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Quantum Jacobi-Trudi Formula for $U_q(B_r^{(1)})$ from Analytic Bethe Ansatz

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1. Introduction

The main message of this note, which is based on the works [KS, KOS] with Y.Ohta and J.Suzuki, is, 'Analytic Bethe ansatz is a character theory of finite dimensional representations of quantum affine algebras'. Analytic Bethe ansatz originates in solvable lattice models in statistical mechanics [B1]. It is a hypothetical prescription to produce an eigenvalue formula for row-to-row transfer matrices of the models. As for its validity, no general proof is known neither any counter example. It was invented by Reshetikhin in [R] by extracting the idea from Baxter's solution of the 8-vertex model [B2]. Let us explain it with a simplest example from $sl(2)$.

Consider the 6-vertex model on a square lattice [B1] with the Boltzmann weights $R_u(\pm,\pm,\pm,\pm) = [2+u]$, $R_u(\pm,\mp,\pm,\mp) = [u]$ and $R_u(\pm,\mp,\mp,\pm) = [2]$, where the local states $+$ or $-$ are ordered anti-clockwise from the left edge of the vertex. The function $[u]$ is defined by

$$[u] = \frac{q^u - q^{-u}}{q - q^{-1}}. \tag{1.1}$$

Here u is a spectral parameter and q is a generic constant (not a root of unity). The Boltzmann weights can be arranged in an R-matrix $R_{W_1,W_1}(u)$ satisfying the Yang-Baxter equation and the model is solvable. Here the indices indicate that it is an intertwiner of the tensor product of the 2-dimensional $U_q(A_1^{(1)})$ module W_1. (We let W_m denote the $m+1$ dimensional irreducible one.) The row-to-row transfer matrix of the 6-vertex model is the $m=1$ case of the following more general matrix

$$T_m(u) = \mathrm{Tr}_{W_m}\left(R_{W_m,W_1}(u-w_1)\cdots R_{W_m,W_1}(u-w_N)\right) \tag{1.2}$$

Here N is the system size, $w_1,\ldots,w_N$ are complex parameters representing the inhomogeneity of local interactions, $m \in \mathbf{Z}_{\geq 0}$. Following the QISM terminology [STF], we say that (1.2) is the row-to-row transfer matrix with the *auxiliary space* W_m that acts on the *quantum space* $W_1^{\otimes N}$. (More precisely, $W_m(u)$ and $\otimes_{j=1}^N W_1(w_j)$, respectively.) Due to the Yang-Baxter equation, $[T_m(u), T_{m'}(u')] = 0$ holds. So they can be simultaneously diagonalized and we shall often write their eigenvalues also by the same symbol $T_m(u)$. One of the main subject in statistical mechanics is to study the spectrum of $T_m(u)$ (especially in the limit $N \to \infty$). Let us quote an eigenvalue formula for $T_1(u)$ [B1];

$$T_1(u) = \frac{Q(u-1)}{Q(u+1)}\phi(u+2) + \frac{Q(u+3)}{Q(u+1)}\phi(u), \tag{1.3a}$$

$$Q(u) = \prod_{j=1}^{n}[u - v_j], \quad \phi(u) = \prod_{j=1}^{N}[u - w_j]. \tag{1.3b}$$

Here, $0 \leq n \leq N/2$ is the number of the $-$ states in the eigenvector, which is preserved under the action of $T_1(u)$. $u_j \in \mathbf{C}$ are any solution of the Bethe ansatz equation (BAE)

$$-\frac{\phi(v_k+1)}{\phi(v_k-1)} = \frac{Q(v_k+2)}{Q(v_k-2)}. \tag{1.4}$$

On the result (1.3-4), one makes a few observations.
(i) The eigenvalue has the "dressed vacuum form (DVF)", which means the following. The "vacuum vector" $+,+,\ldots,+$ is the obvious eigenvector with the vacuum eigenvalue

$$\prod_{j=1}^{N} R_{u-w_j}(+,+,+,+) + \prod_{j=1}^{N} R_{u-w_j}(-,+,-,+) = \phi(u+2) + \phi(u). \tag{1.5}$$

Eq.(1.3) tells that general eigenvalues can still be expressed with the modifying "dress" factors Q/Q which is certainly 1 when $n = 0$. In particular, the number of the terms in $T_1(u)$ is the dimension of the auxiliary space $\dim W_1 = 2$.
(ii) The BAE (1.4) ensures that the eigenvalues are free of poles for finite u. The apparent pole at $u = v_k - 1$ in (1.3a) is spurious as the residues from the two terms cancel due to (1.4). The eigenvalues must actually be pole-free because the local Boltzmann weight, hence the matrix elements of $T_1(u)$ are so.
(iii) Properties inherited from the asymptotic behavior in $|u| \to \infty$ and the first/second inversion relations of the R-matrix (vertex Boltzmann weights). For example, one has

$$(q - q^{-1})^N q^{-N+\sum_j w_j} \lim_{q^u \to \infty} q^{-Nu} T_1(u) = q^{N-2n} + q^{2n-N}, \tag{1.6}$$

which is certainly an expected result from the definition of $T_1(u)$.

The analytic Bethe ansatz is the hypothesis that the postulates (i)-(iii) essentially determine a function of u uniquely and that the so obtained is the actual transfer matrix eigenvalue. As the input data, it only uses the BAE and the special components of the R-matrix (or the vacuum eigenvalue (1.5)) which should be normalized to be an entire function of u. Its validity can only be assured in general by a proper diagonalization, most notably, by the algebraic Bethe ansatz which yields the eigenvectors as well.

In (1.6), one notices already that the RHS is an $sl(2)$ character of the 2-dimensional representation space W_1. Thus $T_1(u)$ is a u-dependent version of it. This view point becomes even more natural if one considers the eigenvalues for general $T_m(u)$ and observes the following functional relations that generalize the usual character identities.

$$\begin{aligned} T_m(u+1)T_m(u-1) &= T_{m+1}(u)T_{m-1}(u) + g_m(u)\mathrm{Id}, \\ g_m(u) &= \prod_{k=0}^{m-1} \phi(u+2k-m)\phi(u+4+2k-m), \end{aligned} \tag{1.7}$$

where $m \geq 0$. Regarding (1.7) as an equation for the eigenvalues one can easily solve it under the initial condition (1.3a) and $T_0(u) = 1$ to find

$$T_m(u) = \Big(\prod_{k=1}^{m-1} \phi(u+m+1-2k)\Big) \sum_{j=0}^{m} \frac{Q(u-m)Q(u+m+2)\phi(u+m+1-2j)}{Q(u+m-2j)Q(u+m+2-2j)}, \tag{1.8}$$

To observe a representation theoretical content, we now set

$$\boxed{1} = \frac{Q(u-1)}{Q(u+1)}\phi(u+2), \quad \boxed{2} = \frac{Q(u+3)}{Q(u+1)}\phi(u), \tag{1.9}$$

where we assume on the LHS that the spectral parameter u is implicitly attached to the single box as well. In this notation (1.3a) reads as $\Lambda_1(u) = \boxed{1} + \boxed{2}$. Moreover, the result (1.8) for general m can be expressed as follows.

