) the set of all fractional ideals of R (resp. O). $\mathbf{A}$ (resp. $\mathbf{A}_O$) is a semigroup with the canonical multiplication.

Definition 3.1. *Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbf{A}$ (resp. $\mathbf{A}_O$) are equivalent if there exists an invertible element λ in $\mathbf{Q}[\zeta]$ such that $\lambda \mathfrak{a} = \mathfrak{b}$.*

Definition 3.2. *The set of all equivalence classes of $\mathbf{A}$ (resp. $\mathbf{A}_O$) will be denoted by $\mathbf{G}$ (resp. $\mathbf{G}_O$). For $\mathfrak{a}$ in $\mathbf{A}$ (resp. $\mathbf{A}_O$), $C(\mathfrak{a})$ denote the ideal class represented by $\mathfrak{a}$.*

$\mathbf{G}$ is called the ideal class semigroup. In view of (2.3), $\mathbf{G}_O$ is a direct product of a finite number of ideal class groups of the algebraic number fields over $\mathbf{Q}$. Therefore $\mathbf{G}_O$ is a finite group and called the ideal class group of O.

Let us now define a map from $\mathbf{G}$ to $G_Z(f)/GL(n,\mathbf{Z})$ as follows (cf. [6] and [12], or p.53 in [8]): For each element $\mathfrak{a}$ in $\mathbf{A}$ with a $\mathbf{Z}$-basis $\{w_1, w_2, \cdots, w_n\}$, define an integral matrix γ by

$$\zeta \mathbf{x} = \gamma \mathbf{x} \text{ where } \mathbf{x} = (w_1, w_2, \cdots, w_n)^T. \tag{3.3}$$

Then $\gamma \in G_Z(f)$. Define a map ϕ: $\mathbf{G} \to G_Z(f)/GL(n,\mathbf{Z})$:

$$\phi(C(\mathfrak{a})) = C(\gamma) \tag{3.4}$$

where $C(\gamma)$ is the class in $G_Z(f)/GL(n,\mathbf{Z})$ represented by γ.

THEOREM 3.3 (LATIMER-MACDUFFEE). *The map ϕ is bijective.*

Proof: Well definedness: Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals belong to the same class C in $\mathbf{G}$. For a $\mathbf{Z}$-basis $\{w_1, w_2, \cdots, w_n\}$(resp. $\{v_1, v_2, \cdots, v_n\}$) of $\mathfrak{a}$ (resp. $\mathfrak{b}$), we define two matrices $\gamma, \delta \in G_Z(f)$ by $\gamma\mathbf{x} = \zeta\mathbf{x}$ and $\delta\mathbf{y} = \zeta\mathbf{y}$ where $\mathbf{x} = (w_1, w_2, \cdots, w_n)^T$ and $\mathbf{y} = (v_1, v_2, \cdots, v_n)^T$. Since $\mathfrak{a}$ is equivalent to $\mathfrak{b}$, there exists an invertible element λ in $\mathbf{Q}[\zeta]$ such that $\lambda\mathfrak{a} = \mathfrak{b}$. Consequently $g(\lambda\mathbf{x}) = \mathbf{y}$ for a suitable g in $GL(n,\mathbf{Z})$. This implies that $\delta = g\gamma g^{-1}$, so $C(\gamma) = C(\delta)$.

Injectivity: Suppose $\phi(C(\mathfrak{a})) = \phi(C(\mathfrak{b})) = C(\gamma)$ for two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbf{A}$ and $\gamma \in G_Z(f)$. Then there are two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\gamma\mathbf{x} = \zeta\mathbf{x}$, $\gamma\mathbf{y} = \zeta\mathbf{y}$ and $\mathfrak{a}$ (resp. $\mathfrak{b}$) is generated by $\mathbf{x}$ (resp. $\mathbf{y}$). By Lemma 3.2, $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent. Hence ϕ is injective.

Surjectivity: Let $C(\gamma)$ be a class in $G_Z(f)/GL(n,\mathbf{Z})$. Again by Lemma 3.2, we can choose a vector $\mathbf{x}$ in R^n such that $\gamma\mathbf{x} = \zeta\mathbf{x}$ and $(\mathbf{x}, \mathbf{x}', \cdots, \mathbf{x}^{(n-1)})$ is invertible. Let $\mathfrak{a}$ be the $\mathbf{Z}$-module generated by the entries of $\mathbf{x}$. Then $\mathfrak{a}$ is nonsingular and $\phi(C(\mathfrak{a})) = C(\gamma)$.

Let us prove the finiteness of the order of $\mathbf{G}$. For this, we need to prove the following two lemmas. Define a map η from $\mathbf{G}$ to $\mathbf{G}_O$ by

$$\eta(C(\mathfrak{a})) = C(O\mathfrak{a}),\ \mathfrak{a} \in \mathbf{A}. \tag{3.5}$$

For a fixed $\mathfrak{a}$ in $\mathbf{A}$, we put

$$M(\mathfrak{a}) = \{\mathfrak{b} \in \mathbf{A} : O\mathfrak{a} = O\mathfrak{b}\}. \tag{3.6}$$

LEMMA 3.4. *The map η defined by (3.5) is surjective, and*

$$\eta^{-1}(C(O\mathfrak{a})) = \{C(\mathfrak{b}) : \mathfrak{b} \in M(\mathfrak{a})\}. \tag{3.7}$$

Proof. A fractional ideal of O is also a fractional ideal of R. Hence the surjectivity of η is obvious. Let us prove (3.7). Let $C(\mathfrak{b})$ be a class in $\eta^{-1}(C(\mathfrak{a}))$. Then there exists α in $\mathbf{Q}[\zeta]$ such that $\alpha O\mathfrak{b} = O\mathfrak{a}$. Put $\mathfrak{q} = \alpha\mathfrak{b}$. Then $C(\mathfrak{q}) = C(\mathfrak{b})$ and $O\mathfrak{a} = O\mathfrak{q}$. Consequently $C(\mathfrak{b})$ belongs to the set in the right hand side of (3.7). Conversely suppose $\mathfrak{b} \in M(\mathfrak{a})$. Then we have $\eta(C(\mathfrak{b})) = C(O\mathfrak{a})$. Hence $C(\mathfrak{b})$ belongs to the set $\eta^{-1}(C(\mathfrak{a}))$. This gives the converse inclusion.

