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Orbits on the flag variety and images of the moment map — For $U(p,q)$ and $Sp(p,q)$ —

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0.1 Introduction

Let $G_{\mathbb{R}}$ be a real classical group whose Cartan subgroups are all connected. It is known that there is a correspondence of irreducible representations of $G_{\mathbb{R}}$ and the orbits of the complexification of a maximal compact subgroup of $G_{\mathbb{R}}$ on the flag variety of the complexification of $G_{\mathbb{R}}$. Essentially, a parametrization of the orbits is given by Matsuki-Oshima [MO]. In this paper, at first, we get an algorithm which gives a representative element of the orbit for the parametrization for the case of $G_{\mathbb{R}} = U(p, n-p)$. We get an algorithm which gives the image of the moment map of a fiber of the conormal bundle of the orbits. For a closed orbit, the image is the associated variety of a representation. By this method, we get the representative element from a parameter directly. Second, we get the similar result for the case of $G_{\mathbb{R}} = Sp(p, n-p)$. For the case of $Sp(p, n-p)$ we define other parameters which is an improvement of the Matsuki-Oshima's parameters. In this definition the set of parameters for $Sp(p,n-p)$ is a subset of the set of parameters of $U(2p, 2n-2p)$. By using the parameter, we can also get a representative element directly. Then we can parametrize the orbit $\mathcal{O}$ for $Sp(p, n-p)$ by the parameter of the orbit for $U(2p, 2n-2p)$ which includes $\mathcal{O}$ by using a suitable realization of the groups.

0.2 Notation

Notation 0.2.1 Let $\mathbb{N}$ denote the set of non negative integers; $\mathbb{N} = \{0,1,2,\ldots\}$. For $n \in \mathbb{N}$, let an $n \times n$ matrix E_{ij} $(1 \leq i,j \leq n)$ denote the matrix unit which has 1 for the (i,j)-entry and 0 for other entries. Let an n-column vector e_i be the vector which has 1 for the i-th entry and 0 for other entries. For a matrix X, let X_{ij} be the (i,j)-entry of X. Let $\mathrm{Mat}(n,m)$ be the set of $n \times m$-matrices. Let $I_n \in \mathrm{Mat}(n,n)$ be the identity matrix, $J_n \in \mathrm{Mat}(n,n)$ satisfy $J_n(e_i) = e_{n+1-i}$:

$$(J_n)_{ij} = \delta_{i,n+1-j},$$

$T_n \in \mathrm{Mat}(n,n)$ satisfy

$$T_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}(n,n).$$

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For $A \in \mathrm{Mat}(n,n)$ and subset $\{i(1), i(2), \ldots, i(m)\}$ of $\{1,\ldots,n\}$, let $A_{(i(1),i(2),\ldots,i(m))}$ be an $m \times m$-matrix which has $A_{i(s)i(t)}$ as (s,t)-entry for $1 \le s,t \le n$. Let $\#S$ denotes cardinality of the finite set S. For m vectors $\{g_1,\ldots,g_n \mid g_i \in \mathbb{C}^m\}$ $(m < n)$ let $\langle g_1,\ldots,g_n\rangle$ be m-dimensional vector space spanned by $\{g_1,\ldots,g_n\}$.

Definition 0.2.2 Let $G_{\mathbb{R}}$ be a real classical Lie group, G be the complexification of $G_{\mathbb{R}}$, $\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of $G_{\mathbb{R}}$, θ be a Cartan involution of $\mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ be the Cartan decomposition corresponding to θ, $\mathfrak{k}$ the complexification of $\mathfrak{k}_{\mathbb{R}}$, K the analytic subgroup of G for $\mathfrak{k}$, and B a Borel subgroup of G.

1 Type AIII

1.1 Clans for AIII type

In this section, we define the parametrization of K-orbits on the flag variety G/B of the type AIII. For the type AIII, we treat the case of $U(p,n-p)$.

We realize a group $U(p,n-p)$ as the group of matrices g in $GL(n,\mathbb{C})$ which leave invariant the Hermitian form

$$x_1\overline{x_1} + \cdots + x_p\overline{x_p} - x_{p+1}\overline{x_{p+1}} - \cdots - x_n\overline{x_n}$$

i.e.

$$U(p,n-p) = \left\{ g \in GL(n,\mathbb{C}) \;\middle|\; {}^t g \begin{pmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{pmatrix} \bar{g} = \begin{pmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{pmatrix} \right\}.$$

Define the numbers of the signature of a Hermitian form. For $U(p,n-p)$, the number of the plus of the Hermitian form is p and the number of the minus of the Hermitian form is $n-p$.

We fix a Cartan involution θ:

$$\theta : x \mapsto \begin{pmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{pmatrix} x \begin{pmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{pmatrix}.$$

Now we have

$$\mathfrak{k} = \left\{ \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} K_{11} \in \mathrm{Mat}(p,p) \\ K_{22} \in \mathrm{Mat}(n-p,n-p) \end{array} \right\} \tag{1}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & P_{12} \\ P_{21} & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} P_{12} \in \mathrm{Mat}(p,n-p) \\ P_{21} \in \mathrm{Mat}(n-p,p) \end{array} \right\} \tag{2}$$

Definition 1.1.1 (clan) [MO] We define a set $\mathcal{C}(G_{\mathbb{R}})$ for $G_{\mathbb{R}}$ of type AIII. A element of the set $\mathcal{C}(G_{\mathbb{R}})$ is an equivalence class of an ordered set $C = (c_1 \ldots c_n)$ of n symbols satisfying the following four conditions.

1. Each symbol c_i is $c_i = +, -$, or an element of $\mathbb{N}$.

2. If $c_i = a \in \mathbb{N}$, then there exists unique $j \neq i$ with $a = c_j$:

$$\#\{\, i \mid c_i = a \,\} = 0 \text{ or } 2 \qquad \text{for any } \ a \in \mathbb{N}.$$

3. If $c_i = a > 1$, then there exists $c_j = a - 1$.

4. The difference of the number of the signature of the symbols coincides with the difference of the number of the signature of the Hermitian form defining the group $G_{\mathbb{R}}$:

$$\#\{\, i \mid c_i = +\} - \#\{\, i \mid c_i = -\} = p - (n-p) = 2p - n.$$

5. A symbols $C = (c_1 \ \ldots \ c_n)$ is equivalent to $D = (d_1 \ \ldots \ d_n)$ if there exists a permutation $\sigma \in \mathfrak{S}_m$, $m := \max\{\, c_i \in \mathbb{N}\,\}$ such that

$$c_i = \begin{cases} \sigma(d_i) & \text{if } d_i \in \mathbb{N} \\ + & \text{if } d_i = + \\ - & \text{if } d_i = - \end{cases}$$

for all $1 \leq i \leq n$. For example, $(2\ 2\ 1\ 1) = (1\ 1\ 2\ 2)$ as clans.

We call the element of $\mathcal{C}(G_{\mathbb{R}})$ *clan* for type AIII.

Definition 1.1.2 (representative indication) For a clan $C = (c_1 \ \ldots \ c_n)$, we define the representative indication $\tilde{C} = (\tilde{c}_1 \ \ldots \ \tilde{c}_n)$ of C, as follows. Suppose $\tilde{c}_i = \tilde{c}_j = a$ for $i < j$ and $\tilde{c}_s = \tilde{c}_t = b$ for $s < t$. If $i < s$, then $a < b$.

Example 1.1.3 The set $\mathcal{C}(U(2,1))$ has six elements represented by the following symbols:

$$\left\{ \begin{matrix} +\ +\ - & , & +\ -\ + & , & -\ +\ + \\ +\ 1\ 1 & , & 1\ 1\ + & , & 1\ +\ 1 \end{matrix} \right\}.$$

Example 1.1.4 The set $\mathcal{C}(U(2,2))$ has 21 elements represented by the following symbols:

$$\left\{ \begin{matrix} +\ +\ -\ - & , & +\ -\ +\ - & , & +\ -\ -\ + & , & -\ +\ +\ - \\ -\ +\ -\ + & , & -\ -\ +\ + & , & 1\ 1\ +\ - & , & 1\ 1\ -\ + \\ +\ 1\ 1\ - & , & -\ 1\ 1\ + & , & +\ -\ 1\ 1 & , & -\ +\ 1\ 1 \\ 1\ +\ 1\ - & , & 1\ -\ 1\ + & , & +\ 1\ -\ 1 & , & -\ 1\ +\ 1 \\ 1\ +\ -\ 1 & , & 1\ -\ +\ 1 & , & 1\ 1\ 2\ 2 & , & 1\ 2\ 1\ 2 \\ 1\ 2\ 2\ 1 & & & & & & \end{matrix} \right\}.$$

We define a graph which has clans as verteces and in which an edge connecting two clans is labelled by i for some $1 \leq i \leq n$.

