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About Commutant Algebra of Cartan-Type Lie Superalgebra $W(n)$

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Introduction.

Undoubtedly, Weyl's classical reciprocity law is very important in Lie theory([7]). Let us review the main points briefly. Let $GL(V)$ be the general linear group on a vecter space V and consider the m-fold tensor product $V^{\otimes m}$

$$V^{\otimes m} = V \otimes V \otimes \cdots \otimes V$$

as a $GL(V)$-module. Observe that the permutation group $\mathfrak{S}_m$ acts on $V^{\otimes m}$ by permuting the factors in a tensor:

$$s(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{s^{-1}(1)} \otimes v_{s^{-1}(2)} \otimes \cdots \otimes v_{s^{-1}(m)} \qquad (s \in \mathfrak{S}_m, \ \ v_i \in V).$$

SCHUR THEOREM [7, p.130]. *The algebras in* End$(V^{\otimes m})$ *generated by* $GL(V)$ *and by* $\mathfrak{S}_m$ *are mutual commutants: each is the full subalgebra of operators commuting with the other.*

Combining Schur's Theorem with the Double Commutant Theorem [7, Chapter 3], we conclude Weyl's reciprocity law.

Theorem 0.1. *As a* $GL(V) \times \mathfrak{S}_m$*-module, the space* $V^{\otimes m}$ *decomposes as follows:*

$$V^{\otimes m} = \sum_D \sigma_D \otimes J_D,$$

where σ_D *is an irreducible representation of* $GL(V)$ *and* J_D *is an irreducible representation of* $\mathfrak{S}_m$*. The decomposition is such that* σ_D *determines* J_D *and vice versa.*

Using the above results, Weyl obtained all the irreducible polynomial representations of $GL(V)$ and studied invariant theory of the natrual representation of $GL(V)$. Weyl's work is a classical work. For a new algebra, we want to follow Weyl's idea to get the classification of irreducible representation of the algebra. This paper is a kind of attempt in this direction.

In the present paper, we want to consider a Cartan-type Lie *superalgebra* $W(n)$. By definition $W(n)$ is a Lie superalgebra of all the superderivations on a Grassmann algebra $\Lambda(n)$

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of n-variables (see [1]), so $W(n)$ acts on $\Lambda(n)$ naturally. We call it the naturual representation of $W(n)$ and denote it by ψ. To study an analogue of Weyl's reciprocity, the first important thing is to calculate the commutant algebra of the natural representation of $W(n)$ in the m-fold tensor product $\otimes^m\Lambda(n)$ of Grassmann algebra $\Lambda(n)$. Let End $[m]$ be the set of all the maps from $[m]$ to $[m]$, where $[m] = \{1, 2, \cdots, m\}$, and denote the semigroup ring of End $[m]$ by $\mathfrak{E}_m$. There is a natural action of $\mathfrak{E}_m$ on $\otimes^m\Lambda(n)$ and denote the image algebra of this representation by $\mathcal{E}_m$. Also we denote the commutant algebra of $\psi^{\otimes m}(W(n))$ in $\mathrm{End}\otimes^m\Lambda(n)$ by $\mathcal{C}_m$. One of the main results is the identification of the commutant algebra $\mathcal{C}_m$ and the image $\mathcal{E}_m$ of the semigroup ring for the case (1) $m \leq n$ and for the case (2) $n = 1$ and arbitrary m, and finally for the case (3) $n = 2, m = 3$.

Another main result is that the bicommutant algebra of $W(1)$ coincides with the image of its universal enveloping algebra $U(W(1))$. We consider it an analogy of Weyl's reciprocity for $W(1) \times \mathrm{End}[m]$. However, we did not get any decomposition of the (non-semisimple) representation of $W(1) \times \mathrm{End}[m]$ on $\otimes^m\Lambda(1) \simeq \Lambda(m)$ yet, and it is a future subject of ours.

We divide this article into five sections. In Section 1, the basic definitions are introduced. In Section 2, we give some lemmas and in Section 3 the commutant algebra $\mathcal{C}_m$ is calculated for the case $m \leq n$. In Section 4, we get the commutant algebra and the bicommutant algebra for the case $n = 1$ and arbitrary m. In Section 5 we obtain the commutant algebra for the case $n = 2, m = 3$.

1. Lie superalgebra $W(n)$ and its natural representation.

Let $\Lambda(n)$ be the Grassmann algebra over $\mathbb{C}$ in n-variables $\{\xi_1, \xi_2, \cdots, \xi_n\}$. If we set $\deg \xi_i = 1$, then $\Lambda(n)$ becomes a $\mathbb{Z}$ -graded algebra. Let Λ_k be the space of k-homogeneous elements of $\Lambda(n)$:

$$\Lambda_k = \langle \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_k} | 1 \leq i_1 < \cdots < i_k \leq n \rangle,$$

where $\langle B \rangle$ denotes the vector space over $\mathbb{C}$ spanned by B. Then $\Lambda(n) = \bigoplus_{k=0}^n \Lambda_k$, a direct sum, where $\Lambda_0 = \mathbb{C}$. We can naturally consider $\Lambda(n)$ as a superalgebra, where even elements are $\Lambda_{\bar{0}} = \bigoplus_{k:\text{even}} \Lambda_k$, and odd elements are $\Lambda_{\bar{1}} = \bigoplus_{k:\text{odd}} \Lambda_k$.

Let A be a superalgebra and $\mathrm{End}_s A$ the set of linear endomorphisms on A of degree s, i.e., $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and $\mathrm{End}_s A = \{X \in \mathrm{End} A \mid X(A_i) \subseteq A_{i+s}, (i \in \mathbb{Z}_2)\}$. A superderivation of degree s ($s \in \mathbb{Z}_2$) of A is an endomorphism $D \in \mathrm{End}_s A$ with the property

$$D(ab) = D(a)b + (-1)^{s \deg a} a D(b)$$

for any homogeneous $a, b \in A$. The space of all the superderivations of degree s is denoted by $\mathrm{Der}_s A$.