LEMMA 3.5. *Let $\mathfrak{a}$ be a fixed fractional ideal of R. Then we have*

$$|M(\mathfrak{a})| \leq (O:R)^n.$$

Proof. Put $m = (O:R)$. Since $1 \in O$ and $mO \subset R$, we have

$$O\mathfrak{a} = O\mathfrak{b} \supset \mathfrak{b} \supset mO\mathfrak{b} = mO\mathfrak{a} \text{ for all } \mathfrak{b} \in M(\mathfrak{a}). \tag{3.8}$$

Let c be the number of all subgroups of the additive group $O\mathfrak{a}/mO\mathfrak{a}$. Then by (3.8), $|M(\mathfrak{a})| \leq c$. On the other hand, by the fundamental theory of finitely generated abelian groups, we have

$$O\mathfrak{a}/mO\mathfrak{a} \cong \mathbf{Z}/(p_1^{m_1}) \oplus \mathbf{Z}/(p_2^{m_2}) \oplus \cdots \oplus \mathbf{Z}/(p_k^{m_k}).$$

Here m_i is a positive integer and p_i is a prime number. Therefore

$$\begin{aligned} |M(\mathfrak{a})| &\leq c \\ &\leq (m_1+1)(m_2+1)\cdots(m_k+1) \\ &\leq (p_1)^{m_1}(p_2)^{m_2}\cdots(p_k)^{m_k} \\ &= (O\mathfrak{a} : mO\mathfrak{a}) = m^n. \end{aligned}$$

Let $\mathfrak{a}$ be a fractional ideal of R. Then the unit group $\mathbf{E}_O$ of O acts on $M(\mathfrak{a})$ by the rule:

$$\mathbf{E}_O \times M(\mathfrak{a}) \ni (\epsilon, \mathfrak{b}) \to \epsilon\mathfrak{b} \in M(\mathfrak{a}). \tag{3.9}$$

LEMMA 3.6. *Let $\mathfrak{a}$ be a nonsingular ideal of R. We define the stabilizer group $\mathbf{E}_\mathfrak{a}$ of $\mathfrak{a}$ by*

$$\mathbf{E}_\mathfrak{a} = \{\epsilon \in \mathbf{E}_O : \epsilon\mathfrak{a} = \mathfrak{a}\}.$$

Then the index $(\mathbf{E}_O : \mathbf{E}_\mathfrak{a})$ as group is finite.

Proof. By Lemma 3.5, the set $M(\mathfrak{a})$ is finite. So the subset $\mathbf{E}_O\mathfrak{a}$ of $M(\mathfrak{a})$ is also finite. Therefore $(\mathbf{E}_O : \mathbf{E}_\mathfrak{a})$ is finite.

TEOREM 3.7. *The order of $\mathbf{G}$ is finite.*

Proof. Let η be the map from $\mathbf{G}$ to $\mathbf{G}_O$. By Lemma 3.4 we have

$$|\mathbf{G}| = |\bigcup_{C(\tilde{\mathfrak{a}}) \in \mathbf{G}_O} \eta^{-1}(C(\tilde{\mathfrak{a}}))| \leq \sum_{C(\tilde{\mathfrak{a}}) \in \mathbf{G}_O} |M(\tilde{\mathfrak{a}})|.$$

From Lemma 3.5 it follows that the order of $\mathbf{G}$ is finite.

§4. Reduction theorem.

For an ideal class $C = C(\mathfrak{a})$ of R, we define a Dirichlet series $\zeta_C(s)$ by

$$\zeta_C(s) = \sum_{\substack{\mathfrak{b}\in C(\mathfrak{a})\\ \mathfrak{b}\subset R}} \frac{1}{(N\mathfrak{b})^s}, \; s \in \mathbf{C}. \tag{4.1}$$

$\zeta_C(s)$ may be called the zeta function of the class C. We shall prove in §7 that $\zeta_C(s)$ is convergent in the complex half plane $\Re(s) > 1$. In this section we shall give a reduction theorem which is useful for investigating with the analytic properties of $\zeta_C(s)$.

Let $\mathbf{A}_C$ be the set of all integral ideals of $C = C(\mathfrak{a})$ and $\breve{\mathfrak{a}}$ the pseudo inverse ideal of $\mathfrak{a}$. The group $\mathbf{E}_\mathfrak{a}$, which is given in Lemma 3.6, stabilizes the set $\breve{\mathfrak{a}}^\times$ of all invertible elements in $\breve{\mathfrak{a}}$. We classify $\breve{\mathfrak{a}}^\times$ by $\mathbf{E}_\mathfrak{a}$-orbits, and denote by $\breve{\mathfrak{a}}^\times/\mathbf{E}_\mathfrak{a}$ the set of all $\mathbf{E}_\mathfrak{a}$-orbits in $\breve{\mathfrak{a}}^\times$. From this, it follows immediately the following lemma.

Lemma 4.1. *The set $\mathbf{A}_C$ is parameterized by*

$$\mathbf{A}_C = \{\lambda\mathfrak{a} : [\lambda] \in \breve{\mathfrak{a}}^\times/\mathbf{E}_\mathfrak{a}\}.$$

Let $\mathfrak{a}$ be the representative of the class C. In the following we assume that $\mathfrak{a}$ is integral. Put $\tilde{\mathfrak{a}} = O\mathfrak{a}$. Since $\tilde{\mathfrak{a}}$ is an ideal of O, $\tilde{\mathfrak{a}}$ has the inverse ideal $\tilde{\mathfrak{a}}^{-1}$. It is easy to see that $\breve{\mathfrak{a}} \subset \tilde{\mathfrak{a}}^{-1}$ and $\tilde{\mathfrak{a}}^{-1}/\breve{\mathfrak{a}}$ is a finite additive group.

Definition 4.1. *Let $\breve{\mathfrak{a}}$ be the pseudo inverse ideal of the representative $\mathfrak{a}$ of C. Denote by B^* the character group of $B = \tilde{\mathfrak{a}}^{-1}/\breve{\mathfrak{a}}$.*

Let $\mathbf{E}_O$ be the unit group of O. $\mathbf{E}_O$ is a direct product of a finite group H_O and a finitely generated free group E_O. Put $E_\mathfrak{a} = \mathbf{E}_\mathfrak{a} \cap E_O$, $H_\mathfrak{a} = \mathbf{E}_\mathfrak{a} \cap H_O$. Let $(\tilde{\mathfrak{a}}^{-1})^\times$ be the set of all invetible elements in $\tilde{\mathfrak{a}}^{-1}$. Then

$$E_O(\tilde{\mathfrak{a}}^{-1})^\times \subset (\tilde{\mathfrak{a}}^{-1})^\times. \tag{4.2}$$

Definition 4.2. *Fix a representative λ for each class $[\lambda]$ in $(\tilde{\mathfrak{a}}^{-1})^\times/E_\mathfrak{a}$, and let χ be a caracter of B. The L-function $L(s:\chi)$ associated to χ is defined by*

$$L(s:\chi) = \sum_{[\lambda]\in(\tilde{\mathfrak{a}}^{-1})^\times/E_\mathfrak{a}} \frac{\chi(\lambda)}{(N(\lambda\tilde{\mathfrak{a}}))^s}$$

where s is a complex number and $N(*)$ is the ideal norm of $(*)$.

We remark that $\chi(\lambda)$ and hence $L(s:\chi)$ depend on the choice of the representatives λ. We choose a representative λ and fix it once and for all.

THEOREM 4.2. *Let $C = C(\mathfrak{a})$ be an ideal class of $\mathbf{G}$. Then the zeta function $\zeta_C(s)$ is expressed as*

$$\zeta_C(s) = \frac{(N\tilde{\mathfrak{a}})^s}{|H_{\mathfrak{a}}|(\tilde{\mathfrak{a}}^{-1} : \check{\mathfrak{a}})(N\mathfrak{a})^s}\{\sum_{\chi \in B^*} L(s : \chi)\}.$$

Proof. The assertion follows from the orthogonality relations (see Theorem 7.3, [5]) on the group B.