Definition 1.1.5 Let $\gamma(U(p, n-p))$ denote the set of all triples (C, C', i), $C, C' \in \mathcal{C}(U(p, n-p))$, $1 \leq i \leq n-1$, satisfying the following conditions.

A clan $C = (c_1 \ldots c_n) \in \mathcal{C}(U(p, n-p))$ satisfies $c_i \neq c_{i+1}$.

If $(c_i, c_{i+1}) = (+,-)$ or $(-,+)$, then $C' = (c_1 \dots c_{i-1} a a c_{i+2} \dots c_n)$.
If $(c_i, c_{i+1}) \neq (+,-)$ nor $(-,+)$, then $C' = (c_1 \dots c_{i-1} c_{i+1} c_i c_{i+2} \dots c_n)$.

Definition 1.1.6 (M-O graph) We define a subset $\Gamma_m(U(p,n-p))$ of $\gamma(U(p,n-p))$ for $m \geq 1$ and a subset $\mathcal{C}_m(U(p,n-p))$ of $\mathcal{C}(U(p,n-p))$ for $m \geq 0$ inductively as follows:

$$\mathcal{C}_0(U(p,n-p)) := \{C = (c_1 \dots c_n) \in \mathcal{C}(U(p,n-p)) \mid c_i = + \text{ or } - \text{ for all } i\} \tag{3}$$

$$\Gamma_m(U(p,n-p)) := \bigcup_{C \in \mathcal{C}_{m-1}(U(p,n-p))} \left\{ \begin{array}{l} (C, C', i) \in \gamma(U(p,n-p)) \mid \\ C' \notin \mathcal{C}_{m'}(U(p,n-p)) \text{ for all } m' < m \end{array} \right\}.$$

$$\mathcal{C}_m(U(p,n-p)) := \{ C' \mid (C, C', i) \in \Gamma_m(U(p,n-p))\} \quad (1 \leq m).$$

The *M-O graph* of $U(p,n-p)$ is a finite oriented graph whose verteces are $\mathcal{C}(U(p,n-p))$ and whose oriented edges are $\Gamma(U(p,n-p)) := \bigcup_{m \in \mathbb{N}} \Gamma_m(U(p,n-p))$.

Remark 1.1.7 *The following conditions are satisfied:*

$$\mathcal{C}(U(p,n-p)) = \bigsqcup_{m \geq 0} \mathcal{C}_m(U(p,n-p)). \tag{4}$$

Notation 1.1.8 Let B_i denote the parabolic subgroup of G for the root $-\alpha_i$ and all positive roots. Let

$$\pi_i : K \backslash G/B \to K \backslash G/B_i$$

be the canonical projection.

Theorem 1.1.9 [MO] *The clans of $U(p,n-p)$ parametrize K-orbits on the flag variety G/B. This parametrization satisfies the following property.*

Suppose clans C and C' correspond to orbits $\mathcal{O}$ and $\mathcal{O}'$ respectively. Then $(C, C', i) \in \Gamma(U(p,n-p))$ if and only if $\dim \mathcal{O} + 1 = \dim \mathcal{O}'$ and $\pi_i(\mathcal{O}) = \pi_i(\mathcal{O}')$.

1.2 The function F

In this section, we define a function from $\mathcal{C}(U(p,n-p))$ to $\mathbb{N}$ which gives a natural number m such that the clan belongs to $\mathcal{C}_m(U(p,n-p))$.

Definition 1.2.1 We define a function F from $\mathcal{C}(U(p,n-p))$ to $\mathbb{N}$ as follows:

$$\begin{aligned} F(C) &= \max\{ a \in \mathbb{N} \mid \tilde{c}_i = a,\ 1 \leq i \leq n \} \\ &\quad + \Sigma_{a \in \mathbb{N}} \#\{ t \in \mathbb{N} \mid \tilde{c}_t = + \text{ or } -,\ i < t < j,\ \tilde{c}_i = \tilde{c}_j = a \} \\ &\quad + \Sigma_{a \in \mathbb{N}} \#\{ t \in \mathbb{N} \mid \tilde{c}_t > \tilde{c}_i = a, \text{ for some } i > t \} \\ &= \#\{ a \in \mathbb{N} \mid c_i = c_j = a \text{ for some } 1 \leq i, j \leq n \} \\ &\quad + \Sigma_{a \in \mathbb{N}, c_i = c_j = a, i < j} (j - i) \\ &\quad - \#\{ (a,b) \in \mathbb{N} \mid c_i = c_j = a, c_s = c_t = b \text{ for } i < s < j < t \} \end{aligned}$$

Where, $(\tilde{c}_1 \dots \tilde{c}_n)$ is the representative indication of clan C.

Proposition 1.2.2 *Suppose $C \in \mathcal{C}(U(p, n-p))$.*

$$\text{If } F(C) = m, \text{ then } C \in \mathcal{C}_m(U(p, n-p)).$$

Notation 1.2.3 For each clan C, let $\mathcal{O}_C$ be the K-orbit on the flag variety G/B corresponding to C.

For $G_{\mathbb{R}} = U(p, n-p)$, $p \leq n-p$, a clan

$$(\underbrace{1\,2\,\ldots\,p}_{p}\,\underbrace{-\,\ldots\,-}_{n-2p}\,\underbrace{p\,\ldots\,2\,1}_{p})$$

corresponds to the open orbit and

$$F(1\,2\,\ldots\,p\,-\,\ldots\,-\,p\,\ldots\,2\,1) = p(n-p).$$

Therefore we have the following corollary.

Corollary 1.2.4 For all $C \in \mathcal{C}(U(p, n-p))$,

$$\mathrm{codim}\mathcal{O}_C = p(n-p) - F(C).$$

1.3 Representatives of K-orbits for clans

From now on fix the Cartan subalgebra $\mathfrak{a}$ and the simple root system

$$\Psi := \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$$

such that the corresponding Borel subalgebra $\mathfrak{b}$ is the set of upper triangular matrices:

$$\mathfrak{b} = \{X \in \mathrm{Mat}(n,n) \mid X_{st} = 0 \text{ if } s > t\}.$$

Then the parabolic subgroup B_i is

$$B_i = \{X \in G \mid X_{st} = 0 \text{ if } s > t \text{ and } (s,t) \neq (i+1, i)\}.$$

Definition 1.3.1 (signed clan) *Signed clans* of a clan $C = (c_1\,\ldots\,c_n)$ are ordered sets $C_{\pm} = (d_1\,\ldots\,d_n)$ of n symbols $+$, $-$, a_+ and a_- satisfies the following two conditions.

1. $d_i = +$ or $-$ if $c_i = +$ or $-$ respectively.
2. $(d_i, d_j) = (a_+, a_-)$ or $(d_i, d_j) = (a_-, a_+)$ if $(c_i, c_j) = (a, a)$ for some $a \in \mathbb{N}$.

Example 1.3.2 All signed clans of $C = (1\,1\,2\,2)$ are

$C = (1_+\,1_-\,2_+\,2_-)$, $C = (1_+\,1_-\,2_-\,2_+)$, $C = (1_-\,1_+\,2_+\,2_-)$, and $C = (1_-\,1_+\,2_-\,2_+)$.

Theorem 1.3.3 *Let $C_\pm = (c_1 \dots c_n)$ be a signed clan of $C \in \mathcal{C}(U(p, n-p))$, the matrix $g(C) = (g_1\ g_2\ \dots\ g_n)$ is a representative of $\mathcal{O}_C$. Here $g_i \in \mathbb{C}_n$ is the column vector defined by the following.*

- *If $c_i = \pm$, then $g_i = e_{\varphi(i)}$.*
- *If $c_i = a_+$, $c_j = a_-$, then*

$$g_i = \frac{1}{\sqrt{2}}(\, e_{\varphi(i)} + e_{\varphi(j)}\,) \quad \text{and} \quad g_j = \frac{1}{\sqrt{2}}(\, -e_{\varphi(i)} + e_{\varphi(j)}\,).$$

Where we can take any $\varphi \in \mathfrak{S}_n$ satisfying the following condition.

$$\begin{array}{ll} 1 \leq \varphi(i) \leq p & \text{if } c_i = + \text{ or } a_+, \\ p+1 \leq \varphi(i) \leq n & \text{if } c_i = - \text{ or } a_-. \end{array} \tag{5}$$