Let $W(n)$ be the set of all superderivations over $\Lambda(n)$, then it becomes naturally a Lie superalgebra. According to the results in [1], every derivation $D \in W(n)$ can be written in the form

$$D = \sum_{i=1}^n P_i \frac{\partial}{\partial \xi_i} \quad (P_i \in \Lambda(n)$$

where $\dfrac{\partial}{\partial \xi_i}$ is a superderivation of degree 1 defined by

$$\frac{\partial}{\partial \xi_i}\xi_j = \delta_{ij}.$$

By definition, the Lie superalgebra $W(n)$ acts on Grassmann superalgebra $\Lambda(n)$ as follows: for $\forall D \in W(n)$ and $\forall \xi_{i_1} \wedge \cdots \wedge \xi_{i_r} \in \Lambda(n)$

$$D(\xi_{i_1} \wedge \cdots \wedge \xi_{i_r}) = \sum_{s=1}^{r}(-1)^{(s-1)\deg D}\xi_{i_1} \wedge \cdots \wedge D(\xi_{i_s}) \wedge \cdots \wedge \xi_{i_r}.$$

We call it the natural representation of $W(n)$, and denote it by ψ.

Let us consider the m-fold tensor product $\otimes^m\Lambda(n)$. Then we have a natural isomorphism as $W(n)$-modules

$$\otimes^m\Lambda(n) \cong \Lambda[\xi_{ij} | 1 \leq i \leq n, 1 \leq j \leq m] =: \Lambda(n,m),$$

where $\Lambda[\xi_{ij}|1 \leq i \leq n, 1 \leq j \leq m]$ is a Grassmann algebra generated by ξ_{ij} $(1 \leq i \leq n, 1 \leq j \leq m)$. In the following, we identify $\otimes^m\Lambda(n)$ with $\Lambda(n,m)$. By means of a tensor product, $W(n)$ is imbedded into $\mathrm{End}(\otimes^m\Lambda(n)) \cong \mathrm{End}\Lambda(n,m)$. More precisely, an element

$$D = \sum_{i=1}^{n} P_i(\xi_1, \cdots, \xi_n)\frac{\partial}{\partial \xi_i} \in W(n)$$

corresponds to an element

$$\psi^{\otimes m}(D) = \sum_{i=1}^{n}\sum_{\alpha=1}^{m} P_i(\xi_{1\alpha}, \cdots, \xi_{n\alpha})\frac{\partial}{\partial \xi_{i\alpha}} \in \mathrm{Der}\Lambda(n,m)$$

via the m-fold tensor product $\psi^{\otimes m}$ of ψ.

By definition of commutativity in Lie superalgebra, we say that a and b are *commutative* if $[a,b] = 0$, whence $ab = (-1)^{\deg a \deg b}ba$ if a and b are homogeneous. Let $\mathcal{C}_m$ denote the commutant algebra of $\psi^{\otimes m}(W(n))$ in End $\Lambda(n,m)$:

$$\mathcal{C}_m = \left\{E \in \mathrm{End}(\otimes^m\Lambda(n)) \mid [E,D] = 0, \ \ \forall D \in \psi^{\otimes m}(W(n))\right\}.$$

Then $\mathcal{C}_m$ has a natural $\mathbb{Z}_2$-graded structure, $\mathcal{C}_m = \mathcal{C}_{m,\bar{0}} \oplus \mathcal{C}_{m,\bar{1}}$.

Denote by $[m]$ the set $\{1, 2, \cdots, m\}$ of integers, and put $\mathrm{End}[m] = \{\varphi : [m] \to [m]\}$ the set of all the maps from $[m]$ to itself. By composition of maps, End$[m]$ becomes a semigroup with unit, whose group elements form permutation group $\mathfrak{S}_m$ of degree m. Denote the semigroup ring of End$[m]$ by $\mathfrak{E}_m$. An element $\varphi \in \mathrm{End}[m]$ acts on $\Lambda(n,m)$ as $(\varphi P)(\xi_{i,j}) = P(\xi_{i,\varphi(j)})(P \in \Lambda(n,m))$ and we extend it to $\mathfrak{E}_m$ by linearity (see [2]), thus, we have a representation of $\mathfrak{E}_m$ on $\Lambda(n,m)$. Unforturnately this representation is not faithful in general (*cf* Lemma 3.2). So we denote the image algebra of this representation by $\mathcal{E}_m \subset \mathrm{End}\Lambda(n,m)$.

2. Some general lemmas.

Using above lemmas in the Section 1, we introduce some lemmas.

Lemma 2.1. *The odd subspace of $\mathcal{C}_m$ vanishes, that is $\mathcal{C}_{m,\mathrm{I}} = \{0\}$.*

Proof. Let $D_0 := \sum_{i=1}^{n} \sum_{j=1}^{m} \xi_{ij} \frac{\partial}{\partial \xi_{ij}}$, then $D_0 \in \psi^{\otimes m}(W(n))$. Because $D_0 : \Lambda(n,m)_k \longrightarrow \Lambda(n,m)_k$, and by $ED_0 = D_0 E$, we have $\deg E = 0$. This means $\mathcal{C}_{m,\mathrm{I}} = \{0\}$. Q.E.D.

By the above lemma, we have

$$\mathcal{C}_m = \{E \in \mathrm{End}\Lambda(n,m) \mid ED = DE, \quad \forall D \in \psi^{\otimes m}(W(n))\}.$$

Lemma 2.2. *For arbitrary n and m, we have*

$$\mathcal{E}_m \subseteq \mathcal{C}_m.$$

Proof. By the direct calaulations we get the lemma. Q.E.D.

Lemma 2.3. (1). *Let $D_i := \sum_{\alpha=1}^{m} \frac{\partial}{\partial \xi_{i\alpha}}$ $(1 \le i \le n)$, then $D_i \in \psi^{\otimes m}(W(n))$ and $(D_i)^2 = 0$.*

(2). *Let $D_{ks}^{(2)} := \sum_{\alpha=1}^{m} \xi_{k\alpha} \wedge \xi_{s\alpha} \frac{\partial}{\partial \xi_{s\alpha}}$ $(k \neq s)$, then $(D_{ks}^{(2)})^2 = 0$, and $[D_i, D_{ks}^{(2)}] = D_s^{(1)}$, if $i = k$. where $D_s^{(1)} := \sum_{\alpha=1}^{m} \xi_{s\alpha} \frac{\partial}{\partial \xi_{s\alpha}}$.*

Proof. (1) is a consequence of easy calculations.