$$T_m(u) = \sum_{j=0}^{m} \overbrace{\boxed{1}\ \cdots\ \boxed{1}}^{m-j}\overbrace{\boxed{2}\ \cdots\ \boxed{2}}^{j}. \tag{1.10}$$

Here we interpret the tableau as the product of the m functions (1.9) with the spectral parameter u shifted to $u-m+1, u-m+3, \ldots, u+m-1$ from the left to the right. Notice that the tableaux appearing in (1.10) are exactly the semi-standard ones that label the weight vectors in the $(m+1)$-dimensional irreducible representation W_m of $U_q(\hat{sl}(2))$ (plainly, the spin $\frac{m}{2}$ representation of $sl(2)$). In this sense $T_m(u)$ is an analogue ("Yang-Baxterizations") of the character of the auxiliary space W_m, which may be natural from (1.2). The functional relation (1.7) thereby plays the role of a character identity. Under the BAE (1.4), is $T_m(u)$ pole-free for general $m \geq 1$? This is a crucial check for (1.10) be a correct DVF. To answer it, solve (1.7) keeping $T_1(u)$. The result reads

$$T_m(u) = det \begin{pmatrix} T_1(u-m+1) & g_1(u-m+2) & 0 & \cdots & 0 \\ 1 & T_1(u-m+3) & & & \\ 0 & 1 & \ddots & & 0 \\ \vdots & \vdots & & T_1(u+m-3) & g_1(u+m-2) \\ 0 & 0 & \cdots & 1 & T_1(u+m-1) \end{pmatrix}, \tag{1.11}$$

which expresses T_m in terms of fundamental T_1. Obviously, this reduces to a Jacobi-Trudi formula [M] for Schur functions if the u-dependence is absent (or in the limit $u \to \infty$).

In this sense (1.11) may be called a quantum Jacobi-Trudi formula. It manifestly tells that $T_m(u)$ is pole-free, which is by no means obvious from the expression (1.10). One can also check the character limit $(q-q^{-1})^{mN}q^{-mN+m\sum_j w_j}\lim_{q^u\to\infty}q^{-mNu}T_1(u) = \sum_{j=0}^m q^{(N-2n)(m-2j)}$.

To summarize so far, the functional relation (1.7), the tableau representation (1.10) and the quantum Jacobi-Trudi formula (1.11) are typical features in transfer matrices and analytic Bethe ansatz in solvable lattice models.

2. Bethe ansatz equation

Having seen the $sl(2)$ example, a natural question is a generalization to other algebras. For simplicity, we shall consider vertex models associated with the Yangian $Y(X_r)$ for $X_r = A_r, B_r, C_r, D_r, E_{6,7,8}, F_4$ and G_2. Let $W^{(i)}, 1 \le i \le N$ be a finite dimensional irreducible $Y(X_r)$ module and $P^{(i)}_a(\zeta)$, $1 \le a \le r$ be the characterizing Drinfel'd polynomials [D]. The BAE relevant to the transfer matrices acting on the quantum space $\otimes_{i=1}^N W^{(i)}$ has been conjectured as follows.

$$
\begin{aligned}
-\frac{P_a(v^{(a)}_k + \frac{(\alpha_a|\alpha_a)}{2})}{P_a(v^{(a)}_k - \frac{(\alpha_a|\alpha_a)}{2})} &= \prod_{b=1}^{r}\frac{Q_b(v^{(a)}_k + (\alpha_a|\alpha_b))}{Q_b(v^{(a)}_k - (\alpha_a|\alpha_b))} \quad 1 \le a \le r,\ 1 \le k \le N_a,\\
P_a(\zeta) &= \prod_{i=1}^{N} P^{(i)}_a(\zeta).
\end{aligned}
\tag{2.1}
$$

Here α_a's are the simple roots (normalization $|$ long root $|^2 = 2$), $Q_a(u) = \prod_{j=1}^{N_a}[u - v^{(a)}_j]$ and we understand that $q \to 1$ in (1.1). (On the other hand, for generic q, we suppose that (2.1) is valid if $P^{(i)}_a(\zeta)$ is replaced by a natural q-analogue.) The RHS of the conjecture (2.1) is due to [RW] and the LHS is due to [KOS] and [ST]. It has been formulated purely from the representation theoretical data, the root system and the Drinfel'd polynomial. As for the functional relations, an analogue of (1.7), called T-system, has been proposed for arbitrary X_r in [KNS]. Here we quote it only for B_r below.

$$
\begin{aligned}
T^{(a)}_m(u-1)T^{(a)}_m(u+1) &= T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u)\\
&\qquad \text{for } 1 \le a \le r-2,\\
T^{(r-1)}_m(u-1)T^{(r-1)}_m(u+1) &= T^{(r-1)}_{m+1}(u)T^{(r-1)}_{m-1}(u) + T^{(r-2)}_m(u)T^{(r)}_{2m}(u),\\
T^{(r)}_{2m}(u-\tfrac{1}{2})T^{(r)}_{2m}(u+\tfrac{1}{2}) &= T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u)\\
&\quad + T^{(r-1)}_m(u-\tfrac{1}{2})T^{(r-1)}_m(u+\tfrac{1}{2}),\\
T^{(r)}_{2m+1}(u-\tfrac{1}{2})T^{(r)}_{2m+1}(u+\tfrac{1}{2}) &= T^{(r)}_{2m+2}(u)T^{(r)}_{2m}(u) + T^{(r-1)}_m(u)T^{(r-1)}_{m+1}(u).
\end{aligned}
\tag{2.2}
$$

What are the symbols $T^{(a)}_m(u)$? They stand for the transfer matrix whose auxiliary space is the irreducible $Y(B_r)$ module characterized by the Drinfel'd polynomial $P_b(\zeta) = (\prod_{i=1}^m(\zeta-$

$u+(\alpha_a|\alpha_a)\frac{m+1-2i}{2}))^{\delta_{ba}}$ for $1 \le b \le r$. In particular, $T_1^{(a)}(u), 1 \le a \le r$ is a basic quantity corresponding to the fundamental representations in the sense of [CP].

In the rest of paper we shall exclusively consider the case $X_r = B_r$. See [BR,C] for A_r case. In the next section, we give a combinatorial construction of the DVFs for $T_1^{(a)}(u)$. In section 4, we shall introduce a wide class of the DVFs $T_{\lambda\subset\mu}(u)$ associated with skew Young diagrams $\lambda \subset \mu$. This corresponds to the tensor-like representations. In section 5, we treat the spinor-like case. Combining these results we give a solution to (2.2) in section 6. For simplicity, we shall consider in the rest the case in which the quantum space is formally trivial and set the LHS of the BAE (2.1) to -1. This corresponds to considering the dress part only, which does not lose the essential features. To recover the vacuum part for a given LHS is easy.