Remark 1.3.4 *The matrix $g(C)$ is real and orthogonal:*

$$g(C)^{-1} = {}^t g(C).$$

Remark 1.3.5 *We regard the flag variety G/B as the set $(V_1, V_2, \dots, V_n)$ of sequences of vector spaces $V_i \subset \mathbb{C}^n$ $(1 \leq \forall i \leq n)$ such that $V_i \subset V_{i+1}$ for all $1 \leq i \leq n-1$, $V_n = \mathbb{C}^n$, and $\dim V_i = i$ for all $1 \leq i \leq n$. Let $g = (g_1 g_2 \cdots g_n)$ and $g_i \in \mathbb{C}^n$ for $1 \leq i \leq n$. Let gB be an element of G/B. We regard the element gB as the set of sequence $(V_1, V_2, \dots, V_n)$ such that $V_i \subset \mathbb{C}^n$ is spanned by i-vectors $g_1, g_2, \dots, g_{i-1}$, and g_i:*

$$V_i := \langle g_1, \dots, g_i \rangle.$$

Notation 1.3.6 Let $V_+ \subset \mathbb{C}^n$ be a p-dimensional vector space spanned by $\{e_1, \dots, e_p\}$. Let $V_- \subset \mathbb{C}^n$ be an $(n-p)$-dimensional vector space spanned by $\{e_{p+1}, \dots, e_n\}$.

$$\begin{aligned} V_+ &:= \langle e_1, \dots, e_p \rangle, \\ V_- &:= \langle e_{p+1}, \dots, e_n \rangle. \end{aligned}$$

Remark 1.3.7 *The vector spaces V_+ and V_- are preserved by the action of K. They satisfy $\mathbb{C}^n = V_+ \oplus V_-$.*

Lemma 1.3.8 Let $(V_1, \dots, V_n)$ be the flag $g(C)B$. By definition of $g(C)$, we have the following condition.

$$\begin{aligned} \dim(V_i \cap V_+) = {} & \#\{\, j \mid c_j = + \ \text{ for } \ j \leq i \,\} \\ & + \#\{\, a \mid c_j = c_l = a,\ j \neq l \ \text{ for } \ \max(j,l) \leq i \,\}, \\ \dim(V_i \cap V_-) = {} & \#\{\, j \mid c_j = - \ \text{ for } \ j \leq i \,\} \\ & + \#\{\, a \mid c_j = c_l = a,\ j \neq l \ \text{ for } \ \max(j,l) \leq i \,\}, \end{aligned}$$

$$\begin{aligned}\dim((V_i+V_-)\cap V_+) = \ & \#\{\, j \mid c_j = + \ \text{ for } \ j \le i\,\} \\ & +\#\{\, a \mid c_j = c_l = a, \ j \ne l \ \text{ for } \ \min(j,l) \le i\,\}, \\ \dim(V_j \cap (V_i+V_-)\cap V_+) = \ & \#\{\, j \mid c_j = + \ \text{ for } \ j \le s\,\} \\ & +\#\left\{ a \ \middle| \ \begin{array}{ll} c_j = c_l = a, \ j \ne l \ \text{ for } & \max(j,l) \le i, \\ & \min(j,l) \le s \end{array} \right\}.\end{aligned}$$

Because V_+ and V_- are preserved by action of K,

$$\begin{aligned}\dim(kV_i \cap V_+) &= \dim(V_i \cap k^{-1}V_+) \\ &= \dim(V_i \cap V_+)\end{aligned}$$

for all $k \in K$ and so on. Therefore we have the following Lemma.

Lemma 1.3.9 For a flag $(V_1, \dots, V_n) \in G/B$, $\dim(V_i \cap V_+)$, $\dim(V_i \cap V_-)$, and $\dim(V_s \cap (V_i + V_-) \cap V_+)$ are invariants by an action of K.

1.4 Driving matrix for $U(p, n-p)$

In this section we get an algorithm to get the image of the moment map of a fiber of the conormal bundle of the K-orbit which corresponds to each clan.

Suppose $C = (c_1 \dots c_n)$ is an element of $\mathcal{C}(U(p, n-p))$. Fix a signed clan of C. From now on, we regard the signed clan as a clan C. The matrix $g(C)$ is the representative element which is given in Theorem 1.3.3, we describe the algorithm to get the image of the moment map $g(C)\mathfrak{b}^\perp g(C)^{-1} \cap \mathfrak{p}$. Where $\mathfrak{b}^\perp$ is a subalgebra of $\mathfrak{g}$ that is orthogonal to $\mathfrak{b}$:

$$\mathfrak{b}^\perp = \{\, b \in \mathfrak{g} \mid \beta(b, b') = 0 \ \text{ for all } \ b' \in B\,\}$$

Let $\beta(\,\cdot\,,\,\cdot\,)$ be a natural inner product on $\mathfrak{g}$.

Definition 1.4.1 For a clan $C = (c_1 \dots c_n) \in \mathcal{C}(U(p, n-p))$, give an $n \times n$-matrix $Dri(C)$ as follows. As (j,i)-entry, $Dri(C)$ has zero or x_{st} for some $1 \le s, t \le n$ as the following steps.

- For $1 \le i, j \le n$, $Dri(C)_{ji} = x_{ji}$ if (c_i, c_j) satisfies one of the following conditions.
    1. $(c_i, c_j) = (+, -)$ or $(-, +)$ for $i < j$.
    2. $(c_i, c_j) = (+, a_-)$ with $c_s = a_+$ and $i < \min(j, s)$.
    3. $(c_i, c_j) = (-, a_+)$ with $c_s = a_-$ and $i < \min(j, s)$.
    4. $(c_i, c_j) = (a_-, +)$ with $c_s = a_+$ and $\max(i, s) < j$.
    5. $(c_i, c_j) = (a_+, -)$ with $c_s = a_-$ and $\max(i, s) < j$.
    6. $(c_i, c_j) = (a_+, b_-)$ with $(c_s, c_t) = (a_-, b_+)$, $\max(i, s) < \max(j, t)$, and $\min(i, s) < \min(j, t)$.
    7. $(c_i, c_j) = (a_-, b_+)$ with $(c_s, c_t) = (a_+, b_-)$ and $\max(i, s) < \min(j, t)$.

- For $1 \leq i,j \leq n$, $Dri(C)_{ts} = -x_{ji}$ if $(c_s, c_t) = (a_-, b_+)$ with $(c_i, c_j) = (a_+, b_-)$ and $\min(i,s) < \min(j,t) < \max(i,s) < \max(j,t)$.

- Other entries are zero.

We call $Dri(C)$ a *driving matrix* of C for $U(p, n-p)$.

Theorem 1.4.2 *For a clan $C \in \mathcal{C}(U(p, n-p))$, the image of the moment map is as follows:*

$$\begin{aligned} g(C)\mathfrak{b}^{\perp}g(C)^{-1} \cap \mathfrak{p} &= \{\varphi Dri(C)^t\varphi \mid x_{ij} \in \mathbb{C}\} \\ &= \{Dri(C)_{(\varphi^{-1}(1),\ldots,\varphi^{-1}(n))} \mid x_{ij} \in \mathbb{C}\}. \end{aligned}$$

We can regard $Dri(C)$ as an element of

$$(g(C)\mathfrak{b}^{\perp}g(C)^{-1} \cap \mathfrak{p})_{(\varphi(1),\ldots,\varphi(n))}.$$

That is

$$Dri(C)_{ij} = (g(C)\mathfrak{b}^{\perp}g(C)^{-1} \cap \mathfrak{p})_{\varphi(i)\varphi(j)}.$$

Since $g(C) = (e_{\varphi(1)} \ldots e_{\varphi(n)})$ if $C \in \mathcal{C}_0(U(p, n-p))$, we have the following lemma.

Lemma 1.4.3 If a clan C corresponds to a closed orbit, then

$$\{\, Dri(C) \mid x_{ij} \in \mathbb{C}\} = \mathfrak{b}^{\perp} \cap ({}^t g(C)\,\mathfrak{p}\, g(C)).$$

Example 1.4.4 For $C = (-++)$, we have

$$\left\{ \left(\begin{array}{ccc} 0 & 0 & 0 \\ x_{2,1} & 0 & 0 \\ x_{3,1} & 0 & 0 \end{array} \right) \middle| \; x_{ij} \in \mathbb{C} \right\} = \mathfrak{b} \cap ({}^t g(C)\mathfrak{p} g(C)).$$

1.5 Signed Young diagram

Definition 1.5.1 We define a *signed Young diagram* to be a Young diagram in which every box is labelled with a $+$ or $-$ sign in such a way that signs alternate across rows and they need not alternate down columns. Two signed Young diagram are regarded as equal if and only if one can be obtained from the other by interchanging rows of equal length.

Proposition 1.5.2 *For $G_{\mathbb{R}} = U(p, n-p)$ nilpotent orbits in $\mathfrak{p}$ are parametrized by signed Young diagram which has p boxes labeled $+$ and $(n-p)$ boxes labeled $-$.*

An element A of $\mathfrak{p}$ satisfies

$$\begin{aligned} AV_+ &\subset V_-, \\ AV_- &\subset V_+. \end{aligned}$$

Definition 1.5.3 The signed Young diagram of a nilpotent orbit is as follows. Take an element A of the orbit.