(2). By calculation, we can get the first equality. For the second one, we have

$$\begin{aligned}
&[D_i, D_{ks}^{(2)}] \\
&= \sum_{\alpha=1}^{m} \frac{\partial}{\partial \xi_{i\alpha}} \Big(\sum_{\beta=1} \xi_{k\beta} \wedge \xi_{s\beta} \frac{\partial}{\partial \xi_{s\beta}} \Big) + \sum_{\beta=1}^{m} \xi_{k\beta} \wedge \xi_{s\beta} \frac{\partial}{\partial \xi_{s\beta}} \sum_{\alpha=1}^{m} \frac{\partial}{\partial \xi_{i\alpha}} \\
&= \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \frac{\partial}{\partial \xi_{i\alpha}} (\xi_{k\beta} \wedge \xi_{s\beta}) \frac{\partial}{\partial \xi_{s\beta}}.
\end{aligned}$$

Put $i = k$, then we get the result. Q.E.D.

Using the above lemma, we obtain the following lemmas.

Lemma 2.4. *D_i is an exact operator, i.e.,*

$$0 \longrightarrow \Lambda_{p_1 \cdots m \cdots p_n} \xrightarrow{D_i} \Lambda_{p_1 \cdots m-1 \cdots p_n} \xrightarrow{D_i} \cdots \xrightarrow{D_i} \Lambda_{p_1 \cdots 1 \cdots p_n} \longrightarrow 0$$

is an exact sequence, if $\exists p_h \neq 0$, where

$$\Lambda_{p_1 \cdots p_i \cdots p_n} = \langle (\xi_{1i_1} \wedge \cdots \wedge \xi_{1i_{p_1}}) \wedge \cdots \wedge (\xi_{nj_{p_1}} \wedge \cdots \wedge \xi_{nj_{p_n}}) \mid \forall i_1, \cdots\cdots, \forall j_1, \cdots\cdots \rangle.$$

Lemma 2.5. *If $E \in \mathcal{C}_m$, then $E(\Lambda_{p_1 \cdots p_n}) \subseteq \Lambda_{p_1 \cdots p_n}$. Furthermore, we have*

$$E(\Re(D_i)) \subseteq \mathcal{R}(D_i), \quad E(\Re(D_{ip}^{(2)})) \subseteq \mathcal{R}(D_{ip}^{(2)}),$$

where $\Re(X)$ denote the range of operator X.

3. Commutant algebra of $\psi^{\otimes m}(W(n))$ (the case $m \leq n$).

Let $(\psi, \Lambda(n))$ be the natural representation of $W(n)$ and $(\psi^{\otimes m}, \Lambda(n,m))$ its m-fold tensor product. Denote by $U(W(n))$ the universal enveloping algebra of $W(n)$, then we have

Lemma 3.1. *Put* $\xi(\alpha) = (\xi_{1\alpha}, \cdots, \xi_{n\alpha})$. *Then the subalgebra* $\psi^{\otimes m}(U(W(n)))$ *of* $\mathrm{End}\Lambda(n,m)$ *is generated by*

$$\sum_{1 \leq \alpha_1, \cdots, \alpha_k \leq m} P_1(\xi(\alpha_1)) \cdots P_k(\xi(\alpha_k)) \frac{\partial^k}{\partial \xi_{b_1 \alpha_1} \cdots \partial \xi_{b_k \alpha_k}},$$

where $1 \leq b_1, \cdots, b_k \leq n$ *are indices,* P_i *is a Grassmannian polynomial in* n*-variables, and* $P_i(\xi(\alpha)) = P_i(\xi_{1\alpha}, \cdots, \xi_{n\alpha})$.

Proof. Let $P_i \in \Lambda(n)$ be homogeneous and fix $1 \leq b_1, b_2 \leq n$. Then it holds that

$$\begin{aligned}
&\sum_{\alpha_1=1}^{m} P_1(\xi(\alpha_1)) \frac{\partial}{\partial \xi_{b_1,\alpha_1}} \sum_{\alpha_2=1}^{m} P_2(\xi(\alpha_2)) \frac{\partial}{\partial \xi_{b_2,\alpha_2}} \\
&= \sum_{\alpha_1,\alpha_2} \left\{ P_1(\xi(\alpha_1)) \frac{\partial}{\partial \xi_{b_1,\alpha_1}} P_2(\xi(\alpha_2)) \frac{\partial}{\partial \xi_{b_2,\alpha_2}} + (-1)^{\deg P_2} P_1(\xi(\alpha_1)) P_2(\xi(\alpha_2)) \frac{\partial^2}{\partial \xi_{b_1\alpha_1} \partial \xi_{b_2\alpha_2}} \right\}.
\end{aligned}$$

By definition of $\psi^{\otimes m}$, we know that the left hand side and the first term in the right hand side of the above formula are in $\psi^{\otimes m}(U(W(n)))$, whence

$$\sum_{\alpha_1,\alpha_2} P_1(\xi(\alpha_1)) P_2(\xi(\alpha_2)) \frac{\partial^2}{\partial \xi_{b_1\alpha_1} \partial \xi_{b_2\alpha_2}} \in \psi^{\otimes m}(U(W(n))).$$

We can complete the proof of the lemma by using induction on k. Q.E.D.

Lemma 3.2. *If* $m \leq n$, *then the representation of* $\mathfrak{E}_m$ *on* $\Lambda(n,m)$ *is faithful, hence we have*

$$\dim \mathcal{E}_m = \dim \mathfrak{E}_m = m^m.$$

Proof. Put

$$\mathcal{N}_k := \{ \varphi \mid \varphi \in \mathrm{End}[m], \ \sharp \mathrm{Im}(\varphi) = k \} = \{ \varphi_h^k \mid h = 1, 2, \cdots, n_k \},$$

where $n_k = \sharp \mathcal{N}_k$. Since the semigroup ring $\mathfrak{E}_m$ is generated by $\mathcal{N}_k$'s, so is $\mathcal{E}_m$.

