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Boundary value problems on Hermitian symmetric spaces

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Abstract

In this article the images of the Poisson transform on the degenerate series representations on various boundaries of a Hermitian symmetric spaces are considered. We characterize the images by means of differential equations.

1 Introduction

Throughout this article we consider representations of non-compact real semisimple Lie groups. Let G denote a non-compact real semisimple Lie group with finite center and K be a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space of the non-compact type.

1.1 Backgrounds

Studies on realizations of representations and those of intertwining operators are important theme in representation theory and harmonic analysis on homogeneous spaces. In several cases representations are realized on solution spaces of differential equations or images of intertwining operators are characterized by differential equations. These differential equations naturally come from left ideals of the universal enveloping algebra of the complexification of the Lie algebra of G.

I give some examples;

(1) **Realization of discrete series.** Schmid [19] realized a discrete series representation as a kernel of a first order differential operator between certain vector bundles over G/K.

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(2) **Szegö mapping.** Knapp and Wallach [14] realized a discrete series representation as the image of the Szegö mapping from a principal series representation to smooth sections of a vector bundle over G/K. The image is contained in the kernel of the Schmid operator. Barchini [1] studied Szegö mapping from the cohomologically induced representations to smooth sections of a vector bundles over G/K.

(3) **Realization of irreducible highest weight modules.** Davidson, Enright, and Stanke [3] realized an irreducible unitary highest weight module as the kernel of homogeneous differential operators between certain homogeneous vector bundles over G/K. Their result can be considered as a generalization of the result of Schmid. Davidson and Stanke [4] proved that the representation is characterized as the image of the Szegö mapping from a principal series representation to a vector bundle over G/K.

(4) **the Penrose transform.** Several people studied the Penrose transform from representations that are realized on Dolbeau cohomology spaces to smooth sections of vector bundles on G/K (cf. [13]). In a case, when the representation is an irreducible unitary highest weight module, H. Sekiguchi [20] showed that the image of the Penrose transform is characterized as the kernel of a homogeneous differential operator between certain vector bundles over G/K.

(5) **the Radon transform.** Characterizations of the images of the Radon transform by differential equations have been studied by Gonzalez, Ishikawa [10], and Oshima [17].

(6) **Helgason's conjecture.** For generic parameter, the Poisson transform gives an isomorphism of a space of hyperfunction sections of spherical principal series representation to a joint eigenspace of invariant differential operators. This fact was conjectures by Helgason [6] and proved by Kashiwara et al [12].

(7) **the Hua operators.** Hua [8] defined the Poisson integrals of functions on the Shilov boundary of a bounded symmetric domain of tube type and showed that the Poisson integrals belong to the kernel of certain second order differential operators (the Hua operators). Conversely, Johnson and Korányi [11] showed that the Hua operators characterize the image of the Poisson integrals.

(8) **Hypergeometric functions.** Radial parts of differential equations in examples above give interesting classes of holonomic systems of differential equations (cf. [5], [7], [9], [15], [17], [18], [26]).

1.2 Problem

There are several interesting connections between (1)-(8). In this article we mainly consider the Poisson transform and the Hua operators ((6) and (7) above).

Let P_E be a standard parabolic subgroup of G and consider the spaces of hyperfunction sections of degenerate series representations on the boundary component G/P_E of G/K. The Poisson transform gives an intertwining operator from a degenerate series representation to the space of functions on G/K. It is known that the image can be characterized by differential equations (cf. [18]). We consider the following problem;

Determine explicitly the differential equations that characterize the image of the Poisson transform from the boundary G/P_E.

If $P_E = P$, a minimal parabolic subgroup, and the parameter is generic, then the answer is given by (6).

If G/P_E is the Shilov boundary of a bounded symmetric domain of tube type and the Poisson integrals are harmonic functions, then the answer is given by Johnson and Korányi [11] and Lassalle [16] ((7) above).

In this article we consider this problem for boundaries of Hermitian symmetric spaces. We also consider generalizations to homogeneous line bundles over G/K. Though several statements hold for general Hermitian symmetric spaces G/K and for the universal covering group of G, we explain the results for $G = Sp(n, \mathbb{R})$ and give remarks for general cases.