1. The number of boxes which are in a j-column ($\forall j \leq i$) and labeled $+$ is $\dim(\ker(A^i|_{V_+}))$:
$$\Sigma_{j=1}^{i} \# \{ (s,j) \mid (s,j)\text{-box labelled with } + \text{ sign.} \} = \dim(\ker(A^i|_{V_+})).$$

2. The number of boxes which are in a j-column ($\forall j \leq i$) and labeled $-$ is $\dim(\ker(A^i|_{V_-}))$:
$$\Sigma_{j=1}^{i} \# \{ (s,j) \mid (s,j)\text{-box labelled with } - \text{ sign.} \} = \dim(\ker(A^i|_{V_-})).$$

Proposition 1.5.4 *Let*

$$\begin{aligned}
\varphi V_+ &:= (e_{\varphi(1)} \dots e_{\varphi(n)})^{-1} V_+ = \langle e_{\varphi^{-1}(1)}, \dots, e_{\varphi^{-1}(p)} \rangle, \\
\varphi V_- &:= (e_{\varphi(1)} \dots e_{\varphi(n)})^{-1} V_- = \langle e_{\varphi^{-1}(p+1)}, \dots, e_{\varphi^{-1}(n)} \rangle,
\end{aligned}$$

then,

$$\begin{aligned}
\dim(\ker((Dri(C))^i|_{\varphi V_+})) &= \dim(\ker(A^i|_{V_+})), \\
\dim(\ker((Dri(C))^i|_{\varphi V_-})) &= \dim(\ker(A^i|_{V_-})).
\end{aligned}$$

Example 1.5.5 For a clan $C = (+11-+)$, let choose a signed clan $(+\ 1_+\ 1_-\ -+)$ We get a permutation, a driving matrix, and two vector spaces as follows:

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}, \quad Dri(C) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ x_{4,1} & x_{4,2} & 0 & 0 & 0 \\ 0 & 0 & x_{5,3} & x_{5,4} & 0 \end{pmatrix},$$

$$\varphi V_+ = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ a_5 \end{pmatrix} \middle| \ a_i \in \mathbb{C} \right\}, \quad \varphi V_- = \left\{ \begin{pmatrix} 0 \\ 0 \\ b_3 \\ b_4 \\ 0 \end{pmatrix} \middle| \ b_i \in \mathbb{C} \right\}.$$

Calculate the kernel..

$$Dri(C) \begin{pmatrix} a_1 \\ a_2 \\ b_3 \\ b_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_{31} a_1 \\ x_{41} a_1 + x_{42} a_2 \\ x_{53} b_3 + x_{54} b_4 \end{pmatrix} =: v_1,$$

$$
\begin{aligned}
V/\ker(Dri(C)) &= \langle a'_{13}, a'_{12} \mid a'_{ij} \in V_+ \rangle \oplus \langle b'_{12} \mid b'_{12} \in V_- \rangle, \\
\ker(Dri(C)) &= \langle a'_{11} \mid a'_{11} \in V_+ \rangle \oplus \langle b'_{11} \mid b'_{11} \in V_- \rangle.
\end{aligned}
$$

We have $(\dim\ker(Dri(C))|_{V_+}, \dim\ker(Dri(C))|_{V_-}) = (1,1)$.

$$
Dri(C)\begin{pmatrix} 0 \\ 0 \\ x_{31}a_1 \\ x_{41}a_1 + x_{42}a_2 \\ x_{53}b_3 + x_{54}b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (x_{53}x_{31} + x_{54}x_{41})a_1 + x_{54}x_{42}a_2 \end{pmatrix} =: v_2,
$$

$$
\begin{aligned}
V/\ker(Dri(C))^2 &= \langle a'_{23} \mid a'_{23} \in V_+ \rangle, \\
\ker(Dri(C))^2 &= \langle a'_{21}, a'_{22} \mid a'_{ij} \in V_+ \rangle \oplus V_-.
\end{aligned}
$$

We have $(\dim\ker(Dri(C))^2|_{V_+}, \dim\ker(Dri(C))^2|_{V_-}) = (2,2)$.

$$
Dri(C)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (x_{53}x_{31} + x_{54}x_{41})a_1 + x_{54}x_{42}a_2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

$$
\begin{aligned}
V/\ker(Dri(C))^3 &= \{0\}, \\
\ker(Dri(C))^3 &= V_+ \oplus V_-.
\end{aligned}
$$

We have

$$
(\dim\ker(Dri(C))^3|_{V_+}, \dim\ker(Dri(C))^3|_{V_-}) = (3,2).
$$

Therefore the signed Young diagram for the clan C is

$$
\begin{array}{|c|c|c|}
\hline
+ & - & + \\
\hline
- & + & \multicolumn{1}{c}{} \\
\cline{1-2}
\end{array}\ .
$$

We can have an element A' of $\{Dri(C)\}$ as follows:

$$
A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.
$$

The (i,j)-entry of A' means $(\varphi(i), \varphi(j))$- entry of A that is an element of the image of the moment map. For example, 1 that is $(3,1)$-entry means $(\varphi(3), \varphi(1)) = (4,1)$-entry of A. That is

$$
A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.
$$

2 Type CII

In this section, we treat the type CII. For the type CII, we use group $G_{\mathbb{R}} = Sp(p, n-p)$. At first, we will change the realization of $U(2p, 2q)$.

2.1 Changing the realization of $U(2p, 2q)$

In order to include $Sp(p, n-p)$ to $U(2p, 2n-2p)$, we change the realization of $U(2p, 2n-2p)$. Let

$$U(2p, 2n-2p) = \left\{ g \in GL(2n, \mathbb{C}) \,\middle|\, {}^t g \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2n-2p} & 0 \\ 0 & 0 & I_p \end{pmatrix} \bar{g} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2n-2p} & 0 \\ 0 & 0 & I_p \end{pmatrix} \right\}.$$

Let mark all notation subscript AIII, if they are defined as the case of $G_{\mathbb{R}} = U(2p, 2n-2p)$. For example, $\mathfrak{b}_{\mathrm{AIII}}$ are the Borel subalgebra for $U(2p, 2n-2p)$. Fix a Cartan involution

$$\theta : x \to \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2n-2p} & 0 \\ 0 & 0 & I_p \end{pmatrix} x \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2n-2p} & 0 \\ 0 & 0 & I_p \end{pmatrix}.$$

Then

$$\mathfrak{k}_{\mathrm{AIII}} = \left\{ \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \,\middle|\, \begin{array}{l} K_{11},\ K_{13},\ K_{31},\ K_{33} \in \mathrm{Mat}(p,p) \\ K_{22} \in \mathrm{Mat}(2n-p, 2n-p) \end{array} \right\}$$

$$\mathfrak{p}_{\mathrm{AIII}} = \left\{ \begin{pmatrix} 0 & P_{12} & 0 \\ P_{21} & 0 & P_{23} \\ 0 & P_{32} & 0 \end{pmatrix} \,\middle|\, \begin{array}{l} P_{12}, P_{32} \in \mathrm{Mat}(p, 2n-2p) \\ P_{21}, P_{23} \in \mathrm{Mat}(2n-2p, p) \end{array} \right\}$$

and $\mathfrak{b}_{\mathrm{AIII}}$ is the set of upper triangular matrices. Under this realization, we have the following proposition.

Proposition 2.1.1 *Suppose C be a clan of $U(2p, 2n-2p)$ and $(c_1 \dots c_{2n})$ a signed clan of C. Then the matrix $g(C) = (g_1\ g_2\ \dots\ g_{2n})$ is a representative element of $\mathcal{O}_{C\,\mathrm{AIII}}$ Here g_i is the following column vector.*

- *If $c_i = +$ or $-$, then $g_i = e_{\varphi(i)}$.*
- *If $(c_i, c_j) = (c_i, c_j)$, then*

$$g_i = \frac{1}{\sqrt{2}}(e_{\varphi(i)} + e_{\varphi(j)}) \quad \text{and} \quad g_j = \frac{1}{\sqrt{2}}(-e_{\varphi(i)} + e_{\varphi(j)}).$$

Where, $\varphi \in \mathfrak{S}_n$ satisfy the following condition. If $c_i = +$ or a_+, then

$$1 \le \varphi(i) \le p \qquad \text{or} \qquad 2n-p+1 \le \varphi(i) \le 2n.$$

If $c_i = -$ or a_-, then

$$p+1 \le \varphi(i) \le 2n-p.$$

2.2 Generalized clans of the type CII

In this section, we define generalized clans for the type CII. They also parametrize K-orbits on the flag variety G/B. The generalized clans are improvement of Matsuki-Oshima's clans [MO]. The number of symbols of the generalized clan is the size of matrices of the element of group G. By using generalized clan , we can get a representative element of the K-orbits on G/B directly. The set of generalized clans of the type CII are subset of clans of the type AIII. For the type CII, we treat the case of $G_{\mathbb{R}} = Sp(p, n-p)$. We can parametrize K-orbits of $G_{\mathbb{R}} = Sp(p, n-p)$ by generalized clan which coincide the clan of K-orbit of $G_{\mathbb{R}} = U(2p, 2n-2p)$ which includes the K-orbit of $G_{\mathbb{R}} = Sp(p, n-p)$ by good realization of groups.