Suppose

$$\sum_{k=1}^{m} \sum_{h=1}^{n_k} c_h^k \varphi_h^k \mid_{\Lambda(n,m)} = 0 \quad (c_h^k \in \mathbb{C}, \varphi_h^k \in \mathcal{N}_k).$$

We want to prove $c_h^k = 0$. By the assumption and the condition $m \leq n$, we have

$$\sum_{k=1}^{m} \sum_{h=1}^{n_k} c_h^k \varphi_h^k (\xi_{11} \wedge \xi_{12} \wedge \cdots \wedge \xi_{1m} \wedge \prod_{i=2}^{m} \xi_{ii}) = 0.$$

Note that φ_h^k $(1 \le k \le m-1)$ kills the vector in the left hand side. So we obtain $c_h^m = 0$ from the formula

$$\sum_h c_h^m \xi_{1\varphi_h^m(1)} \wedge \xi_{1\varphi_h^m(2)} \wedge \cdots \wedge \xi_{1\varphi_h^m(m)} \wedge \prod_{i=2}^m \xi_{i\varphi_h^m(i)} = 0.$$

We prove the result by induction on k. Assume that $c_h^t = 0$ $(t \ge k+1)$. Let us consider the case $t = k$. We have

$$\sum_t \sum_h c_h^t \varphi_h^t(\xi_{11} \wedge \xi_{1i_2} \wedge \cdots \wedge \xi_{1i_k} \wedge \prod_{i=2}^m \xi_{ii}) = 0,$$

where $1 < i_2 < i_3 < \cdots < i_k$. So by the induction hypothesis, we get

$$\begin{aligned}&\sum_h c_h^k \varphi_h^k(\xi_{11} \wedge \xi_{1i_2} \wedge \cdots \wedge \xi_{1i_k} \wedge \prod_{i=2}^m \xi_{ii})\\ &= \sum_h c_h^k \xi_{1\varphi_h^k(1)} \wedge \xi_{1\varphi_h^k(i_2)} \wedge \cdots \wedge \xi_{1\varphi_h^k(i_k)} \wedge \prod_{i=2}^m \xi_{2\varphi_h^k(i)} = 0.\end{aligned}$$

For any two elements $\varphi_{h_1}^k, \varphi_{h_2}^k$ of $\mathcal{N}_k$, obviously,

$$\begin{aligned}&\xi_{1\varphi_{h_1}^k(1)} \wedge \xi_{1\varphi_{h_1}^k(i_2)} \wedge \cdots \wedge \xi_{1\varphi_{h_1}^k(i_k)} \wedge \prod_{i=2}^m \xi_{i\varphi_{h_1}^k(i)}\\ &= \xi_{1\varphi_{h_2}^k(1)} \wedge \xi_{1\varphi_{h_2}^k(i_2)} \wedge \cdots \wedge \xi_{1\varphi_{h_2}^k(i_k)} \wedge \prod_{i=2}^m \xi_{i\varphi_{h_2}^k(i)} \ne 0,\end{aligned}$$

if and only if

$$(\varphi_{h_1}^k(1), \varphi_{h_1}^k(2), \cdots, \varphi_{h_1}^k(m)) = (\varphi_{h_2}^k(1), \varphi_{h_2}^k(2), \cdots, \varphi_{h_2}^k(m)), \text{ i.e., } \varphi_{h_1}^k = \varphi_{h_2}^k.$$

By the linear independence of the elements

$$\{\xi_{1\varphi_h^k(1)} \wedge \xi_{1\varphi_h^k(i_2)} \wedge \cdots \wedge \xi_{1\varphi_h^k(i_k)} \wedge \prod_{i=2}^m \xi_{i\varphi_h^k(i)} (\ne 0) | \varphi_h^k \in \mathcal{N}_k\},$$

we obtain $c_h^k = 0$. By induction, we complete the proof. Q.E.D.

Now we can prove the following

Theorem 3.3. *If $m \le n$, then the commutant algebra $\mathcal{C}_m$ of $\psi^{\otimes m}(W(n))$ coincides with the representation image $\mathcal{E}_m$ of the semigroup ring $\mathfrak{E}_m$ of the permutation semigroup* End$[m]$*:*

$$\mathcal{C}_m = \mathcal{E}_m.$$

Proof. Take an $E \in \mathcal{C}_m$. For Grassmannian polynomials $P_1, \cdots, P_m$ in n-variables, put

$$X(P_1, P_2, \cdots, P_m) = \sum_{1 \le \alpha_1, \cdots, \alpha_m \le m} P_1(\xi(\alpha_1)) \cdots P_m(\xi(\alpha_m)) \frac{\partial^m}{\partial \xi_{1\alpha_1} \cdots \partial \xi_{m\alpha_m}},$$

which is in $\psi^{\otimes m}(U(W(n)))$ by Lemma 2.1. Then we have

$$\begin{aligned} E(P_1(\xi(1))P_2(\xi(2)) \cdots P_m(\xi(m))) &= EX(P_1, P_2, \cdots, P_m)(\xi_{11} \wedge \xi_{22} \wedge \cdots \wedge \xi_{mm}) \\ &= X(P_1, P_2, \cdots, P_m)E(\xi_{11} \wedge \xi_{22} \wedge \cdots \wedge \xi_{mm}). \end{aligned}$$

Since $\Lambda(n, m)$ is generated by $\{P_1(\xi(1)) \cdots P_m(\xi(m)) | P_i \in \Lambda(n)\}$, E is completely determined by $E(\xi_{11} \wedge \xi_{22} \wedge \cdots \wedge \xi_{mm})$. On the other hand, Euler operators

$$\sum_{\alpha=1}^{m} \xi_{j\alpha} \frac{\partial}{\partial \xi_{j\alpha}} \quad (1 \le j \le n)$$

are in $\psi^{\otimes m}(W(n))$, and

$$\begin{aligned} \sum_{\alpha=1}^{m} \xi_{j\alpha} \frac{\partial}{\partial \xi_{j\alpha}} E(\xi_{11} \wedge \cdots \wedge \xi_{mm}) &= E\Big(\sum_{\alpha=1}^{m} \xi_{j\alpha} \frac{\partial}{\partial \xi_{j\alpha}} (\xi_{11} \wedge \cdots \wedge \xi_{mm})\Big) \\ &= \begin{cases} E(\xi_{11} \wedge \cdots \wedge \xi_{mm}) & \text{if } 1 \le j \le m, \\ 0 & \text{if } m < j \le n. \end{cases} \end{aligned} \tag{2.1}$$

This means that if $1 \le j \le m$, then $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is an eigenvector of the Eular operator with eigenvalue 1 and if $m + 1 \le j \le n$, then $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is in the kernel of the Euler operator. So $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is degree 1 in $(\xi_{j1}, \cdots, \xi_{jm})$ if $1 \le j \le m$ and degree 0 in $(\xi_{j1}, \cdots, \xi_{jm})$ if $m + 1 \le j \le n$. Hence we obtain

$$E(\xi_{11} \wedge \cdots \wedge \xi_{mm}) = \sum_{1 \le j_1, \cdots, j_m \le m} a_{j_1 \cdots j_m} (\xi_{1j_1} \wedge \cdots \wedge \xi_{mj_m}).$$

So dim $\mathcal{C}_m$ is less than or equal to m^m.