3. DVFs for $T_1^{(a)}(u)$

We introduce the set J and the order $\prec$ in it as

$$J = \{1, 2, \ldots, r, 0, \bar{r}, \ldots, \bar{1}\}, \tag{3.1a}$$

$$1 \prec 2 \prec \cdots, \prec r \prec 0 \prec \bar{r} \prec \cdots, \prec \bar{1}. \tag{3.1b}$$

For $a \in J$, define the function $z(a; u)$ by

$$\begin{aligned}
z(a;u) &= \frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)} \qquad 1 \le a \le r,\\
z(0;u) &= \frac{Q_r(u+r-2)Q_r(u+r+1)}{Q_r(u+r)Q_r(u+r-1)},\\
z(\bar{a};u) &= \frac{Q_{a-1}(u+2r-a-2)Q_a(u+2r-a+1)}{Q_{a-1}(u+2r-a)Q_a(u+2r-a-1)} \qquad 1 \le a \le r,
\end{aligned} \tag{3.2}$$

where we have set $Q_0(u) = 1$. $z(a,u)$ corresponds to a weight in the vector representation. For $(\xi_1, \cdots, \xi_r) \in \{\pm\}^r$, define the function $sp(\xi_1, \ldots, \xi_r; u)$ by the following recursion relation with respect to r and the initial condition $r = 1$.

$$\begin{aligned}
sp(+,+,\xi_3,\ldots,\xi_r;u) &= \tau^Q sp(+,\xi_3,\ldots,\xi_r;u),\\
sp(+,-,\xi_3,\ldots,\xi_r;u) &= \frac{Q_1(u+r-\frac{5}{2})}{Q_1(u+r-\frac{1}{2})}\tau^Q sp(-,\xi_3,\ldots,\xi_r;u),\\
sp(-,+,\xi_3,\ldots,\xi_r;u) &= \frac{Q_1(u+r+\frac{3}{2})}{Q_1(u+r-\frac{1}{2})}\tau^Q sp(+,\xi_3,\ldots,\xi_r;u+2),\\
sp(-,-,\xi_3,\ldots,\xi_r;u) &= \tau^Q sp(-,\xi_3,\ldots,\xi_r;u+2).
\end{aligned} \tag{3.3a}$$

$$\begin{aligned}
sp(+;u) &= \frac{Q_1(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})},\\
sp(-;u) &= \frac{Q_1(u+\frac{3}{2})}{Q_1(u+\frac{1}{2})},
\end{aligned} \tag{3.3b}$$

In (3.3a) τ^Q is the operation $Q_a \to Q_{a+1}$, namely,

$$\begin{aligned}\tau^Q F(Q_1(u+x_1^1), Q_1(u+x_2^1), \ldots, Q_2(u+x_1^2), Q_2(u+x_2^2), \ldots) \\ = F(Q_2(u+x_1^1), Q_2(u+x_2^1), \ldots, Q_3(u+x_1^2), Q_3(u+x_2^2), \ldots)\end{aligned} \tag{3.4}$$

for any function F.

Now we introduce the meromorphic functions $T^a(u)$ and $T_m(u)$ of u for any $a, m \in \mathbf{Z}_{\geq 0}$ by the following "non-commutative generating series"

$$\begin{aligned}&(1+z(\bar{1};u)X)\cdots(1+z(\bar{r};u)X)(1-z(0;u)X)^{-1}(1+z(r;u)X)\cdots(1+z(1;u)X) \\ &= \sum_{a=0}^{\infty} T^a(u+a-1)X^a, \qquad (3.5a)\\ &(1-z(1;u)X)^{-1}\cdots(1-z(r;u)X)^{-1}(1+z(0;u)X)(1-z(\bar{r};u)X)^{-1}\cdots(1-z(\bar{1};u)X)^{-1} \\ &= \sum_{m=0}^{\infty} T_m(u+m-1)X^m, \qquad (3.5b)\end{aligned}$$

where X is a difference operator with the commutation relation

$$XQ_a(u) = Q_a(u+2)X \quad \text{for any } 1 \leq a \leq r. \tag{3.6}$$

Thus $Xz(a;u) = z(a;u+2)X$ for any $a \in J$. We set $T^a(u) = T_m(u) = 0$ for $a, m < 0$. An immediate consequence of the above definition is

$$\delta_{ij} = \sum_{k=0}^{N} (-)^{i-k} T_{i-k}(u+i+k)T^{k-j}(u+k+j) \tag{3.7a}$$

$$= \sum_{k=0}^{N} (-)^{i-k} T_{i-k}(u-i-k)T^{k-j}(u-k-j) \tag{3.7b}$$

for any $N \geq 0$ and $0 \leq i, j \leq N$. Define $T_1^{(a)}(u)$ for $1 \leq a \leq r$ by

$$\begin{aligned}T_1^{(a)}(u) &= T^a(u) \quad \text{for } 1 \leq a \leq r-1, \\ T_1^{(r)}(u) &= \sum_{\xi_1,\ldots,\xi_r=\pm} sp(\xi_1, \ldots, \xi_r; u).\end{aligned} \tag{3.8}$$

Theorem 3.1. *$T_1^{(r)}(u), T^a(u)$ and $T_m(u) (\forall a, m \in \mathbf{Z})$ are pole-free provided that the BAE (2.1) (LHS set to -1) holds.*

¿From the theorem they are Laurent polynomials in q^u tending to finite functions of q independent of u in the limit $|q^u| \to \infty$ (cf (1.6) with $N = 0$). Thus they are in fact independent of u. This is a rather special situation owing to the fact that we are considering the trivial quantum space. All the DVFs in the sequel will have this property. However they can actually be made non-trivial u-dependent functions upon introduction of the vacuum parts.

For $T_1^{(r)}(u)$ and $T^a(u)$ with $a \leq r-1$, the theorem was proved in [KS] in a more general setting including the vacuum parts. The other cases can be verified quite similarly.

The functions $z(a;u)$ and $sp(\xi_1,\ldots,\xi_r;u)$ are related as follows. Given two sequences $(\xi_1,\ldots,\xi_r)$ and $(\eta_1,\ldots,\eta_r) \in \{\pm\}^r$, we define $i_1 < \cdots < i_k, I_1 < \cdots < I_{r-k} \, (0 \leq k \leq r)$ and $j_1 < \ldots < j_l, J_1 < \ldots < J_{r-l} \, (0 \leq l \leq r)$ by the following.

$$\begin{aligned} &\xi_{i_1} = \cdots = \xi_{i_k} = +, \; \xi_{I_1} = \cdots = \xi_{I_{r-k}} = -, \\ &\eta_{j_1} = \cdots = \eta_{j_l} = -, \; \eta_{J_1} = \cdots = \eta_{J_{r-l}} = +. \end{aligned} \tag{3.9}$$

Then we have

Proposition 3.2. *For any $a \in \mathbf{Z}_{\geq 0}$,*

$$\begin{aligned} &sp(\xi_1,\ldots,\xi_r;u-r+a+\frac{1}{2})sp(\eta_1,\ldots,\eta_r;u+r-a-\frac{1}{2}) \\ &= \prod_{n=1}^{a} z(b_n;u+a+1-2n) \qquad \text{if } k+l \leq a, \end{aligned} \tag{3.10a}$$

where

$$b_n = \begin{cases} i_n & \text{for } 1 \leq n \leq k \\ 0 & \text{for } k < n \leq a-l \\ \overline{j_{a+1-n}} & \text{for } a-l < n \leq a \end{cases}. \tag{3.10b}$$