Let χ be a character of B. For each $\epsilon \in \mathbf{E}_O$, define χ_ϵ by

$$\chi_\epsilon(\alpha) = \chi(\epsilon\alpha),\ \alpha \in \tilde{\mathfrak{a}}^{-1}. \tag{4.3}$$

χ_ϵ is well defined and is a character of the finite additive group $\tilde{\mathfrak{a}}^{-1}/\epsilon^{-1}\check{\mathfrak{a}}$. The orbit classes $(\tilde{\mathfrak{a}}^{-1})^\times/E_a$ is decomposed as follows:

$$(\tilde{\mathfrak{a}}^{-1})^\times/E_a = \bigcup_{[\epsilon]\in E_O/E_a} \bigcup_{[\alpha]\in(\tilde{\mathfrak{a}}^{-1})^\times/E_O} \alpha\epsilon E_a.$$

Therefore

$$L(s : \chi) = \sum_{[\epsilon]\in E_O/E_a} \sum_{[\alpha]\in(\tilde{\mathfrak{a}}^{-1})^\times/E_O} \frac{\chi_\epsilon(\alpha)}{(N\alpha\tilde{\mathfrak{a}})^s}. \tag{4.4}$$

Define an ideal $\tilde{\mathfrak{a}}_i$ of O_i $(1 \le i \le g)$ by

$$\tilde{\mathfrak{a}}_i e_i = \tilde{\mathfrak{a}} e_i. \tag{4.5}$$

Note that $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_1 e_1 \oplus \tilde{\mathfrak{a}}_2 e_2 \oplus \cdots \oplus \tilde{\mathfrak{a}}_g e_g$.

DEFINITION 4.3. *For χ in B^* and ϵ in E_O, define the zeta function $\zeta_i(s : \chi_\epsilon)$ by*

$$\zeta_i(s : \chi_\epsilon) = \sum_{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i} \frac{\chi_\epsilon(\alpha e_i)}{(N\alpha\tilde{\mathfrak{a}}_i)^s}$$

where $\tilde{\mathfrak{b}}_i = (\tilde{\mathfrak{a}}_i)^{-1}$.

From Lemma 2.1 and (4.4) the following lemma can be deduced.

LEMMA 4.3. *With the notations being the same as above, we have*

$$L(s : \chi) = \sum_{[\epsilon]\in E_O/E_a} \prod_{i=1}^{g} \zeta_i(s : \chi_\epsilon).$$

We now state the Abel's summation formula which is used frequently in the analytic number theory (cf. Theorem 1.6, [5]).

LEMMA 4.4. *Let ψ be a function on the interval $(0,\infty)$ of C^1-class. Then for a finite number of comlex sequence $a_1, a_2, \cdots, a_{[t]}$, we have*

$$\sum_{0<m\leq t} a_m\psi(m) = A(t)\psi(t) - \int_1^t A(x)\psi'(x)dx, \ A(x) = \sum_{0<m\leq x} a_m.$$

LEMMA 4.5. *For each positive real number t and $\epsilon \in E_O$, let*

$$A_i(t:\chi_\epsilon) = \sum_{\substack{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i \\ N(\alpha\tilde{\mathfrak{a}}_i)\leq t}} \chi_\epsilon(\alpha e_i).$$

Then we have

$$\sum_{\substack{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i \\ N(\alpha\tilde{\mathfrak{a}}_i)\leq t}} \frac{\chi_\epsilon(\alpha e_i)}{(N(\alpha\tilde{\mathfrak{a}}_i))^s} = A_i(t:\chi_\epsilon)t^{-s} + s\int_1^t A_i(x:\chi_\epsilon)x^{-s-1}dx.$$

Proof. For each positive integer m, put

$$\psi(m) = m^{-s}, \ a_m = \sum_{\substack{[\alpha]\in(\tilde{\mathfrak{b}}_i)^\times/E_i \\ N\alpha\tilde{\mathfrak{a}}_i = m}} \chi_\epsilon(\alpha e_i).$$

Then This lemma follows from Lemma 4.4.

§5. Density of ideals.

We keep the same notations as in the previous section. Let $C = C(\mathfrak{a})$ be the fixed class of the ideal class semigroup of R. For each i, consider the i-th component $\tilde{\mathfrak{a}}_i$ (resp. $\tilde{\mathfrak{b}}_i$) of $\tilde{\mathfrak{a}}$ (resp. $\tilde{\mathfrak{a}}^{-1}$). Let t be a positive real number. Define a subset $T_i(t)$ of $(\tilde{\mathfrak{b}}_i)^\times/E_i$ by

$$T_i(t) = \{[\alpha] \in (\tilde{\mathfrak{b}}_i)^\times/E_i : |N\alpha| \leq \frac{t}{N\tilde{\mathfrak{a}}_i}\}. \tag{5.1}$$

Observe that

$$|A_i(t:\chi_\epsilon)| \leq |T_i(t)| \tag{5.2}$$

where χ is a character of $B = \tilde{\mathfrak{a}}^{-1}/\tilde{\mathfrak{a}}$ and $\epsilon \in E_O$. Especially if χ_ϵ is trivial on $\tilde{\mathfrak{b}}_i e_i$, then

$$A_i(t:\chi) = |T_i(t)|. \tag{5.3}$$

We aim to evaluate the limit:

$$\lim_{t\to\infty} \frac{|T_i(t)|}{t}.$$

Let $k_i^{(1)}, k_i^{(2)}, \cdots, k_i^{(n_i)}$ be all conjugates of k_i over $\mathbf{Q}$. We assume that

$$k_i^{(j)}\,(1 \leq j \leq r_i) \textit{ are real,}$$

$$k_i^{(j)}\,(r_i + 1 \leq j \leq r_i + c_i) \textit{ are complex and}$$

$$k_i^{(r_i+c_i+j)}\,(1 \leq j \leq c_i) \textit{ is the complex conjugate of } k_i^{(r_i+j)}$$

where $n_i = r_i + 2c_i$. Let $(k_i)^\times$ be the set of all invertible elements in k_i. We define a map ℓ^j from $(k_i)^\times$ to the field of real numbers $\mathbf{R}$ by

$$\ell^j \alpha = \begin{cases} \log|\alpha^{(j)}| & 1 \leq j \leq r_i \\ 2\log|\alpha^{(j)}| & r_i + 1 \leq j \leq r_i + c_i \end{cases} \tag{5.4}$$

where $\alpha^{(j)}$ is the j-th conjugate of $\alpha \in k_i$. Then we have

$$\sum_{j=1}^{r_i+c_i} \ell^j \alpha = \log|N_{k_i}\alpha|. \tag{5.5}$$

The unit group $\mathbf{E}_i$ of the field k_i is decomposed into a product: $\mathbf{E}_i = E_i \times H_i$ where E_i is a free group with rank $r_i + c_i - 1$ and H_i is a finite group (see [1] or [4]). Define the regulator $R(\mathbf{E}_i)$ of $\mathbf{E}_i$ as follows:

$$R(\mathbf{E}_i) = \begin{vmatrix} \ell^1\epsilon_1 & \ell^2\epsilon_1 & \cdots & \ell^{r_i+c_i-1}\epsilon_1 \\ \ell^1\epsilon_2 & \ell^2\epsilon_2 & \cdots & \ell^{r_i+c_i-1}\epsilon_2 \\ \vdots & \vdots & \ddots & \vdots \\ \ell^1\epsilon_{r_i+c_i-1} & \ell^2\epsilon_{r_i+c_i-1} & \cdots & \ell^{r_i+c_i-1}\epsilon_{r_i+c_i-1} \end{vmatrix}$$

where $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_{r_i+c_i-1}\}$ is a generator of E_i.