On the other hand, by Lemma 1.2, $\mathcal{C}_m$ contains the subalgebra $\mathcal{E}_m$ and by Lemma 2.2, its dimension is equal to m^m if $m \le n$. Therefore we conclude the theorem. Q.E.D.

4. A kind of Weyl reciprocity for $W(1) \times \mathrm{End}[m]$.

4.1. Commutant algebra of $\psi^{\otimes m}(W(1))$.

In this section, we consider the case $n = 1$. In this case, we get a stronger result which is independent of m. For $n = 1$, it becomes

$$W(1) = \Big\langle \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi} \Big\rangle, \ \deg\Big(\frac{\partial}{\partial \xi}\Big) = 1, \ \deg\Big(\xi \frac{\partial}{\partial \xi}\Big) = 0.$$

For convenience, we use the isomorphism

$$\Lambda(1, m) := \langle \xi_1, \xi_2, \cdots, \xi_m \rangle \cong \Lambda(m).$$

So we have

$$D_{-1} := \psi^{\otimes m}\Big(\frac{\partial}{\partial \xi}\Big) = \sum_{i=1}^{m} \frac{\partial}{\partial \xi_i}, \quad D_0 := \psi^{\otimes m}\Big(\xi \frac{\partial}{\partial \xi}\Big) = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial \xi_i}.$$

Obviously, $D_{-1}(\Lambda_k) \subseteq \Lambda_{k-1}, D_0(\Lambda_k) \subseteq \Lambda_k$ for any k. Further, by Lemma 2.5 we have

Lemma 4.1. *The operator D_{-1} is an exact derivation, i.e., $(D_{-1})^2 = 0$ and the chain complex*

$$0 \longrightarrow \Lambda_m \xrightarrow{D_{-1}} \Lambda_{m-1} \xrightarrow{D_{-1}} \cdots \xrightarrow{D_{-1}} \Lambda_2 \xrightarrow{D_{-1}} \Lambda_1 \xrightarrow{D_{-1}} \Lambda_0 \longrightarrow 0$$

is exact.

By the above lemmas, we can prove the following

Theorem 4.2. *Let $n = 1$ and the notations be as above. Then the commutant algebra $\mathcal{C}_m$ of $\psi^{\otimes m}(W(1))$ coincides with the representation image $\mathcal{E}_m$ of semigroup ring $\mathfrak{C}_m$ of the permutation semigroup* End$[m]$*:*

$$\mathcal{E}_m = \mathcal{C}_m.$$

Proof. By Lemma 1.2, we have $\mathcal{E}_m \subseteq \mathcal{C}_m$, so it is enough to prove $\mathcal{C}_m \subseteq \mathcal{E}_m$. For this purpose, we introduce some notations. For any $E \in \mathcal{C}_m$, put

$$E_k := E\,|_{\Lambda_k}, \quad D_{-1,k} := D_{-1}\,|_{\Lambda_k},$$

$$\Im_k := \{E \in \mathcal{C}_m \mid E_l = 0\ (\forall l > k)\}.$$

Clearly,

$$\mathcal{C}_m = \Im_m \supseteq \Im_{m-1} \supseteq \cdots \supseteq \Im_1 \supseteq \Im_0 = (0),$$

and

$$\mathcal{C}_m \cong (\Im_m/\Im_{m-1}) \oplus (\Im_{m-1}/\Im_{m-2}) \oplus \cdots \oplus \Im_1 \quad \text{(as a vector space)}.$$

We divide the proof into 5 steps.

Step 1. Taking a complementary subspace Λ'_k of $\Re(D_{-1,k+1})$ (here $\Re(D_{-1,k+1})$ is the image space of $D_{-1,k+1}$), we have

$$\Lambda_k = \Re(D_{-1,k+1}) \oplus \Lambda'_k. \tag{3.1}$$

We define a linear operator $P_k : \Im_k/\Im_{k-1} \longrightarrow \mathrm{Hom}(\Lambda'_k, \Lambda_k)$ by

$$P_k(\bar{E}) := E_k\,|_{\Lambda'_k}, \quad \text{for any } \bar{E} \in \Im_k/\Im_{k-1},$$

where $E \in \Im_k$ is a representative of $\bar{E} \in \Im_k/\Im_{k-1}$. This map is well-defined, because if two elements $E, E' \in \mathcal{C}_m$ satisfy $E - E' \in \Im_{k-1}$, then by definition, $(E - E')_k = E_k - E'_k = 0$.

Step 2. We show that P_k is an injection. Assume that $P_k(\bar{E}) = 0$ for an $\bar{E} \in \Im_k/\Im_{k-1}$, then there exists a representative $E \in \Im_k$ such that $E\,|_{\Lambda'_k} = 0$. On the other hand, for any $x \in \Re(D_{-1,k+1})$, there exists $y \in \Lambda_{k+1}$ such that $x = D_{-1}y$, so

$$E_k x = E_k D_{-1} y = D_{-1} E_{k+1} y = 0,$$

whence $E_k\ |_{\Lambda_{k+1}} = 0$. Then we see $E_k = 0$ in total and $E \in \Im_{k-1}$. Thus we obtain $\bar{E} = 0$.

STEP 3. We show P_k is surjective. For any $H \in \mathrm{Hom}_{\mathbf{C}}(\Lambda'_k, \Lambda_k)$, we define $E \in \mathrm{Hom}_{\mathbf{C}}(\Lambda(n), \Lambda(n))$ as follows:

$$\begin{cases} E\ |_{\Lambda_l} = 0 \ \ (l \neq k-1, k), \\ E\ |_{\Re(D_{-1,k+1})} = 0, \quad E\ |_{\Lambda'_{k-1}} = H, \\ E\ |_{\Re(D_{-1,k})} = D_{-1,k}H, \quad E\ |_{\Lambda'_k} = 0. \end{cases}$$

By Lemma 3.1 and the definiton of E, we have $D_0E = ED_0, D_{-1}E = ED_{-1}$ and $E \in \Im_k$. Furthermore, let $\bar{E}$ be the image of E in $\Im_k/\Im_{k-1}$, then

$$P_k(\bar{E}) = E_k\ |_{\Lambda'_k} = H.$$

So P_k is surjective .