When G is a real form of $GL(n, \mathbb{C})$ the answer to the above problem and some related topics are studied by Oshima [17], [18].

2 Notation and preliminary results

2.1 Siegel's upper half plane

Let n be an integer greater than 1. Let

$$G = Sp(n, \mathbb{R}) = \left\{ g \in SL(2n, \mathbb{R})\, ;\, {}^t g J g = J \right\},$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and I_n be the $n \times n$ identity matrix. The group $K = O(2n) \cap Sp(n, \mathbb{R})$ is a maximal compact subgroup of G, which is isomorphic to $U(n)$ by

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K \mapsto a + \sqrt{-1} b \in U(n).$$

We often identify K with $U(n)$ by this isomorphism. Let θ denote the corresponding Cartan involution of G, which is given by $\theta(g) = {}^t g^{-1}$.

Let
$$H_n = \{z \in M_n(\mathbb{C})\,;\, {}^t z = z,\ \mathrm{Im} z > 0\}.$$
Each $g \in G$ gives rise to a holomorphic diffeomorphism
$$z \mapsto (az+b)(cz+d)^{-1}$$
of H_n, where
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
The isotropy subgroup at $\sqrt{-1}I_n$ is K and we have $H_n \simeq G/K$.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. We denote the differential of θ also by θ and let $\mathfrak{p}$ be the -1-eigenspace of θ in $\mathfrak{g}$.

For $l \in \mathbb{Z}$ let τ_l denote the one-dimensional representation of $U(n)$ given by $\tau_l(x) = (\det x)^l$ $(x \in U(n))$ and we denote the corresponding representations of K and $\mathfrak{k}$ by the same notation.

2.2 Root systems

Let E_{ij} denote the $n \times n$ matrix with (i,j)-entry 1 and all other entries being 0, and put $E_i = E_{ii}$ for simplicity. We choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{u}(n) \simeq \mathfrak{k}$ to be the set of diagonal matrices. We define $\varepsilon_i \in \mathfrak{t}_c^*$ by $\varepsilon_i(E_j) = \delta_{ij}$ $(1 \leq i,j \leq n)$. Let Δ denote the root system of $(\mathfrak{g}_c, \mathfrak{t}_c)$ and Δ^+ be the positive system of Δ given by
$$\Delta^+ = \{2\varepsilon_i, \varepsilon_j \pm \varepsilon_k\,;\, 1 \leq i \leq n, 1 \leq j < k \leq n\}.$$
Let Δ_c (resp. Δ_n) denote the set of compact (resp. non-compact) roots. Let $\gamma_i = 2\varepsilon_i$ for $1 \leq i \leq n$. Then $\{\gamma_1, \ldots, \gamma_n\}$ is the maximal set of strongly orthogonal noncompact positive roots. For $\gamma \in \Delta$ let $\mathfrak{g}_\gamma \subset \mathfrak{g}_c$ denote the root space for γ. Let $\mathfrak{p}^\pm = \sum_{\gamma \in \Delta_n^+} \mathfrak{g}_{\pm\gamma}$, where $\Delta_n^+ = \Delta^+ \cap \Delta_n$. Then we have
$$\mathfrak{p}^\pm = \left\{ \begin{pmatrix} X & \pm iX \\ \pm iX & -X \end{pmatrix}\,;\, {}^t X = X \in \mathfrak{gl}(n, \mathbb{C}) \right\},$$
$\mathfrak{p}_c = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] = 0$, and $[\mathfrak{p}^+, \mathfrak{p}^-] = \mathfrak{k}_c$.

We put
$$X_i = \begin{pmatrix} E_i & 0 \\ 0 & -E_i \end{pmatrix} \in \mathfrak{p} \quad (1 \leq i \leq n)$$
and $\mathfrak{a} = \sum_{i=1}^n \mathbb{R} X_i$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. We put $X_0 = X_1 + \cdots + X_r$. Let e_i $(1 \leq i \leq n)$ be the linear form on $\mathfrak{a}$ given by $e_i(X_j) = \delta_{ij}$.