We shall identify $\tilde{\mathfrak{b}}_i$ with the lattice $\mathbf{Z}^{n_i}$ in $\mathbf{R}^{n_i}$. Let $\{w_1, w_2. \cdots, w_{n_i}\}$ be a $\mathbf{Z}$-basis of $\mathfrak{b}_i$. For each $\mathbf{x} = (x_1, x_2, \cdots, x_{n_i})$ in $\mathbf{R}^{n_i}$, we put

$$\alpha(\mathbf{x}) = x_1 w_1 + x_2 w_2 + \cdots + x_{n_i} w_{n_i}. \tag{5.6}$$

Then the map $\mathbf{x} \to \alpha(\mathbf{x})$ from $\mathbf{Q}^{n_i}$ to k_i is bijective. Also through this map, $\tilde{\mathfrak{b}}_i$ can be identified with the lattice $\mathbf{Z}^{n_i}$. Let $w_k^{(j)}, 1 \leq j \leq r_i + 2c_i$ be the algebraic conjugates of w_k. For $\mathbf{x} \in \mathbf{R}^{n_i}$, we put

$$N\alpha(\mathbf{x}) = \prod_{j=1}^{r_i+2c_i} \alpha^{(j)}(\mathbf{x}) \tag{5.7}$$

where $\alpha^{(j)}(\mathbf{x}) = x_1 w_1^{(j)} + x_2 w_2^{(j)} + \cdots + x_{n_i} w_{n_i}^{(j)}$. Note that N is an extension of N_{k_i} to R^{n_i}. Let S be the subset of R^{n_i} defined by

$$S = \{\mathbf{x} \in R^{n_i} : N\alpha(\mathbf{x}) = 0\}.$$

We can define a map Φ: $\mathbf{R}^{n_i} \setminus S \to \mathbf{R}^{r_i+c_i}$:

$$\Phi(\mathbf{x}) = (\ell^1\alpha(\mathbf{x}), \cdots, \ell^{r_i+c_i}\alpha(\mathbf{x})). \tag{5.8}$$

Let $\{\eta_1, \eta_2, \cdots, \eta_{r_i+c_i-1}\}$ be a free basis of E_i. To parameterize $(\tilde{\mathfrak{b}}_i)^\times/E_i$ we choose the following basis for $\mathbf{R}^{r_i+c_i}$:

$$\mathbf{u} = \frac{1}{n_i}(\overbrace{1,1,\ldots,1}^{r_i\text{-times}},\overbrace{2,2,\ldots,2}^{c_i\text{-times}}),\ \mathbf{v}_1 = \Phi(\eta_1), \mathbf{v}_2 = \Phi(\eta_2), \cdots, \mathbf{v}_{r_i+c_i-1} = \Phi(\eta_{r_i+c_i-1}).$$

LEMMA 5.1. *The vectors $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{r_i+c_i-1}$ forms a basis of $\mathbf{R}^{r_i+c_i}$.*

Proof. We put $d = det(\mathbf{u}^T, {\mathbf{v}_1}^T, \cdots, {\mathbf{v}_{r_i+c_i-1}}^T)$. Then it is sufficient to prove that $d \neq 0$. Recalling that

$$\sum_{j=1}^{r_i+c_i} \ell^j \eta_k = \log|N_{k_i}(\eta_k)| = 0,$$

we have immediately $d = \pm R((\mathbf{E}_a)_i)$. This gives $d \neq 0$.

Let $V(= V_i)$ be the subset of $R^{r_i+c_i}$ defined by

$$V = \{u\mathbf{u} + \sum_{k=1}^{r_i+c_i-1} v_k\mathbf{v}_k : u \in \mathbf{R},\ 0 \leq v_k < 1\,(1 \leq k \leq r_i + c_i - 1)\}.$$

We now put

$$P(= P_i) = \Phi^{-1}(V). \tag{5.9}$$

Let $\mathbf{R}^\times$ be the set of non-zero real numbers. It is easy to see that

$$tP \subset P \text{ for all } t \in \mathbf{R}^\times. \tag{5.10}$$

The following lemma asserts that the set $(\tilde{\mathfrak{b}}_i)^\times/E_i$ is parametrized by $P \cap \mathbf{Z}^{n_i}$.

LEMMA 5.2. *The map $\mathbf{x} \to \alpha(\mathbf{x})E_i$ from $P \cap \mathbf{Z}^{n_i}$ to $(\tilde{\mathfrak{b}}_i)^\times/E_i$ is a bijection.*

Proof. First we shall prove the surjectivity. Let αE_i be an element in $(\tilde{\mathfrak{b}}_i)^\times/E_i$. Choose an element $\mathbf{x}$ in $\mathbf{Z}^{n_i}$ satisfying $\alpha(\mathbf{x}) = \alpha$. Put

$$\Phi(\mathbf{x}) = u\mathbf{u} + \sum_{j=1}^{r_i+s_i-1} v_k\mathbf{v}_k,\ v_k = [v_k] + \{v_k\}.$$

where $m_k = [v_k]$ is the Gaussian integral part of v_k and $\{v_k\} = v_k - [v_k]$. Put

$$\eta = \eta_1^{m_1}\eta_2^{m_2}\cdots\eta_{r_i+c_i-1}^{m_{r_i+c_i-1}}.$$

Then $\eta \in E_i$. Put

$$\beta = \alpha\eta^{-1}, \beta = \alpha(\mathbf{y})\, and\, \eta = \alpha(\mathbf{z})\ (\mathbf{y} \in P \cap \mathbf{Z}^{n_i}, \mathbf{z} \in \mathbf{Z}^{n_i}).$$

Then $\Phi(\mathbf{y}) = \Phi(\mathbf{x}) - \Phi(\mathbf{z}) \in V$. So $\mathbf{y} \in P \cap \mathbf{Z}^{n_i}$. From this it follows that the map $\mathbf{x} \to \alpha(\mathbf{x})E_i$ is surjective. Let us prove the injectivity. Suppose $\alpha(\mathbf{y}) = \dot{\alpha}(\mathbf{x})\alpha(\mathbf{z})$ and $\alpha(\mathbf{z}) \in E_i$ for $\mathbf{x}, \mathbf{y} \in P \cap \mathbf{Z}^{n_i}$ and $\mathbf{z} \in \mathbf{Z}^{n_i}$. Since $\Phi(\mathbf{y}) = \Phi(\mathbf{x}) + \Phi(\mathbf{z})$ and $\alpha(\mathbf{z}) \in E_i$, we have $\alpha(\mathbf{z}) = 1$. Thus $\mathbf{y} = \mathbf{x}$ as claimed.

For each positive real number t, we put

$$(5.11) \qquad P(t) = \{\mathbf{x} \in P : |N\alpha(\mathbf{x})| \leq \frac{t}{N\tilde{\mathfrak{a}}_i}\},\ P^* = \{\mathbf{x} \in P : |N\alpha(\mathbf{x})| \leq 1\}.$$

LEMMA 5.3. *Let P^* be the same as in (5.11). Then we have*

$$\lim_{t\to\infty} \frac{|T_i(t)|}{t} = \frac{1}{N\tilde{\mathfrak{a}}_i} vol(P^*)$$

where $vol(P^*)$ is the volume of P^*.