So far, we have the following result:

$$\begin{aligned} \mathcal{C}_m &\cong (\Im_m/\Im_{m-1}) \oplus (\Im_{m-1}/\Im_{m-2}) \oplus \cdots \oplus \Im_1 \\ &\cong \mathrm{Hom}_{\mathbf{C}}(\Lambda'_m, \Lambda_m) \oplus \mathrm{Hom}_{\mathbf{C}}(\Lambda'_{m-1}, \Lambda_{m-1}) \oplus \cdots \oplus \ \mathrm{Hom}_{\mathbf{C}}(\Lambda'_1, \Lambda_1). \end{aligned}$$

STEP 4. Let us prove that the above Λ'_k can be replaced by $\Lambda_{k-1} \wedge \xi_m$, that is,

$$\Lambda_k = \Re(D_{-1,k+1}) \oplus (\Lambda_{k-1} \wedge \xi_m), \tag{3.2}$$

where $k = 1, 2, \cdots, m$ and $D_{1,m+1} = 0$. If $x \wedge \xi_m = D_{-1,k+1}y$ with $x \in \Lambda_{k-1}$, $y \in \Lambda_{k+1}$, then

$$D_{-1,k}(x \wedge \xi_m) = D_{-1,k}D_{-1,k+1}y = 0,$$

and

$$D_{-1,k}(x \wedge \xi_m) = (D_{-1,k}x) \wedge \xi_m - (-1)^{k-1}x.$$

So it holds that

$$x \wedge \xi_m = \Big((-1)^{k-1}(D_{-1,k}x) \wedge \xi_m\Big) \wedge \xi_m = 0.$$

This means that

$$\Re(D_{-1,k+1}) \cap (\Lambda_{k-1} \wedge \xi_m) = (0). \tag{3.3}$$

On the other hand, we have

$$\dim \Lambda'_k = \begin{pmatrix} m-1 \\ m-k \end{pmatrix},$$

which is proved by induction on k. By a direct calculation, we get

$$\dim\,(\Lambda_{k-1} \wedge \xi_m) = \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} = \begin{pmatrix} m-1 \\ m-k \end{pmatrix}.$$

Thus, from (3.1) and (3.3), we obtain the direct sum relation (3.2).

STEP 5. A basis of the space $\Lambda'_k = \Lambda_{k-1} \wedge \xi_m$ is given by

$$\{\xi_{j_1} \wedge \cdots \wedge \xi_{j_{k-1}} \wedge \xi_m \mid 1 \leq j_1 < j_2 < \cdots < j_{k-1} < m\}.$$

For any basis element $\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_k}$ of Λ_k, we define $\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k} \in \mathrm{End}[m]$ as follows:

$$\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k} = \begin{pmatrix} 1 & 2 & \cdots & j_1 & \cdots & j_{k-1} & \cdots & m \\ i_k & i_k & \cdots & i_1 & \cdots & i_{k-1} & \cdots & i_k \end{pmatrix},$$

i.e., $\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k}(j_l) = i_l \ (1 \leq l \leq k-1)$, $\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k}(j) = i_k \ (j \notin \{j_1, \cdots, j_{k-1}\})$. Then we have

$$\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k}(\xi_{j_1} \wedge \cdots \wedge \xi_{j_{k-1}} \wedge \xi_m) = \xi_{i_1} \wedge \cdots \wedge \xi_{i_{k-1}} \wedge \xi_{i_k},$$

and

$$\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k}(\xi_{s_1} \wedge \cdots \wedge \xi_{s_{k-1}} \wedge \xi_m) = 0 \quad \text{for } (s_1, \cdots, s_{k-1}) \neq (j_1, \cdots, j_{k-1}).$$

Therefore the set

$$\{\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k} \mid 1 \leq j_1 < j_2 < \cdots < j_{k-1} \leq m-1, 1 \leq i_1 < i_2 < \cdots < i_k \leq m\}$$

is a basis of $\mathrm{Hom}_{\mathbf{C}}(\Lambda'_k, \Lambda_k)$. So we get a surjection

$$\mathcal{E}_m \longrightarrow \oplus_{k=1}^{m} \mathrm{Hom}_{\mathbf{C}}(\Lambda'_k, \Lambda_k) \simeq \mathcal{C}_m$$

and $\dim \mathcal{C}_m \leq \dim \mathcal{E}_m$. By Lemma 1.2, we have finally $\mathcal{E}_m = \mathcal{C}_m$. Q.E.D.

4.2. The bicommutant algebra.

In the special case where $n = 1$, we also get the bicommutant algebra of $\psi^{\otimes m}(W(1))$. The next Theorem 4.3 states that it is the image of the enveloping algebra $\psi^{\otimes m}(U(W(1)))$. Therefore, in this case, we have an analogue of Schur duality for $W(1) \times \mathrm{End}[m]$.

Theorem 4.3. *The bicommutant algebra of the m-fold tensor product $\psi^{\otimes m}$ of the natural representation ψ of $W(1)$ is equal to the image $\psi^{\otimes m}(U(W(1)))$ of the enveloping algebra.*

Proof. Let $U(W(1))$ be the universal enveloping algebra of $W(1)$. Then by Poincaré-Birkhoff-Witt theorem, $U(W(1))$ is spanned over $\mathbb{C}$ as

$$U(W(1)) = \left\langle \left(\xi \frac{\partial}{\partial \xi}\right)^k, \left(\xi \frac{\partial}{\partial \xi}\right)^k \frac{\partial}{\partial \xi} \mid k = 0, 1, \cdots \right\rangle / \mathbf{C}.$$

Denote the bicommutant algebra by $\mathcal{C}'_m$. Obviously $\psi^{\otimes m}(U(W(1))) \subseteq \mathcal{C}'_m$ holds. Put

$$D_0 = \psi^{\otimes m}(\xi \frac{\partial}{\partial \xi}), \quad D_{-1} = \psi^{\otimes m}(\frac{\partial}{\partial \xi}),$$

then, under the representation $\psi^{\otimes m}$,

$$\psi^{\otimes m}(U(W(1))) = \langle I, D_0^k, D_0^{k-1}D_{-1} \mid k = 1, 2, \cdots \rangle / \mathbf{C}.$$

We prove that $\{I, D_0, \cdots, D_0^m, D_{-1}, D_0D_{-1}, \cdots, D_0^{m-1}D_{-1}\}$ are linearly independent, hence $\dim \psi^{\otimes m}(U(W(n))) \geq 2m+1$.