For any $a \in \mathbf{Z}_{\leq 2r-1}$,

$$\begin{aligned} &sp(\xi_1,\ldots,\xi_r;u-r+a+\frac{1}{2})sp(\eta_1,\ldots,\eta_r;u+r-a-\frac{1}{2}) \\ &= \prod_{n=1}^{2r-1-a} z(b'_n;u+2r-a-2n) \qquad \text{if } k+l \geq a+1, \end{aligned} \tag{3.11a}$$

where

$$b'_n = \begin{cases} J_n & \text{for } 1 \leq n \leq r-l \\ 0 & \text{for } r-l < n \leq r+k-1-a \\ \overline{I_{2r-a-n}} & \text{for } r+k-1-a < n \leq 2r-1-a \end{cases} \tag{3.11b}$$

This enables the evaluation of the product $sp(\xi_1,\ldots,\xi_r;u-r+a+\frac{1}{2})sp(\eta_1,\ldots,\eta_r;u+r-a-\frac{1}{2})$ for any $\{\xi_i\},\{\eta_i\}$ and $a \in \mathbf{Z}$ in terms of z (3.2). For $1 \leq a \leq r-1$, (3.10) is theorem A.1 in [KS]. It is straightforward to extend it to any $a \in \mathbf{Z}_{\geq 0}$. Eq. (3.11) can be derived from (3.10) by replacing a by $2r-1-a$. Note in (3.10b) that $b_1 \prec \cdots \prec b_k \prec b_{k+1} = \cdots = b_{a-l} = 0 \prec b_{a-l+1} \prec \cdots \prec b_a \in J$. A similar inequality holds also for b'_n. Comparing them with (3.5a) and (3.8) we get

Theorem 3.3.

$$T^a(u) + T^{2r-1-a}(u) = T_1^{(r)}(u-r+a+\frac{1}{2})T_1^{(r)}(u+r-a-\frac{1}{2}) \quad \forall a \in \mathbf{Z}. \tag{3.12}$$

This is invariant under the exchange $a \leftrightarrow 2r-1-a$. If $a<0$ or $a>2r-1$, there is in fact only one term on the LHS.

4. Spin-even case

Let $\mu = (\mu_1, \mu_2, \ldots)$, $\mu_1 \geq \mu_2 \geq \cdots \geq 0$ be a Young diagram and $\mu' = (\mu'_1, \mu'_2, \ldots)$ be its transpose. We let d_μ denote the length of the main diagonal of μ. By a skew-Young diagram we mean a pair of Young diagrams $\lambda \subset \mu$. It is depicted by the region corresponding to the subtraction $\mu - \lambda$. See the Fig.4.1 for example.

Fig.4.1

For definiteness, we assume that $\lambda'_{\mu_1} = \lambda_{\mu'_1} = 0$. A Young diagram μ is naturally identified with a skew-Young diagram $\phi \subset \mu$. By an *admissible* tableau b on a skew-Young diagram $\lambda \subset \mu$ we mean an assignment of an element $b(i,j) \in J$ to the (i,j)-th box in $\lambda \subset \mu$ under the following rule: (We locate $(1,1)$ at the top left corner of μ, $(i+1,j)$ and $(i,j+1)$ to the below and the right of (i,j), respectively.)

$$\begin{aligned} &b(i,j) \preceq b(i,j+1), \qquad b(i,j) \prec b(i+1,j) \quad \text{with the exception that} \\ &b(i,j) = b(i,j+1) = 0 \ \text{ is forbidden}, \quad b(i,j) = b(i+1,j) = 0 \ \text{ is allowed.} \end{aligned} \tag{4.1}$$

Without the exception this coincides with the usual definition of the semi-standard Young tableau. Denote by $Atab(\lambda \subset \mu)$ the set of admissible tableaux on $\lambda \subset \mu$.

Given a skew-Young diagram $\lambda \subset \mu$, we define the function $T_{\lambda\subset\mu}(u)$ as the following sum over the admissible tableaux.

$$T_{\lambda\subset\mu}(u) = \sum_{b \in Atab(\lambda\subset\mu)} \prod_{(i,j)\in(\lambda\subset\mu)} z(b(i,j); u + \mu'_1 - \mu_1 - 2i + 2j). \tag{4.2}$$

Comparing this with (3.5) we have

$$T^a(u) = T_{(1^a)}(u): \ \text{single column of length } a, \tag{4.3a}$$

$$T_m(u) = T_{(m)}(u): \ \text{single row of length } m. \tag{4.3b}$$

We also prepare a notation for the single hook,

$$T_{k,l} = T_{(l+1,1^k)}(u). \tag{4.3c}$$

Then we have

Theorem 4.1.

$$T_{\lambda\subset\mu}(u) = det \begin{pmatrix} 0 & \cdots & 0 & R_{11} & \cdots & R_{1d_\mu} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & R_{d_\lambda 1} & \cdots & R_{d_\lambda d_\mu} \\ C_{11} & \cdots & C_{1d_\lambda} & H_{11} & \cdots & H_{1d_\mu} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ C_{d_\mu 1} & \cdots & C_{d_\mu d_\lambda} & H_{d_\mu 1} & \cdots & H_{d_\mu d_\mu} \end{pmatrix}, \tag{4.4a}$$

where

$$\begin{aligned} R_{ij} &= T_{\mu_j - \lambda_i + i - j}(u + \mu'_1 - \mu_1 + \mu_j + \lambda_i - i - j + 1), \\ C_{ij} &= -T^{\mu'_i - \lambda'_j - i + j}(u + \mu'_1 - \mu_1 - \mu'_i - \lambda'_j + i + j - 1), \\ H_{ij} &= T_{\mu'_i - i, \mu_j - j}(u + \mu'_1 - \mu_1 - \mu'_i + \mu_j + i - j). \end{aligned} \tag{4.4b}$$

Two particular cases corresponding to the formal choices $\mu_i = \lambda_i$ or $\mu'_i = \lambda'_i$ for $1 \le i \le d_\lambda = d_\mu$ yield simpler formulae. In these cases, redefining μ_i, μ'_i, λ_i and λ'_i so that $\lambda'_{\mu_1} = \lambda_{\mu'_1} = 0$, we have

$$
\begin{aligned}
T_{\lambda\subset\mu}(u) &= det_{1\le i,j\le\mu_1}(T^{\mu'_i-\lambda'_j-i+j}(u+\mu'_1-\mu_1-\mu'_i-\lambda'_j+i+j-1)), \quad (4.5a)\\
&= det_{1\le i,j\le\mu'_1}(T_{\mu_j-\lambda_i+i-j}(u+\mu'_1-\mu_1+\mu_j+\lambda_i-i-j+1)). \quad (4.5b)
\end{aligned}
$$