Proof. Define a map Φ_t: $\mathbf{R}^{n_i} \to \mathbf{R}^{n_i}$: $\Phi_t(\mathbf{x}) = (\frac{N\tilde{\mathfrak{a}}_i}{t})^{\frac{1}{n_i}}\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^{n_i}$. We have $\Phi_t(P(t)) = P^*$. Therefore the area P^* is meshed by the n_i-dimensional fundamental parallelepipeds with edges in the lattice $\Phi_t(\mathbf{Z}^{n_i})$. On the other hand by Lemma 5.2, $\alpha^{-1}(T_i(t))$ is the set of all lattice in $P(t)$. From the definition of the integration we may conclude that

$$\lim_{t\to\infty} \frac{|T_i(t)|}{t} = \frac{1}{N\tilde{\mathfrak{a}}_i} vol(P^*).$$

The volume of P^* is given by

$$vol(P^*) = \int_{P^*} d\mathbf{x}\ with\ d\mathbf{x} = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n_i}.$$

We can calculate $vol(P^*)$ explicitly (cf. §1.3, Chapt.5, [1]). Then we have the following lemma.

LEMMA 5.4. *With notations as above in force, we have*

$$\lim_{t\to\infty} \frac{|T_i(t)|}{t} = \frac{2^{r_i+c_i}(\pi)^{c_i}|R((\mathbf{E}_a)_i)|}{N\tilde{\mathfrak{b}}_i\, N\tilde{\mathfrak{a}}_i\, \sqrt{|\mathbf{D}_i|}}$$

where $\mathbf{D}_i$ is the discriminant of the field k_i.

For $\mathbf{a}, \mathbf{b}$ ($\mathbf{a} \neq 0$) in $\mathbf{R}^{n_i}$, consider a line $\ell(\mathbf{a} : \mathbf{b}) = \{x\mathbf{a} + \mathbf{b} : x \in \mathbf{R}\}$. Let $\overline{P(t)}$ be the closure of $P(t)$ in $\mathbf{R}^{n_i}$.

LEMMA 5.5. *Let* $\mathbf{a}, \mathbf{b}\,(\mathbf{a} \notin \overline{S})$ *be two elements in* $\mathbf{Z}^{n_i}$. *Denote by* $n(\mathbf{a}:\mathbf{b})$ *the number of connected components of* $\overline{P(t)} \cap \ell(\mathbf{a}:\mathbf{b})$. *Then we have* $n(\mathbf{a}:\mathbf{b}) < 2{n_i}^2(n_i+1)$.

Proof. For $\mathbf{x}$ in $\mathbf{R}^{n_i} \setminus S$, put $\Phi(\mathbf{x}) = u\mathbf{u} + \sum_{k=1}^{r_i+c_i-1} v_k \mathbf{v}_k$. Then u and v_k are expressed as in the forms:

$$\begin{cases} u &= \log|N\alpha(\mathbf{x})|, \\ v_k &= \sum_{j=1}^{r_i+c_i} c_{jk} \log|\alpha^{(j)}(\mathbf{x})| \ (1 \le k \le r_i + c_i - 1), \ c_{jk} \in \mathbf{R}. \end{cases}$$

Define the functions $F_k(\mathbf{x})$ $(1 \le k \le r_i + c_i - 1)$ by

$$F_k(\mathbf{x}) = \sum_{j=1}^{r_i+c_i-1} \frac{1}{2} \log|\alpha^{(j)}(\mathbf{x})|^2.$$

For each nonnegative real number a, we put

$$S_a = \{\mathbf{x} \in \mathbf{R}^{n_i} : |N\alpha(\mathbf{x})| = a\},\ S_{a,k} = \{\mathbf{x} \in \mathbf{R} \setminus S : F_k(\mathbf{x}) = a\}\ (1 \le k \le r_i + c_i - 1).$$

Let $\partial P(t)$ be the boundary of $P(t)$. Then

$$\partial P(t) \subset S \cup S_{\frac{t}{N a_i}} \cup (\cup_{k=1}^{r_i+ci-1}(S_{0,k} \cup S_{1,k})). \tag{5.12}$$

As $\mathbf{a} \notin S$, it follows that

$$|S \cap \ell(\mathbf{a}:\mathbf{b})| \le n_i,\ |S_{\frac{t}{N a_i}} \cap \ell(\mathbf{a}:\mathbf{b})| \le 2n_i. \tag{5.13}$$

Let us prove that for each k, there are the following two cases:
(5.14)
$$Case\,(1) : (\ell(\mathbf{a}:\mathbf{b}) \setminus S) \subset (S_{0,k} \cup S_{1.k}),\ Case\,(2) : |\ell(\mathbf{a}:\mathbf{b}) \cap (S_{0,k} \cup S_{1,k})| \le 4n_i(n_i+1).$$

Put $J = \{x \in \mathbf{R} : x\mathbf{a} + \mathbf{b} \notin S\}$. The first derivative of the function $F_k(x\mathbf{a}+\mathbf{b})$ on J is expressed as

$$\frac{d}{dx} F_k(x\mathbf{a}+\mathbf{b}) = \frac{g_k(x)}{(N\alpha(x\mathbf{a}+\mathbf{b}))^2}$$

where g_k is a polynomial in x and $deg\, g_k \le 2n_i - 1$. Suppose $g_k \equiv 0$ on J. Then $F_k(x\mathbf{a}+\mathbf{b})$ is a constant function on J. Therefore $(\ell(\mathbf{a}:\mathbf{b}) \setminus S) \cap (S_{0,k} \cup S_{1,k}) = \phi$ or $(\ell(\mathbf{a}:\mathbf{b}) \setminus S) \subset (S_{0,1} \cup S_{1,k})$. Suppose that g_k is not identically 0 on J. Then the equation $F_k(x\mathbf{a}+\mathbf{b}) = 0$ has at most $2n_i$ solutions on each connected component of J. Furthermore, by the first inequality of (5.13), J has at most n_i+1 connected components. Hence, if $g_k \not\equiv 0$ on J, then $\ell(\mathbf{a}:\mathbf{b})$ intersects to $S_{0,k} \cup S_{1,k}$ at most $4n_i(n_i+1)$ times. Let us now prove this lemma. If $\ell(\mathbf{a}:\mathbf{b})$ satisfies Case (1) for a number k, then $n(\mathbf{a}:\mathbf{b}) \le n_i + 1$. Suppose $\ell(\mathbf{a}:\mathbf{b})$ stisfies Case (2) for all $k = 1, 2, \cdots, r_i + c_i - 1$. Then by (5.12) and (5.13),

$$n(\mathbf{a}:\mathbf{b}) \le \frac{1}{2}(n_i + 2n_i + 4(r_i + c_i - 1)n_i(n_i+1)) < 2{n_i}^2(n_i+1).$$

§6. Asymptotic formula for $\zeta_i(s:\chi_\epsilon)$.