We realize $G = Sp(p, n-p)$ as the group of matrices g in $GL(2n, \mathbb{C})$ which leave invariant the Hermitian form

$$x_1\overline{x_1} + \cdots + x_p\overline{x_p} - x_{p+1}\overline{x_{p+1}} - \cdots - x_{2n-p}\overline{x_{2n-p}} + x_{2n-p+1}\overline{x_{2n-p+1}} + \cdots + x_{2n}\overline{x_{2n}}$$

and the exterior form

$$x_1 \wedge x_{2n} + x_2 \wedge x_{2n-1} + \cdots + x_n \wedge x_{n+1}.$$

i.e.

$$Sp(p, n-p) = \left\{ g \in GL(2n, \mathbb{C}) \;\middle|\; \begin{array}{c} {}^t g \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \\ {}^t g \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2n-2p} & 0 \\ 0 & 0 & I_p \end{pmatrix} \bar{g} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_{2n-2p} & 0 \\ 0 & 0 & I_p \end{pmatrix} \end{array} \right\}.$$

Then $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{b}$ satisfy

$$\mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}_{\mathrm{AIII}}, \qquad \mathfrak{p} = \mathfrak{g} \cap \mathfrak{p}_{\mathrm{AIII}}, \qquad \text{and} \qquad \mathfrak{b} = \mathfrak{g} \cap \mathfrak{b}_{\mathrm{AIII}}.$$

Definition 2.2.1 We define an order $\leq_c$. We say that $i \leq_c j$ if $0 < i \leq j$, $j \leq 0 \leq i$, or $j \leq i < 0$. In this order, we have $i <_c j < 0 <_c -j <_c -i$ for $0 < i < j$.

Definition 2.2.2 (generalized clan) We define a set $\mathcal{C}(Sp(p, n-p))$ of type CII. An element of $\mathcal{C}(Sp(p, n-p))$ is an equivalence class of an ordered set C of $2n$ symbols

$$C = (\, c_1\, c_2\, \ldots\, c_n\, c_{-n}\, \ldots\, c_{-2}\, c_{-1}\,).$$

satisfying the following five conditions.

1. Each symbols c_i is $c_i = +$, $-$ or an element of $\mathbb{N}$.

2. If $c_i = a \in \mathbb{N}$ then there exist unique $j \neq i$ with $a = c_j$:

$$\#\{\, i \mid c_i = a \,\} = 0 \text{ or } 2 \qquad \text{for any } a \in \mathbb{N}.$$

3. If $c_i = a > 1$, then there exists $c_j = a - 1$.

4. The difference of the number of the signature of the symbols in a symbols coincides with the difference of the number of the signature of the Hermitian form defining the group $G_{\mathbb{R}}$:

$$\#\{\, i \mid c_i = + \,\} - \#\{\, i \mid c_i = - \,\} = 2p - (2n - 2p) = 4p - 2n.$$

5. A symbols $C = (c_1 \ \dots \ c_{-1})$ is equivalent to $D = (d_1 \ \dots \ d_{-1})$ if there exists a permutation $\sigma \in \mathfrak{S}_m$, $m := \max\{\, c_i \in \mathbb{N} \,\}$ such that

$$c_i = \begin{cases} \sigma(d_i) & \text{if } d_i \in \mathbb{N} \\ + & \text{if } d_i = + \\ - & \text{if } d_i = - \end{cases}$$

for all $1 \leq i \leq n$. $(1\,1\ +\ +\,2\,2) = (2\,2\ +\ +\,1\,1)$ as generalized clans .

6. If $c_i = +$, then $c_{-i} = +$.
If $c_i = -$, then $c_{-i} = -$.
If $c_i = c_j \in \mathbb{N}$, then $c_{-i} = c_{-j} \in \mathbb{N}$.

We call an element of $\mathcal{C}(Sp(p, n-p))$ generalized clan.

Definition 2.2.3 (representative indication of a generalized clan) For a generalized clan $C = (c_1 \ \dots \ c_{-1})$, we define the representative indication $\tilde{C} = (\tilde{c}_1 \dots \tilde{c}_{-1})$ as follows. Suppose $\tilde{c}_i = \tilde{c}_j = a$ for $i <_c j$ and $\tilde{c}_s = \tilde{c}_t = b$ for $s <_c t$. If $i <_c s$, then $a < b$.

Remark 2.2.4 *The representative indication of a generalized clan*

$$C = (c_1 \dots c_{-1})$$

in the sense of Definition 2.2.3 is equal to the representative indication of the clan

$$(c_1 \dots c_{2n}) := (c_1 \ \dots \ c_{-1}) = C$$

in the sense of Definition 1.1.2.

Example 2.2.5 The set $\mathcal{C}(Sp(1,1))$ has four elements represented by

$$\left\{ \begin{matrix} + & - & - & + & , & - & + & + & - \\ 1 & 1 & 2 & 2 & , & 1 & 2 & 1 & 2 \end{matrix} \right\}.$$

Example 2.2.6 The set $\mathcal{C}(Sp(2,1))$ has nine elements represented by

$$\left\{ \begin{matrix} + & + & - & - & + & + & , & + & - & + & + & - & + & , & - & + & + & + & + & - \\ + & 1 & 1 & 2 & 2 & + & , & 1 & 1 & + & + & 2 & 2 & , & 1 & + & 1 & 2 & + & 2 \\ + & 1 & 2 & 1 & 2 & + & , & 1 & + & 2 & 1 & + & 2 & , & 1 & 2 & + & + & 1 & 2 \end{matrix} \right\}.$$

Remark 2.2.7 *The set $\mathcal{C}(Sp(p, n-p))$ is subset of $\mathcal{C}(U(2p, 2n-2p))$ and satisfies the following conditions.*

$$\mathcal{C}(Sp(p,n-p)) = \left\{ (c_1 \ldots c_{-1}) \in \mathcal{C}(U(2p, 2n-2p)) \;\middle|\; \begin{array}{l} \text{If } c_i = \pm \text{ then } c_{-i} = \pm. \\ \text{If } c_i = c_j \text{ then } c_{-i} = c_{-j}. \\ \text{If } c_i \in \mathbb{N} \text{ then } c_{-i} \neq c_i. \end{array} \right\}.$$

Remark 2.2.8 *Generalized clans are improvement of Matsuki-Oshima's clans.*

For explaining that generalized clans are improvement of Matsuki-Oshima's clan, we prepare some notations.

Let $\mathfrak{a}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ and Ψ be a simple root system of G:

$$\Psi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

For a simple root system Ψ, take the orthonormal bases $a_1^*, a_2^*, \ldots, a_n^*$ of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ such that

$$\begin{aligned} \alpha_i &= a_i^* - a_{i+1}^* \qquad \text{for } 1 \leq i \leq n-1, \\ \alpha_n &= 2a_n^*. \end{aligned}$$

Notation 2.2.9 Let $a_{-i}^* := -a_i^*$ for $1 \leq i \leq n$.

Since a Cartan involution θ induces an involution of root system, we have the following proposition. The following proposition leads Remark 2.2.8.

Proposition 2.2.10 *A pair $(\mathfrak{a}, \Psi)$ corresponds to a generalized clan*

$$C = (\, c_1\, c_2\, \ldots\, c_n\, c_{-n}\, \ldots\, c_{-2}\, c_{-1}\,)$$

satisfying the following four conditions.