Eq.(4.5a) can be verified, for example, by induction on μ_1, i.e., by showing the same recursive relation for the tableau sum (4.2) as an expansion of the determinant. Then (4.5b) can be shown by using (3.7). Theorem 4.1 is proved from these results by applying Sylvester's theorem on determinants. ¿From (4.5a) and Theorem 3.1 one has

Corollary. *$T_{\lambda\subset\mu}(u)$ is pole-free provided the BAE (2.1) (LHS set to -1) holds.*

The admissibility condition (4.1) leads to the above conclusion although it is by no means obvious in the defining expression (4.2). Despite the exception in (4.1), our formulae (4.4) and (4.5) formally coincide with the classical ones due to Giambelli and Jacobi-Trudi on Schur functions [M] if one drops the u-dependence (or in the limit $|u| \to \infty$). If $\mu'_{i+1} - \lambda'_i > 2r$ for some i, $Atab(\lambda \subset \mu) = \phi$. Correspondingly, one can show that the determinant (4.5a) is vanishing using the fact that $T^a(u)$ factorizes for $a \ge 2r$ due to Theorem 3.3. Henceforth we assume that $\mu'_{i+1} - \lambda'_i \le 2r$ for $1 \le i \le \mu_1$. (We set $\mu'_{\mu_1+1} = -\infty$.)

The $T_{\lambda\subset\mu}(u)$ (4.2) describes the spectrum of the transfer matrix whose auxiliary space is labeled by the skew-Young diagram $\lambda \subset \mu$ and u. Denote the space by $W_{\lambda\subset\mu}(u)$. We suppose it is an irreducible finite dimensional module over $Y(B_r)$ (or $U_q(B_r^{(1)})$ in the trigonometric case) in view that all the terms in (4.2) seem coupling to make the apparent poles spurious under BAE. Now we shall specify the Drinfel'd polynomial $P_a(\zeta)$ [D] that characterizes $W_{\lambda\subset\mu}(u)$ based on some empirical procedure. Our convention slightly differs from the original one in Theorem 2 of [D] in such a way that

$$1 + \sum_{k=0}^{\infty} d_{ik}\zeta^{-k-1} = \frac{P_i(\zeta + \frac{1}{t_i})}{P_i(\zeta - \frac{1}{t_i})}. \tag{4.6}$$

For any $b \in Atab(\lambda \subset \mu)$, the corresponding summand (4.2) has the form

$$\prod_{a=1}^{r} \frac{Q_a(u+x^a_1)\cdots Q_a(u+x^a_{i_a})}{Q_a(u+y^a_1)\cdots Q_a(u+y^a_{i_a})}, \tag{4.7}$$

where x^a_j, y^a_j and i_a are specified from b. This summand carries the B_r-weight

$$wt(b) = \sum_{a=1}^{r}\Big(\frac{t_a}{2}\sum_{j=1}^{i_a}(y^a_j - x^a_j)\Big)\Lambda_a \tag{4.8}$$

in the sense that $\lim_{q^u\to\infty}(4.7) = q^{-2(wt(b)|\sum_{a=1}^r N_a\alpha_a)}$. ($\Lambda_a$: a-th fundamental weight.) ¿From $Atab(\lambda \subset \mu)$, take such b_0 that $wt(b_0)$ is highest with respect to the root system. In our case, such b_0 is unique and given as follows. Fill the left most column of $\lambda \subset \mu$ from the top to the bottom by assigning the first $\mu'_1 - \lambda'_1$ letters from the sequence $1, 2, \ldots, r, 0, 0, \ldots$. Given the $(i-1)$-th column, the i-th column is built from the top to the bottom by taking the first $\mu'_i - \lambda'_i$ letters from the sequence $1, 2, \ldots, r, \overbrace{0, \ldots, 0}^{k}, \overline{r}, \overline{r-1}, \ldots, \overline{1}$, where $k = \max(0, \min(\lambda'_{i-1} - \lambda'_i, \mu'_i - \lambda'_i - r))$. (We set $\lambda'_0 = +\infty$.) See the example in Fig.4.2.

Fig.4.2.

It turns out that (4.7) for the 'highest' term b_0 can be expressed uniquely in the form

$$\prod_{a=1}^{r} \prod_{j=1}^{M_a} \frac{Q_a(u + z_j^a - \frac{1}{t_a})}{Q_a(u + z_j^a + \frac{1}{t_a})} \tag{4.9}$$

for some M_a and $\{z_j^a | 1 \le j \le M_a\}$ up to the permutations of z_j^a's for each a. We then propose that the Drinfel'd polynomial $P_a^{W_{\lambda\subset\mu}(u)}(\zeta)$ for $W_{\lambda\subset\mu}(u)$ is given by

$$P_a^{W_{\lambda\subset\mu}(u)}(\zeta) = \prod_{j=1}^{M_a} (\zeta - u - z_j^a) \quad 1 \le a \le r. \tag{4.10}$$

In our case, it reads explicitly as follows.

$$\begin{aligned} P_a^{W_{\lambda\subset\mu}(u)}(\zeta) = & \prod_{\substack{1 \le i \le \mu_1 \\ \mu'_i - \lambda'_i = a}} (\zeta - u - \mu'_1 + \mu_1 + 1 + a + 2\lambda'_i - 2i) \\ \times & \prod_{\substack{1 \le i \le \mu_1 - 1 \\ \mu'_{i+1} - \lambda'_i = 2r - a}} (\zeta - u - \mu'_1 + \mu_1 + 2 + a + 2\lambda'_i - 2i) \quad 1 \le a \le r - 1, \end{aligned} \tag{4.11a}$$

$$\begin{aligned} P_r^{W_{\lambda\subset\mu}(u)}(\zeta) = & \prod_{\substack{1 \le i \le \mu_1 \\ \lambda'_i + r \le \mu'_i \le \lambda'_{i-1} + r}} (\zeta - u - \mu'_1 + \mu_1 + 2\mu'_i - 2i - r + \frac{3}{2}) \\ \times & \prod_{\substack{1 \le i \le \mu_1 \\ \mu'_{i+1} \le \lambda'_i + r \le \mu'_i}} (\zeta - u - \mu'_1 + \mu_1 + 2\lambda'_i - 2i + r + \frac{1}{2}), \end{aligned} \tag{4.11b}$$

where we have set

$$\mu'_{\mu_1+1} = -\infty, \quad \lambda'_0 = \infty. \tag{4.12}$$

We will call the irreducible finite dimensional $Y(B_r)$ module spin-even (resp. spin-odd) if and only if the characterizing Drinfel'd polynomial $P_r(\zeta)$ is even (resp. odd) degree. The one in (4.11b) is even for any skew-Young diagram $\lambda \subset \mu$. For example, in the case of the single column or row (4.3), (4.11) reads

$$P_a^{W_{(1^c)}(u)}(\zeta) = \begin{cases} (\zeta - u)^{\delta_{ac}} & 1 \le c < r \\ ((\zeta - u + c - r + \frac{1}{2})(\zeta - u - c + r - \frac{1}{2}))^{\delta_{ar}} & c \ge r \end{cases}, \tag{4.13a}$$

$$P_a^{W_{(m)}(u)}(\zeta) = ((\zeta - u + m - 1)(\zeta - u + m - 3) \cdots (\zeta - u - m + 1))^{\delta_{a1}}. \tag{4.13b}$$