Let $C = C(\mathfrak{a})$ be an ideal class of R with a representative $\mathfrak{a} \subset R$. Put $\tilde{\mathfrak{a}} = O\mathfrak{a}$, and denote by $\tilde{\mathfrak{a}}^{-1}$ the inverse ideal of the ideal $\tilde{\mathfrak{a}}$ of O. Let $\tilde{\mathfrak{a}}^{-1} = \tilde{\mathfrak{b}}_1 e_1 + \tilde{\mathfrak{b}}_2 e_2 + \cdots + \tilde{\mathfrak{b}}_g e_g$ be the decomposition of $\tilde{\mathfrak{a}}^{-1}$ by the fractional ideals $\tilde{\mathfrak{b}}_i$ of O_i. By Lemma 4.3 and Lemma 5.2, we can consider the L-functions $L(s:\chi)\,(\chi \in B^*)$ which have the following properties:

$$(1):\ L(s:\chi) = \sum_{[\epsilon]\in E_O/E_{\mathfrak{a}}} \prod_{i=1}^{g} \zeta_i(s:\chi_\epsilon),$$

$$(6.1)\qquad (2):\ \zeta_i(s:\chi_\epsilon) = \sum_{\mathbf{a}\in P_i\cap \mathbf{Z}^{n_i}} \frac{\chi_\epsilon(\alpha(\mathbf{a})e_i)}{(N\alpha(\mathbf{a})\tilde{\mathfrak{a}}_i)^s}\ (1 \le i \le g).$$

In this section we shall prove that the zeta function $\zeta_i(s:\chi_\epsilon)$ is holomorphic in the half plane $\Re(s) > 1$, and calculate the value:

$$\lim_{\sigma\to 1+0} (\sigma - 1)\zeta_i(\sigma:\chi_\epsilon).$$

The follwing lemma, which is well known (cf. (2.1.2), [13]), plays a crucial role in calculating this value.

LEMMA 6.1. *Suppose ψ is a function on $\mathbf{R}$ of C^1-class. Then for each closed interval $[a,b] \subset \mathbf{R}$, we have*

$$\sum_{a<m\le b} \psi(m)$$
$$= \int_a^b \psi(x)dx + \int_a^b (x - [x] - \frac{1}{2})\psi'(x)dx - (a - [a] - \frac{1}{2})\psi(a) - (b - [b] - \frac{1}{2})\psi(b)$$

where $[x]$ is the Gaussian integral part of x.

LEMMA 6.2. *Let p and q be two positive integers satisfying $\frac{q}{p} < 1$. For the function $\psi(x) = e^{2\pi\sqrt{-1}\frac{q}{p}x}$, we have*

$$\Big|\sum_{a<m\le b} \psi(m)\Big| \le \frac{q}{\pi p}\sum_{\nu=1}^{\infty} \frac{1}{\nu^2 - (\frac{q}{p})^2} + 1 + \frac{p}{\pi q}.$$

Proof. We shall apply Lemma 6.1 to the function ψ. By the Fourier expansion theorem, we have

$$x - [x] - \frac{1}{2} = -\frac{1}{\pi}\sum_{\nu=1}^{\infty} \frac{\sin 2\pi\nu x}{\nu}\ \textit{for } x \notin \mathbf{Z}.$$

Therefore

$$\int_a^b (x-[x]-\frac{1}{2})\psi'(x)dx = -2\sqrt{-1}\frac{q}{p}\int_a^b \sum_{\nu=1}^{\infty} \frac{\sin 2\pi\nu x}{\nu}\psi(x)dx.$$

Since the series $\sum_{\nu=1}^{\infty} \frac{\sin 2\pi\nu x}{\nu}$ is uniformly convergent on each closed interval $I \subset [a,b]\setminus \mathbf{Z}$, the summation and the integration are interchanged. Consequently

$$\begin{aligned}\int_a^b (x-[x]-\frac{1}{2})\psi'(x)dx &= -2\sqrt{-1}\frac{q}{p}\sum_{\nu=1}^{\infty}\int_a^b \frac{\sin 2\pi\nu x}{\nu}\psi(x)dx \\ &= \frac{q}{p}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_a^b (e^{2\pi\sqrt{-1}(\frac{q}{p}-\nu)x} - e^{2\pi\sqrt{-1}(\frac{q}{p}+\nu)x})dx.\end{aligned}$$

Therefore

$$\left|\int_a^b (x-[x]-\frac{1}{2})\psi'(x)dx\right| \le \frac{q}{\pi p}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2-(\frac{q}{p})^2}.$$

Hence by Lemma 6.1 we have our assertion.

DEFINITION 6.1. *We say that* $\mathbf{a} = (a_1, a_2, \cdots, a_{n_i})$ *in* $\mathbf{Z}^{n_i}$ *is primitive* if the greatest common divisor of $a_1, a_2, \cdots, a_{n_i}$ is equal to 1.

LEMMA 6.3. *Let $V = V_i$ and $P = P_i$ be the same as in (5.9). Suppose χ_ϵ is nontrivial on $\tilde{\mathfrak{b}}_i e_i$. Then there exists a primitive element* $\mathbf{a}$ *in* $P \cap \mathbf{Z}^{n_i}$ *such that* $\chi_\epsilon(\alpha(\mathbf{a})e_i) \neq 1$.

Proof. Denote by V^0 the set of all interior points in V. Put $P^0 = \Phi^{-1}(V^0)$. Then $P^0 \subset P$ and P^0 is open in $\mathbf{R}^{n_i}$. Therefore $P^0 \cap \mathbf{Z}^{n_i} \neq \phi$. Let $\mathbf{b}$ be a primitive element in $P^0 \cap \mathbf{Z}^{n_i}$. Choose a $\mathbf{Z}$-basis $\{\mathbf{b}_1 = \mathbf{b}, \mathbf{b}_2, \cdots, \mathbf{b}_{n_i}\}$ of $\mathbf{Z}^{n_i}$. We shall prove that $P^0 \cap \mathbf{Z}^{n_i}$ contains a $\mathbf{Z}$-basis of $\mathbf{Z}^{n_i}$. Since P^0 is open in $\mathbf{R}^{n_i}$, there exists a (sufficiently small) positive irrational number δ such that $\mathbf{b}_1 + \delta\mathbf{b}_2 \in P^0$. Let $U(\subset P^0)$ be a n_i-dimensional open ball in $\mathbf{R}^{n_i}$ centered at $\mathbf{b}_1 + \delta\mathbf{b}_2$. Put

$$C(U) = \{x\mathbf{x} : x \in \mathbf{R}^\times, \mathbf{x} \in U\}.$$

By (5.10), $C(U)$ is an open cone in $\mathbf{R}^{n_i}$. Furthermore, it is easy to see that $C(U) \subset P^0$. On the other hand, by a theorem of continued fractions (cf. Theorem 7.9, [11]), there exists an infinite sequence $(p_m, q_m) \in \mathbf{Z}^2$ $(m = 1, 2, \cdots)$ such that

$$(1): \ 0 \le p_m, 0 < q_m < q_{m+1},$$
$$(2): \ p_{m+1}q_m - q_{m+1}p_m = \pm 1,$$
$$(3): \ \lim_{m\to\infty}\frac{p_m}{q_m} = \delta.$$

Put $\mathbf{v}_m = q_m\mathbf{b}_1 + p_m\mathbf{b}_2$. By (3), the series of angles θ_m between the two vectors $\mathbf{v}_m$ and $\mathbf{b}_1 + \delta\mathbf{b}_2$ converges to 0. Consequently, $\mathbf{v}_m, \mathbf{v}_{m+1} \in C(U) \subset P^0$ for a sufficiently large number m. Put $\mathbf{b}'_1 = \mathbf{v}_m$, $\mathbf{b}'_2 = \mathbf{v}_{m+1}$. Then $\mathbf{b}'_j \in P^0 \cap \mathbf{Z}^{n_i}$ for $j = 1, 2$. On the other hand, from (2) it follows that $\{\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}_3, \cdots, \mathbf{b}_{n_i}\}$ is a $\mathbf{Z}$-basis of $\mathbf{Z}^{n_i}$. By the same arguments as above, we may conclude that $P^0 \cap \mathbf{Z}^{n_i}$ contains a $\mathbf{Z}$-basis of $\mathbf{Z}^{n_i}$. Hence we can choose a primitive element $\mathbf{a}$ in $\mathbf{Z}^{n_i}$ satisfying $\chi_\epsilon(\alpha(\mathbf{a})e_i) \neq 1$.