Let Σ denote the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$ and Σ^+ be the positive system of Σ given by

$$\Sigma^+ = \{2e_i, e_j \pm e_k\,;\, 1 \le i \le n, 1 \le j < k \le n\}.$$

Let W denote the Weyl group of Σ. Put $\alpha_i = e_i - e_{i+1}\,(1 \le i \le n-1)$ and $\alpha_n = 2e_n$. Then the set of simple roots is

$$\Psi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Let $\{H_1, \ldots, H_n\}$ be the basis of $\mathfrak{a}$ that is dual to Ψ. For $\alpha \in \Sigma$ let $\mathfrak{g}^\alpha \subset \mathfrak{g}$ be the root space for α. For $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ we have $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Sigma$. We put $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} \alpha$. For any $\lambda \in \mathfrak{a}_c^*$ let A_λ be the element of $\mathfrak{a}_c$ determined by $B(H, A_\lambda) = \lambda(H)$ for all $H \in \mathfrak{a}$, where B denotes the Killing form of $\mathfrak{g}_c$. For λ, $\mu \in \mathfrak{a}_c^*$ we put $\langle\lambda, \mu\rangle = B(A_\lambda, A_\mu)$. Since $\{e_1, \ldots, e_n\}$ forms a basis of $\mathfrak{a}^*$, any $\lambda \in \mathfrak{a}_c^*$ can be written $\lambda = \sum_{i=1}^n \lambda_i e_i$ $(\lambda_i \in \mathbb{C})$. We identify $\mathfrak{a}_c^*$ with $\mathbb{C}^n$ by $\lambda \mapsto (\lambda_1, \ldots, \lambda_n)$. In this identification, $\rho = (n, n-1, \ldots, 1)$.

2.3 Parabolic subgroups

Let A be the analytic subgroups of G corresponding to $\mathfrak{a}$. Let $\mathfrak{n}^+ = \sum_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$ and $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$. Let N^+ and N^- be the corresponding analytic subgroups of G. Let M be the centralizer of $\mathfrak{a}$ in K. Then $P = MAN^+$ is a minimal parabolic subgroup of G.

For each subset $E \subset \Psi$ let P_E be the corresponding standard parabolic subgroup of G with the Langlands decomposition $P_E = M_E A_E N_E^+$ such that $A_E \subset A$. Let $\mathfrak{a}(E)$ be the orthogonal complement of $\mathfrak{a}_E = \mathrm{Lie}(A_E)$ in $\mathfrak{a}$ with respect to B. Let $\langle E\rangle$ be the set of linear spans of E in Σ, and W_E be the subgroup of W that is generated by the reflections with respect to the roots in E. Throughout this article we consider the following subsets of Ψ and corresponding parabolic subgroups of G;

$$\Xi = \Psi \setminus \{\alpha_n\},$$
$$\Theta_k = \{\alpha_i\,;\, n-k+1 \le i \le n\}\ (k = 0, \ldots, n).$$

Then we have

$$\mathfrak{a}_\Xi = \mathbb{R}X_0, \qquad \mathfrak{a}(\Xi) = \sum_{i=1}^{n-1} \mathbb{R}(X_i - X_{i+1}),$$

$$\mathfrak{a}_{\Theta_k} = \textstyle\sum_{i=1}^{n-k} \mathbb{R}X_i, \qquad \mathfrak{a}(\Theta_k) = \displaystyle\sum_{i=n-k+1}^{n} \mathbb{R}X_i,$$

and $M_\Xi \simeq GL(n,\mathbb{R})$, $M_{\Theta_k} \simeq \{\pm 1\}^{n-k} \times Sp(k,\mathbb{R})$. The space $G/P \simeq K/M$ is the Furstenberg boundary and G/P_Ξ is the Shilov boundary of G/K. The set of $n\times n$ real symmetric matrices is an open dense subset of the Shilov boundary. Notice that the pairs $(K, K\cap M_\Xi) \simeq (U(n), O(n))$ and $(K, K\cap M_{\Theta_{n-1}}) \simeq (U(n), \{\pm 1\}\times U(n-1))$ are Gelfand pairs.