1. *$\theta a_i^* = a_i^*$ if and only if $c_i = +$ or $-$.*
2. *Suppose $\theta a_i^* = a_i^*$ and $\theta a_j^* = a_j^*$. If the root $a_i^* - a_j^*$ is a compact root, then $c_i = c_j$. Then, if*

   $$\mathfrak{g}(\mathfrak{a}, a_i^* - a_j^*) \subset \mathfrak{sp}(p, \mathbb{C}) \oplus \{0\} \subset \mathfrak{sp}(p, \mathbb{C}) \oplus \mathfrak{sp}(n-p, \mathbb{C}) = \mathfrak{k},$$

   then $c_i = c_j = +$. If

   $$\mathfrak{g}(\mathfrak{a}, a_i^* - a_j^*) \subset \{0\} \oplus \mathfrak{sp}(n-p, \mathbb{C}) \subset \mathfrak{sp}(p, \mathbb{C}) \oplus \mathfrak{sp}(n-p, \mathbb{C}) = \mathfrak{k},$$

   then $c_i = c_j = -$.
3. *Suppose $\theta a_i^* = a_i^*$ and $\theta a_j^* = a_j^*$. If the root $a_i^* - a_j^*$ is a non-compact root, then $c_i \neq c_j$.*
4. *$\theta a_i^* = a_j^*$ $(i \neq j)$ if and only if $c_i = c_j = a$ for some $a \in \mathbb{N}$.*

2.3 M-O graph of type CII with generalized clans

In this section, we define a graph which has generalized clans as verteces and in which edge connecting two generalized clans is labelled by an element of $\{1,\ldots,n\}$. At first, we define a subset $\gamma(Sp(p,n-p))$ of

$$\{ (C,C',i) \mid C,C' \in \mathcal{C}(Sp(p,n-p)),\ 1 \leq i \leq n \}.$$

Definition 2.3.1 Let $\gamma(Sp(p,n-p))$ denote a set of all triples (C,C',i), $C,C' \in \mathcal{C}(Sp,(p,n-p))$, $1 \leq i \leq n$, satisfying the following conditions.

A clan $C=(c_1\ldots c_{-1}) \in \mathcal{C}(Sp(p,n-p))$ satisfy $c_i \neq c_{i+1}$ or $c_n \neq c_{-n}$,

If $c_n \neq c_{-n}$, then

$$C' = (c_1 \ldots c_{n-1}c_{-n}c_nc_{-n+1}\ldots c_{-1}).$$

If $(c_i,c_{i+1}) = (+,-)$ or $(-,+)$, then

$$C' = (c_1 \ldots c_{i-1}aac_{i+2}\ldots c_{-i-2}bbc_{-i+1}\ldots c_{-1})$$

If $(c_i,c_{i+1}) \neq (+,-)$ nor $(-,+)$, then

$$C' = (c_1 \ldots c_{i-1}c_{i+1}c_ic_{i+2}\ldots c_{-i-2}c_{-i}c_{-i-1}c_{-i+1}\ldots c_{-1}).$$

Definition 2.3.2 We define a subset $\Gamma_m(Sp(p,n-p))$ $(m \geq 1)$ of $\gamma(Sp(p,n-p))$ and a subset $\mathcal{C}_m(Sp(p,n-p))$ $(m \geq 0)$ of $\mathcal{C}(Sp(p,n-p))$ inductively as follows:

$$\mathcal{C}_0(Sp(p,n-p)) := \{ C=(c_1\ldots c_{-1}) \in \mathcal{C}(Sp(p,n-p)) \mid c_i = + \text{ or } - \text{ for all } i \}.$$

$$\Gamma_m(Sp(p,n-p)) := \bigcup_{C \in \mathcal{C}_{m-1}(Sp(p,n-p))} \left\{ \begin{array}{l} (C,C',i) \in \gamma(Sp(p,n-p)) \mid \\ C' \notin \mathcal{C}_{m'}(Sp(p,n-p)) \text{ for all } m' < m \end{array} \right\}$$

$$\mathcal{C}_m(Sp(p,n-p)) := \{ C' \mid (C,C',i) \in \Gamma_m(Sp(p,n-p))\} \quad (m \geq 1).$$

The *M-O graph* of $Sp(p,n-p)$ is a finite oriented graph whose verteces are $\mathcal{C}(Sp(p,n-p))$ and whole oriented edges are $\Gamma(Sp(p,n-p)) := \bigcup_{m\in\mathbb{N}} \Gamma_m(Sp(p,n-p))$.

The definition of M-O graph is deduce the following remark.

Remark 2.3.3 *We have the following conditions:*

$$\mathcal{C}(Sp(p,n-p)) = \bigsqcup_{m\geq 0} \mathcal{C}_m(Sp(p,n-p)),$$

Theorem 2.3.4 [MO] *The generalized clans of $Sp(p,n-p)$ parametrize K-orbits on the flag variety G/B. This parametrization satisfies the following property:*

Suppose generalized clans C and C' are parameter of orbits $\mathcal{O}$ and $\mathcal{O}'$ respectively. Then $(C,C',i) \in \Gamma(Sp(p,n-p))$ if and only if $\dim \mathcal{O}+1 = \dim \mathcal{O}'$ and $\pi_i(\mathcal{O}) = \pi_i(\mathcal{O}')$.

2.4 The function F_{CII}

In this section, we define a function $\mathcal{C}(Sp(p, n-p))$ to $\mathbb{N}$ which gives a natural number m such that the generalized clan belongs to $\mathcal{C}_m(Sp(p, n-p))$.

Definition 2.4.1 We define a function F_{CII} from $C(U(p, n-p))$ to $\mathbb{N}$ such that

$$F_{\mathrm{CII}}(C) = \frac{1}{2}(F(C) + \frac{1}{2}\#\{a \in \mathbb{N} \mid \tilde{c}_i = \tilde{c}_j = a,\ i <_c 0 <_c j\}). \tag{6}$$

Where F is the function which is defined in the Definition 1.2.1. Let $(\tilde{c}_1 \ldots \tilde{c}_{-1})$ is a representative indication of C and $F(C)$ is the value with regarding C as a clan of $U(2p, 2n-2p)$.

By definition, we get the following proposition.

Proposition 2.4.2 *If $(C, D, n) \in \Gamma(Sp(p, n-p))$, then $(C, D, n) \in \Gamma(U(p, n-p))$. If $(C, D, i) \in \Gamma(Sp(p, n-p))$ and $i \neq n$, then there exists a generalized clan $C' \in \mathcal{C}(U(2p, 2n-2p))$ such that both of (C, C', i) and $(C', D, 2n+1-i)$ are elements of $\Gamma(U(2p, 2n-2p))$.*

Proposition 2.4.3 *For generalized clan $C \in C(Sp(p, n-p))$, if $F_{CII}(C) = m$, then $C \in \mathcal{C}_m(Sp(p, n-p))$.*

Corollary 2.4.4 For $G_{\mathbb{R}} = Sp(p, n-p)$, $p \leq n-p$, a generalized clan

$$C' = (\underbrace{1\,2\,\ldots\,(2p)}_{2p}\,\underbrace{-\,\ldots\,-}_{2n-4p}\,\underbrace{(2p-1)\,(2p)\,\ldots\,5\,6\,3\,4\,1\,2}_{2p})$$

corresponds to open orbit and

$$F(C') = 2p(n-p).$$

Therefore we have

$$\mathrm{codim}\mathcal{O}_C = 2p(n-p) - F(C)$$

for all $C \in \mathcal{C}(Sp(p, n-p))$.

2.5 Representative element of K-orbit for generalized clans

Let B_i denote the parabolic subgroup of G for the root $-\alpha_i$ and all positive roots. Let $\pi_i : K\backslash G/B \to K\backslash G/B_i$ be the canonical projection.

Notation 2.5.1 For a generalized clan $C = (c'_1 \ldots c'_{-1})$, let $C_{\pm} = (c_1 \ldots c_{-1})$ be a signed clan such that $(c_i, c_j) = (a_+, a_-)$ if $c'_i = c'_j = a \in \mathbb{N}$ for $i <_c j$ and $|i| < |j|$, and $(c_i, c_j) = (a_-, a_+)$ if $c'_i = c'_j = a \in \mathbb{N}$ for $i <_c j$ and $|i| > |j|$,

we define a map $\varphi' : \{1, \ldots, n, -n, \ldots, -1\} \to \{1, \cdots, 2n\}$ satisfying the following conditions:

- If $i \neq j$, then $\varphi'(i) \neq \varphi'(j)$.
- φ' satisfies

$$\begin{array}{rcll} 1 & \leq \; \varphi'(i) \; \leq & p & \text{if } i <_c 0 \text{ and } c_i = + \text{ or } a_+, \\ p+1 & \leq \; \varphi'(i) \; \leq & n & \text{if } i \leq_c 0 \text{ and } c_i = - \\ & & & \text{or if } (c_i, c_j) = (a_-, a_+) \text{ for } j <_c 0, \\ \multicolumn{3}{l}{\varphi'(i) := 2n+1-\varphi'(-i)} & \text{otherwise.} \end{array}$$

Remark 2.5.2 *We can regard φ' as an element of $\mathfrak{S}_{2n}$ such that $\varphi'(2n+1-i) := \varphi'(-i)$ for $1 \leq i \leq n$. Then φ' satisfies the conditions of φ in Proposition 2.1.1.*

Theorem 2.5.3 *Suppose $C_\pm = (c_1 \dots c_{-1})$ be a signed generalized clan of $Sp(p, n-p)$. Then the matrix*

$$g(C) = (g_1\, g_2\, \dots\, g_{2n}) \in Mat(2n, 2n)$$

is a representative of $\mathcal{O}_C$. Here g_i is the following column vector.