As a B_r module, the $Y(B_r)$ module $W_{\lambda\subset\mu}(u)$ decomposes as

$$W_{\lambda\subset\mu}(u) \simeq \sum_{\eta}\Big(\sum_{\kappa,\nu} LR^{\mu}_{\lambda\nu} LR^{\nu}_{(2\kappa)'\eta}\Big)\pi_{O(2r+1)}(V_\eta), \tag{4.14}$$

which is u-independent. Here $LR^{\mu}_{\lambda\nu}$ etc denote the Littlewood-Richardson coefficients for the universal character ring Λ of GL type introduced in [KT]. The sums run over all the Young diagrams η, ν and $\kappa = (\kappa_1, \kappa_2, \ldots)$, where $(2\kappa)'$ stands for the transpose of $2\kappa = (2\kappa_1, 2\kappa_2, \ldots)$. $\pi_{O(2r+1)}(V_\eta)$ is the image of the specialization homomorphism [KT]. It is equal to (± 1 or 0) "times" the irreducible B_r module V_{η^*} with the highest weight labeled by the Young diagram η^* with $(\eta^*)'_1 \le r$. They are determined according to the equality $\pi_{O(2r+1)}(\chi(\eta)) = (\pm 1 \text{ or } 0) \times \chi(\eta^*)$ at the character level [KT].

5. Spin-odd case

Consider the following subset $Spin \subset Atab((1^r))$.

$$\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_r \\ \hline \end{array} \in Spin \Leftrightarrow \begin{cases} i_1 \prec \cdots \prec i_r \in J, \\ 0 \text{ is not contained}, \\ \text{only one of } i \text{ and } \bar{i} \text{ is contained for any } 1 \le i \le r. \end{cases} \tag{5.1}$$

There is a bijection $\iota : Spin \to \{(\xi_1, \ldots, \xi_r) \mid \xi_j = \pm\}$ sending (5.1) with $1 \preceq i_1 \prec \cdots \prec i_k \preceq r \prec \bar{r} \preceq i_{k+1} \prec \cdots \prec i_r \preceq \bar{1}$ to such $(\xi_1, \ldots, \xi_r)$ that $\xi_{i_1} = \cdots = \xi_{i_k} = +$, $\xi_{\overline{i_{k+1}}} = \cdots = \xi_{\overline{i_r}} = -$, where we interpret $\bar{k} = i$ if $k = \bar{i}$. Thus the latter of (3.8) can also be written as $T^{(r)}_1(u) = \sum_{b\in Spin} sp(\iota(b); u)$.

For a skew-Young diagram $\lambda \subset \mu$ with $\mu'_1 - \lambda'_1 \ge r$, hatch the bottom r boxes in the leftmost column, which we call an L-hatched skew-Young diagram $\lambda \subset \mu$. See Fig.5.1.

Fig.5.1.

Consider a tableau b on it, namely, a map b : L-hatched $\lambda \subset \mu \to J$. We call a tableau b on an L-hatched $\lambda \subset \mu$ *L-admissible* if and only if all of the following three conditions are valid. ($n = \mu'_2 - (\mu'_1 - r)$ and see Fig. 5.2 for the definitions of i_l and j_l.)

$$\begin{aligned} &\text{(i) hatched part } \in Spin, \text{ and (4.1) for non-hatched part,} \\ &\text{(ii) } j_0 \prec i_1, \\ &\text{(iii) } i_1 \preceq j_1, \ldots, i_n \preceq j_n \text{ or there exists } k \in \{1, \ldots, n\} \text{ such that} \\ &\qquad i_1 \preceq j_1, \ldots, i_{k-1} \preceq j_{k-1} \text{ and } \bar{r} \preceq j_k \prec i_k \preceq \bar{1}. \end{aligned} \tag{5.2}$$

Here (ii) is void when $\mu'_1 - \lambda'_1 = r$ and so is (iii) for $n = 0$.

Fig.5.2.

Denote by $Atab_L(\lambda \subset \mu)$ the set of L-admissible tableaux on the L-hatched $\lambda \subset \mu$. We note that $Atab(\lambda \subset \mu) \not\subseteq Atab_L(\lambda \subset \mu)$ nor $Atab(\lambda \subset \mu) \not\supseteq Atab_L(\lambda \subset \mu)$. Given an L-hatched skew-Young diagram $\lambda \subset \mu$, we define the function $S^L_{\lambda\subset\mu}(u)$ by

$$\begin{aligned} S^L_{\lambda\subset\mu}(u) = &\sum_{b\in Atab_L(\lambda\subset\mu)} sp(\iota(\text{hatched part}); u) \\ &\times \prod_{(i,j)\in \text{ non hatched part of } (\lambda\subset\mu)} z(b(i,j); u + 2\mu'_1 - r - 2i + 2j - \frac{3}{2}). \end{aligned} \tag{5.3}$$

We have an L $\leftrightarrow$ R (left vs. right) dual of these definitions as follows. For a skew-Young diagram $\lambda \subset \mu$ with $\mu'_{\mu_1} \geq r$ (remember we assumed $\lambda'_{\mu_1} = 0$), hatch the top r boxes in the rightmost column, which we call an R-hatched skew-Young diagram $\lambda \subset \mu$. See Fig.5.3.

Fig.5.3.

Consider a tableau b : R-hatched $\lambda \subset \mu \to J$. We call a tableau b on an R-hatched $\lambda \subset \mu$ *R-admissible* if and only if all of the following three conditions are valid. ($n = r - \lambda'_{\mu_1 - 1}$ and see Fig. 5.4 for the definitions of i_l and j_l.)

$$\begin{aligned}&\text{(i) hatched part } \in Spin, \text{ and (4.1) for non-hatched part,}\\ &\text{(ii) } i_1 \prec j_0,\\ &\text{(iii) } j_1 \preceq i_1, \ldots, j_n \preceq i_n \text{ or there exists } k \in \{1, \ldots, n\} \text{ such that}\\ &\qquad j_1 \preceq i_1, \ldots, j_{k-1} \preceq i_{k-1} \text{ and } 1 \preceq i_k \prec j_k \preceq r,\end{aligned} \tag{5.4}$$

where (ii) is void when $\mu'_{\mu_1} = r$ and so is (iii) for $n = 0$.

Fig.5.4.