LEMMA 6.4. *Let $A_i(t:\chi_\epsilon)$ be the function given in Lemma 4.6. Suppose χ_ϵ is nontrivial on $\tilde{\mathfrak{b}}_i e_i$. Then there exists a positive constant K such that*

$$|A_i(t:\chi_\epsilon)| \leq Kt^{\frac{n_i-1}{n_i}} \ for\, all\ t>0.$$

Proof. Since $\chi_\epsilon \not\equiv 1$ on $\tilde{\mathfrak{b}}_i e_i$, Lemma 6.3 implies that there exists a primitive element $\mathbf{a} \in P(t) \cap \mathbf{Z}^{n_i}$ such that $\chi(\alpha(\mathbf{a})e_i) \neq 1$. Therefore $\chi_\epsilon(\alpha(\mathbf{a})) = e^{2\pi\sqrt{-1}\frac{q}{p}}$ for two suitable positive integers p, q $(q < p)$. Let $\{\mathbf{a} = \mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_{n_i}\}$ be a $\mathbf{Z}$-basis of $\mathbf{Z}^{n_i}$ and W be the $n_i - 1$ dimensional subspace of $\mathbf{R}^{n_i}$ generated by $\{\mathbf{b}_2, \mathbf{b}_3, \cdots, \mathbf{b}_{n_i}\}$. Define a projection map $\varpi : \mathbf{R}^{n_i} \to W$:

$$\varpi(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_{n_i}\mathbf{b}_{n_i}) = x_2\mathbf{b}_2 + \cdots + x_{n_i}\mathbf{b}_{n_i}.$$

Put $S(t) = \varpi(P(t))$. $S(t)$ is bounded. Let $\mathbf{b}$ be each element in $S(t) \cap \mathbf{Z}^{n_i}$. Then $\varpi^{-1}(\{\mathbf{b}\})$ is a line which is parallel to the vector $\mathbf{a}$. Let $n(\mathbf{b})$ be the number of the connected components of $\varpi^{-1}(\{\mathbf{b}\}) \cap P(t)$. By Lemma 5.5 we have

$$n(\mathbf{b}) < 2{n_i}^2(n_i+1). \tag{6.2}$$

Hence there exist a finite number of the intervals $(a_j, b_j](1 \leq j \leq n(\mathbf{b}))$ such that

$$P(t) \cap \mathbf{Z}^{n_i} \cap \varpi^{-1}(\{\mathbf{b}\}) = \bigcup_{j=1}^{n(\mathbf{b})} \{u\mathbf{a} + \mathbf{b} : u \in (a_j, b_j] \cap \mathbf{Z}\}.$$

Since

$$P(t) \cap \mathbf{Z}^{n_i} = \bigcup_{\mathbf{b} \in \varpi(P(t) \cap \mathbf{Z}^{n_i})} P(t) \cap \varpi^{-1}(\{\mathbf{b}\}) \cap \mathbf{Z}^{n_i},$$

we have

$$A_i(t:\chi_\epsilon) = \sum_{\mathbf{b} \in \varpi(P(t)\cap\mathbf{Z}^{n_i})} \sum_{j=1}^{n(\mathbf{b})} \sum_{\substack{a_j < m \leq b_j \\ m \in \mathbf{Z}}} \chi_\epsilon(\alpha(m\mathbf{a})e_i)\chi_\epsilon(\alpha(\mathbf{b})e_i).$$

Hence, by Lemma 6.2 and (6.2), there exists a positive constant K_1 such that

$$|A_i(t:\chi_\epsilon)| \leq K_1 |\varpi(P(t) \cap \mathbf{Z}^{n_i})| \ for\ all\ t > 0. \tag{6.3}$$

Since

$$\lim_{t\to\infty} \frac{|\varpi(P(t)\cap\mathbf{Z}^{n_i})|}{t^{\frac{n_i-1}{n_i}}} = vol(\varpi(P(1))),$$

the assertion follows from (6.3).

Lemma 6.5. *The zeta function $\zeta_i(s:\chi_\epsilon)$ is holomorphic in the half plane where $\Re(s)>1$. Furthermore, we have*

$$\lim_{\sigma\to 1+0}(\sigma-1)\zeta_i(\sigma:\chi_\epsilon)=\begin{cases}\kappa_i(C) & \chi_\epsilon\equiv 1 \text{ on } \tilde{\mathfrak{b}}_i e_i\\ 0 & \text{otherwise}\end{cases}$$

where

$$\kappa_i(C)=\frac{2^{r_i+c_i}\pi^{c_i}|R(E_i)|}{N\tilde{\mathfrak{a}}_i\, N\tilde{\mathfrak{b}}_i\sqrt{|\mathbf{D}_i|}}.$$

Proof. We first prove that the series $\zeta_i(s:\chi_\epsilon)$ is holomorphic in the complex half plane $\Re(s)>1$. By Lemma 4.6 we have

$$\sum_{\substack{[\alpha]\in(\mathfrak{b}_i)^\times/B_i\\ N(\alpha\tilde{\mathfrak{a}}_i)\le t}}\frac{\chi_\epsilon(\alpha e_i)}{(N(\alpha\tilde{\mathfrak{a}}_i))^s}=A_i(t:\chi_\epsilon)t^{-s}+s\int_1^t A_i(x:\chi_\epsilon)x^{-s-1}dx.$$

Since $|A_i(t:\chi_\epsilon)|\le|T_i(t)|$, Lemma 5.4 implies that $\frac{|A_i(t:\chi_\epsilon)|}{t}$ is bounded on the half line: $t\ge 1$. Therefore $\lim_{t\to\infty}A_i(t:\chi_\epsilon)t^{-s}=0$. Hence we have

$$\zeta_i(s:\chi_\epsilon)=s\int_1^\infty A_i(x:\chi_\epsilon)x^{-s-1}dx. \tag{6.4}$$

Let b be the upper bound of $\frac{|T_i(t)|}{t}$ $(1\le t)$. Then we have

$$\left|\int_1^\infty A_i(x:\chi_\epsilon)x^{-s-1}dx\right|\le b\int_1^\infty x^{-\sigma}dx<\infty$$

where $\sigma=\Re(s)$. Consequently by (6.4), $\zeta_i(s:\chi_\epsilon)$ is a convergent series in the half plane $\Re(s)>1$.