In fact, assume that

$$\sum_{i=0}^{m} k_i D_0^i + \sum_{i=1}^{m} k_{m+i} D_0^{i-1} D_{-1} = 0,$$

then

$$(\sum_{i=0}^{m} k_i D_0^i + \sum_{i=1}^{m} k_{m+i} D_0^{i-1} D_{-1})(\xi_1 \wedge \cdots \wedge \xi_r) = 0.$$

Take $r = 0, 1, \cdots, m$, then we get $k_0 = 0$ and

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^m \\ \vdots & \vdots & \ddots & \vdots \\ m & m^2 & \cdots & m^m \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (m-1) & \cdots & (m-1)^{m-1} \end{pmatrix} \begin{pmatrix} k_{m+1} \\ k_{m+2} \\ k_{m+3} \\ \vdots \\ k_{2m} \end{pmatrix} = 0.$$

So we have $k_1 = k_2 = \cdots = k_{2m} = 0$.

Second, we prove that

$$\dim C'_m \leq 2m+1.$$

For this purpose, let Q_k be the projection from $\Lambda(m)$ onto Λ_k, then

$$C'_m = \oplus_{i,j=0}^{m} Q_i C'_m Q_j.$$

For any $E \in C'_m$, we have $E = \sum_{i,j=0}^{m} E_{ij}$, where $E_{ij} = Q_i E Q_j$, and E_{ij} is essentially a linear operator from Λ_j to Λ_i. For any multi-index $(j_1, \cdots, j_k)$, take a $\varphi \in \mathfrak{S}_m$ such that

$$\varphi(j_t) = t \quad (1 \leq t \leq k).$$

By $E\varphi = \varphi E$, we have

$$E(\xi_{j_1} \wedge \cdots \wedge \xi_{j_k}) = \varphi^{-1} E(\xi_1 \wedge \cdots \wedge \xi_k).$$

So the value $E(\xi_{j_1} \wedge \cdots \wedge \xi_{j_k})$ is determined uniquely by $E(\xi_1 \wedge \cdots \wedge \xi_k)$.

Let us now consider E_{ij}.

1°. THE CASE E_{ij} $(j < i)$. Put

$$E_{ij}(\xi_1 \wedge \cdots \wedge \xi_j) = \sum_{1 \le t_1 < \cdots < t_i \le m} c_{t_1 \cdots t_i} \xi_{t_1} \wedge \cdots \wedge \xi_{t_i}.$$

Take a $\varphi \in \mathrm{End}[m]$ such that $\varphi(k) = k\,(1 \le k \le j), \varphi(k) = j\,(j < k \le m)$. Then, because of $E_{ij}\varphi = \varphi E_{ij}$,

$$E_{ij}(\xi_1 \wedge \cdots \wedge \xi_j) = \sum_{1 \le t_1 < \cdots < t_i \le m} c_{t_1 \cdots t_i} \xi_{\varphi(t_1)} \wedge \cdots \wedge \xi_{\varphi(t_i)} = 0,$$

whence $E_{ij} = 0$.

2°. THE CASE E_{ij} $(j \ge i + 2)$. Put

$$E_{ij}(\xi_1 \wedge \cdots \wedge \xi_j) = \sum_{1 \le t_1 < \cdots < t_i \le m} c_{t_1 \cdots t_i} \xi_{t_1} \wedge \cdots \wedge \xi_{t_i}.$$

For any $1 \le k_1 < k_2 < \cdots < k_i \le m, 1 \le k \le m$, and $k \notin \{k_1, \cdots, k_i\}$, take $\varphi \in \mathrm{End}[m]$ such that $\varphi(k_l) = k_l\,(1 \le l \le i), \varphi(t) = k\,(t \notin \{k_1, \cdots, k_i\})$. Then $\varphi(\xi_1 \wedge \cdots \wedge \xi_j) = 0$, and it holds that

$$\begin{aligned} 0 = E_{ij}\varphi(\xi_1 \wedge \cdots \wedge \xi_j) &= \varphi E_{ij}(\xi_1 \wedge \cdots \wedge \xi_j) \\ &= \sum_{1 \le t_1 < \cdots < t_i \le m} c_{t_1 \cdots t_i} \xi_{\varphi(t_1)} \wedge \cdots \wedge \xi_{\varphi(t_i)}. \end{aligned}$$

Therefore we obtain $c_{k_1 \cdots k_i} = 0$, whence $E_{ij} = 0$.

According to the above facts, an operator $E \in C'_m$ is determined completely by the values

$$E_{0,0}(1), E_{1,1}(\xi_1), \cdots, E_{m,m}(\xi_1 \wedge \cdots \wedge \xi_m),$$

and

$$E_{1,0}(\xi_1), E_{2,1}(\xi_1 \wedge \xi_2), \cdots, E_{m,m-1}(\xi_1 \wedge \cdots \wedge \xi_m).$$

Further, if we take

$$\varphi = \begin{pmatrix} 1 & 2 & \cdots & r & r+1 & \cdots & j & \cdots & m \\ 1 & 2 & \cdots & r & r & \cdots & r & \cdots & r \end{pmatrix},$$

then φ maps $\Lambda_r(m)$ into the one-dimensional subspace $\mathbb{C}\,\xi_1 \wedge \cdots \wedge \xi_r$. So by $E\varphi = \varphi E$, we get

$$E_{r,r}(\xi_1 \wedge \cdots \wedge \xi_r) \in \mathbb{C}\,\xi_1 \wedge \cdots \wedge \xi_r,$$
$$E_{r,r-1}(\xi_1 \wedge \cdots \wedge \xi_{r-1}) \in \mathbb{C}\,\xi_1 \wedge \cdots \wedge \xi_r.$$

Therefore, it holds that

$$\dim C'_m \le 2m + 1.$$

In conclusion, we have $\dim C'_m = 2m + 1$ and $C'_m = \psi^{\otimes m}(U(W(1)))$. This completes the proof of the theorem. Q.E.D.