- *If $c_i = \pm$, then $g_i = e_{\varphi'(i)}$.*
- *If $(c_i, c_j) = (a_+, a_-)$ for some $a \in \mathbb{N}$ and $i <_c 0$, then*

$$g_i = \frac{1}{\sqrt{2}}(e_{\varphi'(i)} + e_{\varphi'(j)}) \quad \text{and} \quad g_j = \frac{1}{\sqrt{2}}(-e_{\varphi'(i)} + e_{\varphi'(j)}).$$

- *If $(c_i, c_j) = (a_+, a_-)$ for some $a \in \mathbb{N}$ and $0 <_c i$, then*

$$g_i = \frac{1}{\sqrt{2}}(e_{\varphi'(i)} + e_{\varphi'(j)}) \quad \text{and} \quad g_j = \begin{cases} \frac{1}{\sqrt{2}}(e_{\varphi'(i)} - e_{\varphi'(j)}) & \text{if } j <_c 0 \\ \frac{1}{\sqrt{2}}(-e_{\varphi'(i)} + e_{\varphi'(j)}) & \text{if } 0 <_c j. \end{cases}$$

Where φ' satisfies the conditions in Notation 2.5.1.

Proposition 2.5.4 *The matrix $g(C)$ which we got in Theorem 2.5.3 is also a representative of K_{AIII}-orbit on the flag variety G_{AIII}/B_{AIII} which corresponds to the clan $(c_1 \dots c_{2n}) := (c_1 \dots c_{-1}) = C$.*

Remark 2.5.5 *If $C \in \mathcal{C}(Sp(p, n-p))$, then*

$$Kg(C)B \subset K_{AIII}g(C)B_{AIII}.$$

Therefore for C, $C' \in \mathcal{C}(Sp(p, n-p))$, if $C \neq C'$ then $Kg(C)B \cap Kg(C')B = \emptyset$.
Outline of proof of Theorem 2.5.3. If $(C, C', i) \in \Gamma(Sp(p, n-p))$, then

$$\begin{array}{ccc} K_{AIII}g(C)B_{AIII} & \cap & K_{AIII}g(C')B_{AIII} = \emptyset \\ \cup & & \cup \\ Kg(C)B & & Kg(C')B. \end{array}$$

Therefore

$$Kg(C)B \cap Kg(C')B = \emptyset.$$

$$\begin{array}{ccc} \pi_{i\mathrm{AIII}}\pi_{2n+1-i\mathrm{AIII}}(K_{\mathrm{AIII}}g(C)B_{\mathrm{AIII}}) & = & \pi_{i\mathrm{AIII}}\pi_{2n+1-i\mathrm{AIII}}(K_{\mathrm{AIII}}g(C')B_{\mathrm{AIII}}) \\ \cup & & \cup \\ \pi_i(Kg(C)B) & & \pi_i(Kg(C')B) \end{array}$$

We also have

$$\pi_{i\mathrm{AIII}}\pi_{2n+1-i\mathrm{AIII}}(K_{\mathrm{AIII}}g(C)B_{\mathrm{AIII}}) \cap \pi_{i\mathrm{AIII}}\pi_{2n+1-i\mathrm{AIII}}(K_{\mathrm{AIII}}g(D)B_{\mathrm{AIII}}) = \emptyset.$$

for $D \neq C, C'$.

If $(C, C', i) \in \Gamma(Sp(p, n-p))$ $i \neq n$, then there exist just one K-orbit which is not $Kg(C)B$ and the projection of which by π_i is equal to $\pi_i(Kg(C)B)$. It should be $\pi_i(Kg(C')B)$:

$$\pi_i(Kg(C)B) = \pi_i(Kg(C')B).$$

Now we know that one of the following two conditions is satisfied:

$$Kg(C)B \subset \overline{Kg(C')B}, \tag{7}$$

$$Kg(C')B \subset \overline{Kg(C)B}. \tag{8}$$

By the condition

$$K_{\mathrm{AIII}}g(C)B_{\mathrm{AIII}} \subset \overline{K_{\mathrm{AIII}}g(C')B_{\mathrm{AIII}}}, \tag{9}$$

we get condition (7).

2.6 Driving matrix for $Sp(p, n-p)$

In this section we get an algorithm to get the image of the moment map of a fiber of the conormal bundle of the K-orbit which corresponds to each generalized clan .

Suppose $C = (c_1 \ldots c_{-1})$ is an element of $\mathcal{C}(Sp(p, n-p))$. From now on, the signed clan $C_\pm$ can be regarded as generalized clan C. Because we define $C_\pm$ uniquely, the matrix $g(C)$ is the representative element which is given in Theorem 2.5.3. we describe the algorithm to get the image of the moment map $g(C)\mathfrak{b}^\perp g(C)^{-1} \cap \mathfrak{p}$. Where $\mathfrak{b}^\perp$ is orthogonal of $\mathfrak{b}$. Because

$$\begin{aligned} g(C)\mathfrak{b}^\perp g(C)^{-1} \cap \mathfrak{p} &= g(C)(\mathfrak{b}^\perp_{\mathrm{AIII}} \cap \mathfrak{g})g(C)^{-1} \cap (\mathfrak{p}_{\mathrm{AIII}} \cap \mathfrak{g}) \\ &= (g(C)\mathfrak{b}^\perp_{\mathrm{AIII}}g(C)^{-1} \cap \mathfrak{p}_{\mathrm{AIII}}) \cap \mathfrak{g} \end{aligned}$$

and because we get $g(C)\mathfrak{b}^\perp_{\mathrm{AIII}}g(C)^{-1} \cap \mathfrak{p}_{\mathrm{AIII}}$ in Theorem 1.4.2, we get the following definition and theorem.

Definition 2.6.1 Give a $2n \times 2n$-matrix $Dri(C)$ as follows.

As (j, i)-entry, $Dri(C)$ has $x_{s,t}$ some $1 \leq s, t \leq n$ or zero as the following steps.

- For $i,j \leq n$ $(1 \leq_c i,j <_c 0)$, (j,i)-entry is x_{ji} and $(2n+1-i, 2n+1-j)$-entry is $-x_{j,i}$ if (c_i, c_j) satisfies one of the following conditions.

  1. $(c_i, c_j) = (\pm, \mp)$, $i < j$.
  2. $(c_i, c_j) = (\pm, a_\mp)$ with $c_s = a_\pm$ for some $i <_c \min_c(j,s)$.
  3. $(c_i, c_j) = (a_\pm, \mp)$ with $c_s = a_\mp$ for some $\max_c(i,s) <_c j$.
  4. $(c_i, c_j) = (a_\pm, b_\mp)$ with $(c_s, c_t) = (a_\mp, b_\pm)$ for some $\max_c(i,s) <_c \min_c(j,t)$.

- For $i,j \leq n$ $(1 \leq_c i \leq_c -j \leq_c j)$, (j,i)-entry and $(2n+1-j, 2n+1-i)$-entry are $x_{j,i}$ if (c_i, c_j) satisfies one of the following conditions.

  1. $(c_i, c_j) = (\pm, \mp)$, $i < j$.
  2. $(c_i, c_j) = (\pm, a_\mp)$ with $c_s = a_\pm$ for some $i <_c \min_c(j,s)$.
  3. $(c_i, c_j) = (a_\pm, \mp)$ with $c_s = a_\mp$ for some $\max_c(i,s) <_c j$.
  4. $(c_i, c_j) = (a_\pm, b_\mp)$ with $(c_s, c_t) = (a_\mp, b_\pm)$ for some $\max_c(i,s) <_c \min_c(j,t)$.

- For $1 \leq i,j \leq n$ $(1 \leq_c i <_c j <_c 0)$, (j,i)-entry and $(2n+1-s, 2n+1-t)$-entry are $x_{j,i}$. (t,s)-entry and $(2n+1-i, 2n+1-j)$-entry are $-x_{j,i}$ if (c_i, c_j) satisfies the following condition.

  1. $(c_i, c_j) = (a_+, b_-)$ with $(c_s, c_t) = (a_-, b_+)$ for $i <_c \min_c(j,t) <_c s <_c \max_c(j,t)$.

- For $1 \leq i,j,s,t \leq n$ $(i <_c t <_c -j <_c -s <_c n)$, (j,i)-entry and $(2n+1-i, 2n+1-j)$-entry are $x_{j,i}$. (t,s)-entry and $(2n+1-s, 2n+1-t)$-entry are $-x_{j,i}$ if (c_i, c_j) satisfies the following condition.

  1. $(c_i, c_j) = (a_+, b_-)$ with $(c_s, c_t) = (a_-, b_+)$ for $i <_c \min_c(j,t) <_c s <_c \max_c(j,t)$.

- Other entries are zero.

Where, $\max_c$ and $\min_c$ gives a maximal number and a minimal number by order $<_c$ respectively.