Finally we prove the asymptotic formula of $\zeta_i(s:\chi_\epsilon)$. Assume that χ_ϵ is nontrivial on $\tilde{\mathfrak{b}}_i e_i$. By Lemma 6.5 and (6.4) we have

$$|\zeta_i(s:\chi_\epsilon)|\le K|s|\int_1^\infty x^{-\sigma-\frac{1}{n_i}}dx.$$

Therefore $\lim_{s\to 1}(s-1)\zeta_i(s:\chi_\epsilon)=0$.

Let us consider the case: χ_ϵ is trivial on $\tilde{\mathfrak{b}}_i e_i$.

From Lemma 5.4 it follows that

$$\lim_{t\to\infty}\frac{|T_i(t)|}{t}=\kappa_i(C).$$

Put

$$\frac{A_i(t:1)}{t} \equiv \frac{|T_i(t)|}{t} = \kappa_i(C) + R(t). \tag{6.5}$$

Then for each positive real number δ, there exists a number N_0 such that

$$|R(t)| \leq \delta \textit{ for all } t \geq N_0.$$

On the other hand

$$\begin{aligned}
&(\sigma-1)\zeta_i(\sigma:1) - \kappa_i(C) \\
=&(\sigma-1)\{\sigma \int_1^\infty A_i(x:1)x^{-\sigma-1}dx - \int_1^\infty \kappa_i(C)x^{-\sigma}dx\} \\
=&(\sigma-1)\{\kappa_i(C) + \sigma \int_1^\infty R(x)x^{-\sigma}dx\}.
\end{aligned}$$

Let R_0 be the upper bound of the set: $\{|R(x)| : x \in [1, N_0]\}$. Then we have

$$\begin{aligned}
&|(\sigma-1)\zeta_i(\sigma:1) - \kappa_i(C)| \\
\leq&(\sigma-1)\kappa_i(C) + \sigma(\sigma-1)\{R_0 \int_1^{N_0} \frac{1}{x}dx + \delta \int_{N_0}^\infty x^{-\sigma}dx\} \\
=&(\sigma-1)\{\kappa_i(C) + \sigma R_0 \log N_0\} + \sigma\delta(N_0)^{-\sigma+1}.
\end{aligned}$$

Therefore

$$\overline{\lim_{\sigma\to 1+0}}|(\sigma-1)\zeta_i(\sigma:1) - \kappa_i(C)| \leq \delta$$

for all positive real numbers δ. This implies that

$$\lim_{\sigma\to 1+0}(\sigma-1)\zeta_i(\sigma:1) = \kappa_i(C).$$

This completes our proof of the lemma.

§7. Main theorems.

Let ζ be a regular element of $GL(n,\mathbf{Z})$ with the characteristic polynomial $f(X) = f_1(X)f_2(X)\cdots f_g(X)$ and $R = \mathbf{Z}[\zeta]$ the ring generated by ζ over $\mathbf{Z}$. Let r_i (resp. $2c_i$) be the number of real (resp. complex) roots of $f_i(X)$. Put $r = \sum_{i=1}^g r_i$ and $c = \sum_{i=1}^g c_i$. The following lemma is a direct consequence of Lemma 6.5.

Lemma 7.1. *The function $L(s:\chi)$ is holomorphic in the complex half plane where $\Re(s) > 1$. Furthermore, we have*

$$\lim_{\sigma\to 1+0}(\sigma-1)^g L(\sigma:\chi) = \begin{cases} \frac{2^{r+c}\pi^c(E_O:E_a)|R(\mathbf{E}_O)|}{N\mathfrak{a}^{-1}N\mathfrak{a}\prod_{i=1}^g\sqrt{|\mathbf{D}_i|}} & \chi = 1 \\ 0 & \chi \neq 1. \end{cases}$$

Theorem 7.2. *Let $\zeta_C(s)$ be the zeta function of an ideal class $C = C(\mathfrak{a})$, which is defined by*

$$\zeta_C(s) = \sum_{\substack{\mathfrak{b}\in C(\mathfrak{a})\\ \mathfrak{b}\subset R}} \frac{1}{(N\mathfrak{b})^s}.$$

Then we have

$$(1): \zeta_C(s) \text{ is holomorphic in the complex half plane } \Re(s) > 1, \text{ and}$$

$$(2): \lim_{\sigma\to 1+0}(\sigma-1)^g\zeta_C(\sigma) = 2^{r+c}(\pi)^c\frac{(\mathbf{E}_O : \mathbf{E}_\mathfrak{a})|R(\mathbf{E}_O)|}{N(\check{\mathfrak{a}})N\mathfrak{a}|H_O|\sqrt{|Nf'(\zeta)|}}.$$

Proof. By Theorem 4.2 and Lemma 7.1 the assertion of (1) is obvious. Furthermore,

$$\lim_{\sigma\to 1+0}(\sigma-1)^g\zeta_C(\sigma) = \frac{(\mathbf{E}_O : \mathbf{E}_\mathfrak{a})|R(\mathbf{E}_O)|}{|H_\mathfrak{a}|(\tilde{\mathfrak{a}}^{-1} : \check{\mathfrak{a}})\,N\mathfrak{a}\,N\tilde{\mathfrak{a}}^{-1}\prod_{i=1}^g\sqrt{|\mathbf{D}_i|}}.$$

It is easy to see that $Nf'(\zeta) = (O : R)^2\prod_{i=1}^g\mathbf{D}_i$ and $(\tilde{\mathfrak{a}}^{-1} : \check{\mathfrak{a}})N\tilde{\mathfrak{a}}^{-1} = (O : R)N\check{\mathfrak{a}}$. Hence the theorem follows.

Theorem 7.3. *We define a Dirichlet series $\zeta_R(s)$ by*

$$\zeta_R(s) = \sum_{\mathfrak{b}}\frac{a(\mathfrak{b})}{(N\mathfrak{b})^s}$$

where $\sum_\mathfrak{b}$ runs over all nonsingular ideals of R and $a(\mathfrak{b})$ the constant defined by $a(\mathfrak{b}) = \frac{N\check{\mathfrak{b}}N\mathfrak{b}}{(\mathbf{E}_O:\mathbf{E}_\mathfrak{b})}$. Then we have

$$\lim_{\sigma\to 1+0}(\sigma-1)^g\zeta_R(\sigma) = N_f 2^{r+c}\pi^c\frac{|R(\mathbf{E}_O)|}{|H_O|\sqrt{|Nf'(\zeta)|}}$$

where N_f is the number of orbits of $GL(n,\mathbf{Z})$ on $G_Z(f)$.

Proof. Let $\mathfrak{b}$ be an integral ideal in $C(\mathfrak{a})$. Then there exists an invertible element λ in $\mathbf{Q}[\zeta]$ such that $\lambda\mathfrak{a} = \mathfrak{b}$. Since $\check{\lambda\mathfrak{a}} = \lambda^{-1}\check{\mathfrak{a}}$, we have $N\check{\mathfrak{b}}N\mathfrak{b} = N\check{\mathfrak{a}}N\mathfrak{a}$. Also since λ is invertible, we have $\mathbf{E}_\mathfrak{b} = \mathbf{E}_\mathfrak{a}$. Therefore $a(\mathfrak{b}) = a(\mathfrak{a})$. From this it follows that $\zeta_R(s) = \sum_{C(\mathfrak{a})\in\mathbf{G}} a(\mathfrak{a})\zeta_C(s)$. Hence by Theorem 3.3 and Theorem 7.2 we have our assertion.

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