The content in §§1 − 4 will be published in [3] and [4].

5. Commutant Algebra of $\psi^{\otimes m}(W(2))$

For the case $n = 2$ and $m = 3$, since the rank and the dimensions are small, we can calculate out the commutant algebra C_3 explicitly as shown here (cf, [6]). At first, by the definition of $\psi^{\otimes 3}$, we have

$$\psi^{\otimes 3}(W(2)) = \langle D_i, D_{ij}, D_{12i} \mid i,j = 1,2\rangle/\mathbf{C},$$

where

$$D_i = \sum_{\alpha=1}^{3} \frac{\partial}{\partial \xi_{i\alpha}}, \quad D_{ij} = \sum_{\alpha=1}^{3} \xi_{i\alpha}\frac{\partial}{\partial \xi_{j\alpha}}, \quad D_{12i} = \sum_{\alpha=1}^{3} \xi_{1\alpha} \wedge \xi_{2\alpha}\frac{\partial}{\partial \xi_{i\alpha}}.$$

Using the notations in Section 2 for the case $W(2)$, we have $\Lambda_{p,q} = \Lambda_p[\xi_{1i} \mid 1 \le i \le 3] \otimes \Lambda_q[\xi_{2j} \mid 1 \le j \le 3]$, where $\Lambda_p[\xi_{1i} \mid 1 \le i \le 3]$ (resp. $\Lambda_q[\xi_{2j} \mid 1 \le j \le 3]$) is a subspace of all the Grassmannian polynomials of homogeneous degree p (resp. q) generated by $\{\xi_{1i} \mid 1 \le i \le 3\}$ (resp. $\{\xi_{2j} \mid 1 \le j \le 3\}$). Then we have a direct sum decomposition

$$\otimes^3\Lambda(2) = \otimes_{p,q=0}^{3}\Lambda_{p,q}.$$

By lemmas in Section 2, we have the following results.

Lemma 5.1. *For any $E \in C_3$, we obtain $E(\Lambda_{p,q}) \subset \Lambda_{p,q}$.*

If we set $E_{pq} := E\mid_{\Lambda_{pq}}$, then $E = \oplus_{p,q=0}^{3} E_{pq}$.

Lemma 5.2. *In the algebra $\psi^{\otimes 3}(W(2))$ we have the following relations:*

$$(D_i)^2 = 0, \quad (D_{12i})^2 = 0 \quad (i = 1,2),$$
$$(D_1D_{122} + D_{122}D_1) = [D_1, D_{122}] = D_{22},$$
$$(D_2D_{121} + D_{121}D_2) = [D_2, D_{121}] = D_{11}.$$

Lemma 5.3. *For any $E \in \mathrm{End}(\otimes^3\Lambda(2))$, if E belongs to C_3, then*

$$E_{pq}(\Re(D_i)) \subseteq \Re(D_i), \quad E_{pq}(\Re(D_{12i})) \subseteq \Re(D_{12i}) \quad (p,q = 0,1,2,3; i = 1,2).$$

Lemma 5.4. *Any $E \in C_3$ is completely determined by its components E_{22} and E_{11}.*

Proof. Obviously, $\Lambda_{33} = \langle \xi_{11} \wedge \xi_{12} \wedge \xi_{13} \wedge \xi_{21} \wedge \xi_{22} \wedge \xi_{23}\rangle$, one-dimensional, and

$$E(\xi_{11} \wedge \xi_{12} \wedge \xi_{13} \wedge \xi_{21} \wedge \xi_{22} \wedge \xi_{23}) = ED_{121}D_{122}(-\xi_{11} \wedge \xi_{12} \wedge \xi_{21} \wedge \xi_{22})$$
$$= -D_{121}D_{122}E(\xi_{11} \wedge \xi_{12} \wedge \xi_{21} \wedge \xi_{22}).$$

Hence, E_{33} is determined by E_{22}. Because of $D_{122}(\Lambda_{22}) = \Lambda_{32}$, and $D_{12}(\Lambda_{22}) = \Lambda_{31}$, we see that E_{32} and E_{31} are also determined by E_{22}.

For Λ_{21}, we have $\Lambda_{21} = \Re(D_1) \oplus \Re(D_{122})$ by Lemma 2.2, so E_{21} is determined by E_{31} and E_{11}. For $\Lambda_{30}, \Lambda_{20}$, and Λ_{10}, because $D_2(\Lambda_{i1}) = \Lambda_{i0}$ $(i = 1,2,3)$, the maps E_{30}, E_{20} and E_{10} are determined by E_{31}, E_{21} and E_{11}. Further E_{00} is determined by E_{11}.

By the same argument, we see that E_{pq} $(p < q)$ is also determined by E_{22} and E_{11}. So the lemma is proved. Q.E.D.

Lemma 5.5. *For E_{22} and E_{11}, we have*

$$\dim\{E\,|_{\Lambda_{22}} \in \mathrm{End}\Lambda_{22} \mid E \in \mathcal{C}_3\} \leq 11,$$

$$\dim\{E\,|_{\Lambda_{11}} \in \mathrm{End}\Lambda_{11} \mid E \in \mathcal{C}_3\} \leq 13.$$

Lemma 5.6. *For $n = 2$ and $m = 3$, we have*

$$\dim \mathcal{E}_3 = 24.$$

The proofs of Lemmas 5.5, and 5.6 consist of solving certain systems of linear equations and we omit them. From Lemma 2.1, we know that $\mathcal{E}_3 \subseteq \mathcal{C}_3$ and using the above facts on dimensions, we get the following theorem.

Theorem 5.7. *Let $n = 2$ and $m = 3$. Then the commutant algebra $\mathcal{C}_3$ of $\psi^{\otimes 3}(W(2))$ coincides with the representation image $\mathcal{E}_3$ of the semigroup ring $\mathfrak{E}_3$ of the permutation semigroup* End[3]*: $\mathcal{C}_3 = \mathcal{E}_3$. Moreover the dimension of $\mathcal{C}_3$ is equal to 24.*

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