Denoting by $Atab_R(\lambda \subset \mu)$ the set of R-admissible tableaux on the R-hatched $\lambda \subset \mu$, we define

$$\begin{aligned}S^R_{\lambda\subset\mu}(u) = &\sum_{b \in Atab_R(\lambda\subset\mu)} sp(\iota(\text{hatched part}); u)\\ &\times \prod_{(i,j)\in \text{ non hatched part of } (\lambda\subset\mu)} z(b(i,j); u - 2\mu_1 + r - 2i + 2j + \frac{3}{2}).\end{aligned} \tag{5.5}$$

The first main results in this section is

Theorem 5.1.

$$S^L_{\lambda\subset\mu}(u) = det_{1\leq i,j\leq \mu_1}(S^L_{ij}) \tag{5.6a}$$

$$= det_{1\leq i,j\leq \mu'_2}(\overline{S}^L_{ij}), \tag{5.6b}$$

where

$$S^L_{ij} = \begin{cases} T^{\mu'_j - \lambda'_i + i - j}(u + 2\mu'_1 - \mu'_j - \lambda'_i + i + j - r - \frac{5}{2}) & j \geq 2\\ T^{(r)}_1(u + 2i - 2 + 2(\mu'_1 - \lambda'_i - r)) & j = 1\end{cases}, \tag{5.7a}$$

$$\overline{S}^L_{ij} = \begin{cases} T_{\mu_i - \lambda_j - i + j}(u + 2\mu'_1 + \mu_i + \lambda_j - i - j - r - \frac{1}{2}) & 1 \leq j \leq \lambda'_1\\ \mathcal{H}^L_{\mu_i + \lambda'_1 - i}(u + 2\mu'_1 - 2\lambda'_1 - 2r) & j = \lambda'_1 + 1\\ T_{\mu_i - i + j - 1}(u + 2\mu'_1 + \mu_i - i - j - r + \frac{1}{2}) & j > \lambda'_1 + 1\end{cases}, \tag{5.7b}$$

$$\mathcal{H}^L_m(u) = \sum_{l=0}^{m} (-1)^l T^{(r)}_1(u + 2l) T_{m-l}(u + m + r + l - \frac{1}{2}). \tag{5.7c}$$

¿From (5.6a), (5.7a,c) and Theorem 4.1, $\mathcal{H}^L_m(u)$ is equal to the L-hatched hook $S^L_{(m+1,1^{r-1})}(u)$.

For an R-hatched diagram $\lambda \subset \mu$, let $\xi \subset \eta$ be the sub-diagram obtained by removing the rightmost column of $\lambda \subset \mu$. See Fig. 5.5.

Fig. 5.5.

Thus for example $\eta_i = \mu_1 - 1$ for $1 \leq i \leq \mu'_{\mu_1} - \lambda'_{\mu_1 - 1}$. Then another main result in this section is the R-hatched version of the previous theorem as follows.

Theorem 5.2.

$$S^R_{\lambda\subset\mu}(u) = det_{1\leq i,j\leq \mu_1}(\mathcal{S}^R_{ij}) \tag{5.8a}$$

$$= det_{1\leq i,j\leq \eta'_1}(\overline{\mathcal{S}}^R_{ij}), \tag{5.8b}$$

where

$$\mathcal{S}^R_{ij} = \begin{cases} T^{\mu'_j - \lambda'_i + i - j}(u - 2\mu_1 - \mu'_i - \lambda'_j + i + j + r + \frac{1}{2}) \ j \leq \mu_1 - 1 \\ T^{(r)}_1(u - 2\mu_1 - 2\mu'_i + 2i + 2r) \ j = \mu_1 \end{cases}, \tag{5.9a}$$

$$\overline{\mathcal{S}}^R_{ij} = \begin{cases} T_{\eta_i - \xi_j - i + j}(u - 2\lambda'_{\mu_1 - 1} - 2\eta_1 + \eta_i + \xi_j - i - j + r + \frac{1}{2}) i \neq \eta'_1 - \mu'_1 + \mu'_{\mu_1} \\ \mathcal{H}^R_{\eta_i - \xi_j - i + j}(u - 2\mu'_{\mu_1} + 2r) \ i = \eta'_1 - \mu'_1 + \mu'_{\mu_1} \end{cases} \tag{5.9b}$$

$$\mathcal{H}^R_m(u) = \sum_{l=0}^{m} (-1)^l T^{(r)}_1(u - 2l) T_{m-l}(u - m - r - l + \frac{1}{2}). \tag{5.9c}$$

¿From (5.8a), (5.9a,c) and Theorem 4.1, one sees that $\mathcal{H}^R_m(u)$ is equal to the R-hatched "dual hook" $S^R_{(m^{r-1})\subset((m+1)^r)}(u)$. ¿From Theorems 3.1, 5.1 and 5.2, we have

Corollary. *$S^L_{\lambda\subset\mu}(u)$ and $S^R_{\lambda\subset\mu}(u)$ are pole-free provided that the BAE (2.1) (LHS set to -1) holds.*

Following a similar argument to the previous section, we propose the Drinfel'd polynomials corresponding to the auxiliary spaces $W^L_{\lambda\subset\mu}(u)$ and $W^R_{\lambda\subset\mu}(u)$ of $S^L_{\lambda\subset\mu}(u)$ and $S^R_{\lambda\subset\mu}(u)$, respectively.

$$P_a^{W^L_{\lambda\subset\mu}(u)}(\zeta) = P_a^{W_{\lambda\subset\mu}(u + \mu'_1 + \mu_1 - r - \frac{3}{2})}(\zeta) \quad 1 \leq a \leq r - 1, \tag{5.10a}$$

$$\begin{aligned} P_r^{W^L_{\lambda\subset\mu}(u)}(\zeta) &= \frac{1}{\zeta - u + 1} P_r^{W_{\lambda\subset\mu}(u + \mu'_1 + \mu_1 - r - \frac{3}{2})}(\zeta) \\ &= \prod_{\substack{2\leq i\leq \mu_1 \\ \lambda'_i + r \leq \mu'_i \leq \lambda'_{i-1} + r}} (\zeta - u + 3 + 2(\mu'_i - i - \mu'_1)) \\ &\times \prod_{\substack{1\leq i\leq \mu_1 \\ \mu'_{i+1} \leq \lambda'_i + r \leq \mu'_i}} (\zeta - u + 2 + 2(\lambda'_i - i - \mu'_1 + r)), \end{aligned} \tag{5.10b}$$

$$P_a^{W^R_{\lambda\subset\mu}(u)}(\zeta) = P_a^{W_{\lambda\subset\mu}(u - \mu'_1 - \mu_1 + r + \frac{3}{2})}(\zeta) \quad 1 \leq a \leq r - 1, \tag{5.11a}$$

$$
\begin{aligned}
P_r^{W^R_{\lambda\subset\mu}(u)}(\zeta) &= \frac{1}{\zeta-u-1} P_r^{W_{\lambda\subset\mu}(u-\mu_1'-\mu_1+r+\frac{3}{2})}(\zeta) \\
&= \prod_{\substack{1\le i\le\mu_1 \\ \lambda_i'+r\le\mu_i'\le\lambda_{i-1}'+r}} (\zeta-u+2(\mu_i'-i+\mu_1-r)) \\
&\times \prod_{\substack{1\le i\le\mu_1-1 \\ \mu_{i+1}'\le\lambda_i'+r\le\mu_i'}} (\zeta-u-1+2(\lambda_i'-i+\mu_1)),
\end{aligned}
\tag{5.11b}
$$

where we assume (4.12).