We call $Dri(C)$ the *driving matrix* of C for $Sp(p, n-p)$.

Theorem 2.6.2 *For a generalized clan $C \in \mathcal{C}(Sp(p, n-p))$, the image of the moment map is as follows:*

$$\begin{aligned} g(C)\mathfrak{b}^\perp g(C)^{-1} \cap \mathfrak{p} &= \{\varphi' Dri(C)^t \varphi' \mid x_{ij} \in \mathbb{C}\} & (10) \\ &= \{Dri(C)^t_{(\varphi^{-1}(1), \ldots, \varphi^{-1}(-1))} \mid x_{ij} \in \mathbb{C}\}. & (11) \end{aligned}$$

That is we can regard $Dri(C)$ as an element of

$$(g(C)\mathfrak{b}^\perp g(C)^{-1} \cap \mathfrak{p})_{(\varphi'(1), \ldots, \varphi'(-1))},$$

$$Dri(C)_{ij} = (g(C)\mathfrak{b}^\perp g(C)^{-1} \cap \mathfrak{p})_{\varphi'(i)\varphi'(j)}.$$

If $C \in \mathcal{C}_0(Sp,(p,n-p))$, then $g(C) = (e_{\varphi'(1)} \dots e_{-\varphi'(1)})$. Therefore we have the following lemma.

Lemma 2.6.3 If a generalized clan C corresponds to closed orbit, then

$$\{\, Dri(C) \mid x_{ij} \in \mathbb{C}\,\} = \mathfrak{b}^{\perp} \cap {}^{t}g(C)\,\mathfrak{p}\,g(C).$$

Example 2.6.4 For $C = (-++-)$

$$\left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ x_{2,1} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 \\ 0 & x_{3,1} & -x_{2,1} & 0 \end{pmatrix} \right| x_{ij} \in \mathbb{C} \right\} = \mathfrak{b} \cap ({}^{t}g(C)\mathfrak{p}g(C)).$$

Definition 2.6.5 Let V_+ be a $2p$-dimensional vector space spanned by

$$\{e_1, \dots, e_p, e_{2n+1-p}, \dots, e_{2n}\}.$$

Let $V_- \subset \mathbb{C}^n$ be a $(2n-2p)$-dimensional vector subspace of $\mathbb{C}^n$ spanned by

$$\{e_{p+1}, \dots, e_{2n-p}\}.$$

Theorem 2.6.6 For $G_{\mathbb{R}} = Sp(p, n-p)$ nilpotent orbits in $\mathfrak{p}$ are parametrized by signed Young diagram which has $2p$ boxes labeled $+$ and $2(n-p)$ boxes labeled $-$.

Definition 2.6.7 The signed Young diagram of a nilpotent orbit is as follows. Take a regular element A of the orbit.

1. The number of boxes which are in a j-column $(\forall j \leq i)$ and labeled $+$ is $\dim(\ker(A^i|_{V_+}))$.

2. The number of boxes which are in a j-column $(\forall j \leq i)$ and labeled $-$ is $\dim(\ker(A^i|_{V_-}))$.

Proposition 2.6.8 *Let*

$$\begin{aligned} \varphi' V_+ &:= (e_{\varphi'(1)} \dots e_{\varphi'(2n)})^{-1} V_+ = \langle e_{\varphi'^{-1}(1)}, \dots, e_{\varphi'^{-1}(p)}, e_{\varphi'^{-1}(2n-p+1)}, e_{\varphi'^{-1}(2n)} \rangle \\ \varphi' V_- &:= (e_{\varphi'(1)} \dots e_{\varphi'(-1)})^{-1} V_- = \langle e_{\varphi'^{-1}(p+1)}, \dots, e_{\varphi'^{-1}(2n-p)} \rangle \end{aligned}$$

then,

$$\begin{aligned} \dim(\ker((Dri(C))^i|_{\varphi' V_+})) &= \dim(\ker(A^i|_{V_+})), \\ \dim(\ker((Dri(C))^i|_{\varphi' V_-})) &= \dim(\ker(A^i|_{V_-})). \end{aligned}$$

Where $e_{-i} := e_{2n+1-i}$ *for* $0 < i$.

Corollary 2.6.9 By definition, we have

$$\varphi' V_+ = \langle\, e_i \mid c_i = + \ \text{ or } \ a_+ \ \text{ for some } a \in \mathbb{N} \,\rangle,$$
$$\varphi' V_- = \langle\, e_i \mid c_i = - \ \text{ or } \ a_- \ \text{ for some } a \in \mathbb{N} \,\rangle.$$

Example 2.6.10 For generalized clan $C = (1\,1 + + 2\,2)$, the signed clan is $(1_+ 1_- + + 2_- 2_+)$ we get φ', a driving matrix $Dri(C)$, and two vector spaces V_+ and V_- as follows:

$$\varphi' = \begin{pmatrix} 1 & 2 & 3 & -3 & -2 & -1 \\ 1 & 3 & 2 & 5 & 4 & 6 \end{pmatrix},$$

$$Dri(C) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{3,2} & 0 & 0 & 0 & 0 \\ 0 & x_{4,2} & 0 & 0 & 0 & 0 \\ x_{5,1} & 0 & x_{4,2} & -x_{3,2} & 0 & 0 \\ 0 & x_{5,1} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\varphi' V_+ = \left\{ \begin{pmatrix} a_1 \\ 0 \\ a_3 \\ a_4 \\ 0 \\ a_6 \end{pmatrix} \middle|\; a_i \in \mathbb{C} \right\}, \quad \varphi' V_- = \left\{ \begin{pmatrix} 0 \\ b_2 \\ 0 \\ 0 \\ b_5 \\ 0 \end{pmatrix} \middle|\; b_i \in \mathbb{C} \right\}.$$

Calculate the kernel.

$$Dri(C) \begin{pmatrix} a_1 \\ b_2 \\ a_3 \\ a_4 \\ b_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_{32}b_2 \\ x_{42}b_2 \\ x_{51}a_1 + x_{42}a_3 - x_{32}a_4 \\ x_{51}b_2 \end{pmatrix} =: v_1,$$

$$V/\ker(Dri(C)) \ = \langle a'_{14} \,|\, a'_{14} \in V_+ \rangle \oplus \langle b'_{12} \,|\, b'_{12} \in V_- \rangle$$

$$\ker(Dri(C)) \ = \langle a'_{11}, a'_{12}, a'_{13} \,|\, a'_{ij} \in V_+ \rangle \oplus \langle b'_{11} \,|\, b'_{11} \in V_- \rangle.$$

We have $(\dim\ker(Dri(C))|_{V_+}, \dim\ker(Dri(C))|_{V_-}) = (3, 1)$.

$$Dri(C) \begin{pmatrix} 0 \\ 0 \\ x_{32}b_2 \\ x_{42}b_2 \\ x_{51}a_1 + x_{42}a_3 - x_{32}a_4 \\ x_{51}b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} =: v_2,$$

$$
\begin{aligned}
V/\ker(Dri(C))^2 &= \{0\} \\
\ker(Dri(C))^2 &= V_+ \oplus V_-
\end{aligned}
$$

We have

$$(\dim \ker(Dri(C))^2|_{V_+}, \dim \ker(Dri(C))^2|_{V_-}) = (4, 2).$$

Therefore the signed Young diagram for the generalized clan C is

$$
\begin{array}{|c|c|}
\hline
- & + \\
\hline
+ & - \\
\hline
+ \\
\cline{1-1}
+ \\
\cline{1-1}
\end{array}
\quad .
$$

References

[BB] W. Borho and J. Brylinski, *Differential operators on homogeneous spaces. III,* Invent. Math., **80** (1985), 1–68.

[CM] D. Collingwood and W. McGovern, *Nilpotent orbits in semisimple Lie algebras,* Van Nostrand Reinhold, New York (1993).

[G] D. Garfinkle, *The annihilators of irreducible Harish-Chandra modules for $SU(p,q)$ and other type A_{n-1} groups,* Amer. J. Math, **115** (1993), 305–369.

[M] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups,* J. Math. Soc. Japan **31** (1979), 331-357.

[MO] T. Matsuki and T. Oshima, *Embeddings of discrete series into principal series,* The orbit Method in Representation theory, Progress in Math. **82** (1988), Birkhäuser, 147-175.

[V] D. Vogan, *Irreducible characters of semisimple Lie groups III,* Invent. Math., **71** (1983), 381–417.

[Y1] A. Yamamoto, *Leading exponents of matrix coefficients of Harish-Chandra module and Matsuki-Oshima diagram,* (1993), Master thesis, Univ. of Tokyo, Japan.

[Y2] A. Yamamoto, $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$*-orbits on the flag variety of* $GL(p+q,\mathbb{C})$ *and the image of the moment map* (1993), Master thesis, Univ. of Tokyo, Japan, (In Japanese).