In $Atab_L(\lambda\subset\mu)$ and $Atab_R(\lambda\subset\mu)$, we have considered the hatched part (*Spin* (5.1)) only in the bottom left or top right position. A natural question may be whether it is possible to define a tableau sum that becomes pole-free and contains *Spin* simultaneously in various places in a skew-Young diagram $\lambda\subset\mu$. It is indeed possible to include *Spin* both at the bottom left and the top right. However, we have found only few examples beyond that so far.

6. Solution to the T-system

The functions $T_{\lambda\subset\mu}(u)$ (4.2), $S^L_{\lambda\subset\mu}(u)$ (5.3) and $S^R_{\lambda\subset\mu}(u)$ (5.5) provide the solution to the T-system for B_r (2.2). For $m\in\mathbf{Z}_{\ge 0}$, put

$$
T_m^{(a)}(u) = T_{(m^a)}(u) \quad 1\le a\le r-1, \tag{6.1a}
$$

$$
T_{2m}^{(r)}(u) = T_{(m^r)}(u), \tag{6.1b}
$$

$$
T_{2m+1}^{(r)}(u) = S^L_{((m+1)^r)}(u-m) = S^R_{((m+1)^r)}(u+m). \tag{6.1c}
$$

The latter equality in (6.1c) can be shown easily by using (3.12), (5.7a) and (5.9a). The definition (6.1) includes (3.8). Moreover, from (4.11) and (5.10,11), the Drinfel'd polynomial corresponding to $T_m^{(a)}(u)$ is given by $P_b(\zeta) = \left(\prod_{i=1}^m(\zeta-u+\frac{m+1-2i}{t_a})\right)^{\delta_{ba}}$ for $1\le b\le r$, in agreement with the one said after (2.2).

Theorem 6.1. *$T_m^{(a)}(u)$ defined above satisfies the T-system (2.2).*

Outline of the proof. We use the determinantal expressions (4.5a) and (5.6a). Then the first two equations in (2.2) reduce to the Jacobi identity among the determinants. To prove the third equation, substitute (5.6a) into $T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u)$. Expanding the determinants with respect to the first column, we have

$$
\begin{aligned}
T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u) &= \sum_{i=0}^{m-1}\sum_{j=0}^{m}(-1)^{i+j}R_j^{(m)}R_i^{(m-1)} \\
&\times \left(T^{r+j-i-1}(u-m+i+j+\frac{1}{2}) + T^{r+i-j}(u-m+i+j+\frac{1}{2})\right).
\end{aligned}
$$

Here, $R_j^{(m)}$ denotes the cofactor of $T_1^{(r)}(u-m+2j)$ in $T_{2m+1}^{(r)}(u)$ and we have used (3.12). On taking the j-sum, the $T^{r+j-i-1}(u-m+i+j+\frac{1}{2})$ term vanishes. After taking the

i-sum, the $T^{r+i-j}(u-m+i+j+\frac{1}{2})$ term is non-zero only for $j=0$ or $j=m$. Noting that $R_0^{(m)}=T_{2m}^{(r)}(u+\frac{1}{2})$ and $R_m^{(m)}=T_m^{(r-1)}(u-\frac{1}{2})$, one has the third equation. The last equation in (2.2) can be verified quite similarly. We remark that Theorem 6.1 is valid for $T_m^{(a)}(u)$ defined through any $Q_b(u)$'s, not necessarily solutions to the BAE (2.1).

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Figure Captions.

Figure 4.1: An example of a skew-Young diagram $\lambda \subset \mu$. Here $\mu = (5, 4^2, 1), \lambda = (2, 1), \mu' = (4, 3^3, 1)$ and $\lambda' = (2, 1)$, respectively. The lengths of the main diagonal are given by $d_\mu = 3$ and $d_\lambda = 1$.

Figure 4.2: The way to assign the letters to each box is explained in the text. This is an example for $r = 3, \mu' = (9, 7, 2), \lambda' = (3, 1)$. Notice that zeros are arranged lest they are adjacent horizontally.

Figure 5.1: An example of an L-hatched skew-Young diagram $r = 4, \mu = (4^3, 3, 2, 1^2), \lambda = (3, 1)$.

Figure 5.2: The bottom left part of an L-hatched skew-Young tableau and the assignment of the letters $\{i_l\}$ and $\{j_l\}$ in (5.2).

Figure 5.3: An example of an R-hatched skew-Young diagram $r = 4, \mu = (4^5, 3, 1), \lambda = (3^2, 2, 1)$.

Figure 5.4: The top right part of an R-hatched skew-Young tableau and the assignment of the letters $\{i_l\}$ and $\{j_l\}$ in (5.4).

Figure 5.5: An R-hatched skew-Young diagram for $r = 3$ with $\mu = (5^4, 3, 2, 1), \lambda = (4, 3, 1^2)$. Broken lines are guides to eyes for defining $\eta = (4^3, 3, 2, 1), \xi = (3, 1^2)$.

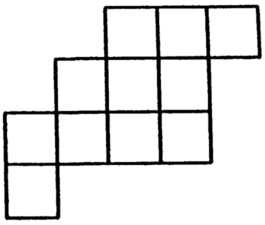

Fig.4.1

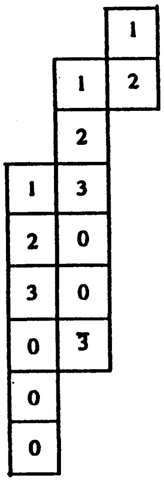

Fig.4.2

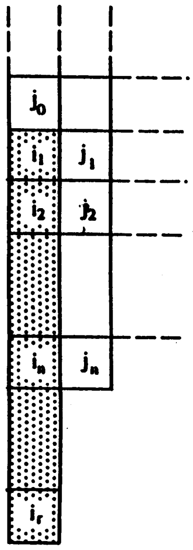

Fig.5.2

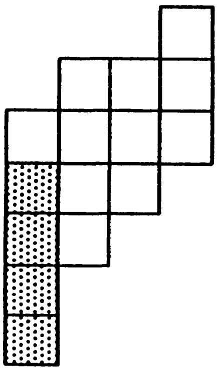

Fig.5.1

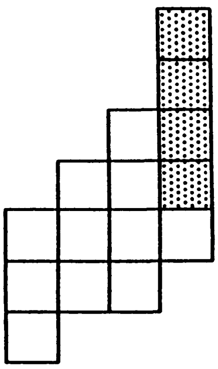

Fig.5.3

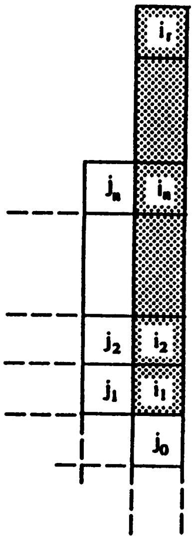

Fig.5.4

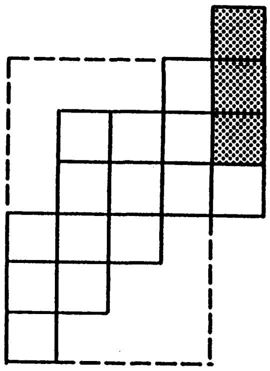

Fig.5.5