2.4 Eigenspaces of invariant differential operators

We review the main result of [22], which gives a characterization of the image of the Poisson transform.

For a real analytic manifold X we denote by $\mathcal{B}(X)$, $\mathcal{A}(X)$ and $C^\infty(X)$ the space of all hyperfunctions, analytic functions and smooth functions on X respectively. Let $\lambda \in \mathfrak{a}_c^*$ and $l \in \mathbb{Z}$. For $\mathcal{F} = \mathcal{B}$, $\mathcal{A}$ or C^∞, we define

$$\mathcal{F}(G/P, L_{\lambda,l}) = \{f \in \mathcal{F}(G)\,;\, f(gman) = e^{(\lambda-\rho)(\log a)}\tau_l(m)^{-1}f(g)$$
$$g \in G,\ m \in M,\ a \in A,\ n \in N^+\}$$

and

$$\mathcal{F}(G/K;\ \tau_l) = \{u \in \mathcal{F}(G)\,;\, u(gk) = \tau_l(k)^{-1}u(g) \text{ for any } g \in G,\ k \in K\}.$$

We define the Poisson transform $\mathcal{P}_{\lambda,l}$ by

$$\mathcal{P}_{\lambda,l}f(g) = \int_K f(gk)\tau_l(k)dk, \quad f \in \mathcal{B}(G/P; L_{\lambda,l}),$$

where dk denotes the invariant measure on K with total measure 1.

Let $\mathbb{D}_l(G/K)$ be the algebra of invariant differential operators on $\mathcal{B}(G/K;\ \tau_l)$. We have the Harish-Chandra isomorphism

$$\gamma_l\ :\ \mathbb{D}(G/K) \xrightarrow{\sim} S(\mathfrak{a}_c)^W,$$

where $S(\mathfrak{a}_c)^W$ denote the set of W-invariant elements in the symmetric algebra $S(\mathfrak{a}_c)$. Let $\mathcal{A}(G/K, \mathcal{M}_{\lambda,l})$ denote the space of all functions in $\mathcal{A}(G/K, \tau_l)$ satisfying the system of differential equations,

$$\mathcal{M}_{\lambda,l}\ :\ Du = \gamma_l(D)(\lambda)u, \quad D \in \mathbb{D}_l(G/K).$$

Put

$$e_{\lambda,l}^{-1} = \prod_{1\le j<k\le n} \Gamma\left(\tfrac{1}{2}(1+\lambda_j+\lambda_k)\right)\Gamma\left(\tfrac{1}{2}(1+\lambda_j-\lambda_k)\right)$$
$$\times \prod_{1\le i\le n} \Gamma\left(\tfrac{1}{2}(\lambda_i+1+l)\right)\Gamma\left(\tfrac{1}{2}(\lambda_i+1-l)\right).$$

Theorem 2.1 [22] *If $\lambda \in \mathfrak{a}_c^*$ and $l \in \mathbb{Z}$ satisfy the conditions*

$$-2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \{1, 2, 3, \dots\} \quad \text{for all } \alpha \in \Sigma^+ \tag{2.1}$$

and

$$e_{\lambda,l} \neq 0, \tag{2.2}$$

then the Poisson transform $\mathcal{P}_{\lambda,l}$ is a bijection of $\mathcal{B}(G/P; L_{\lambda,l})$ onto $\mathcal{A}(G/K, \mathcal{M}_{\lambda,l})$.

Under the conditions of the above theorem, the inverse of the Poisson transform is given by the boundary value map up to a non-zero constant multiple.

2.5 the partial Poisson transforms

We recall the partial Poisson transforms (cf. [22]). Let E be a subset of Ψ. Put $\rho_E = \rho|_{\mathfrak{a}_E}$. Let $\lambda \in \mathfrak{a}_c^*$ and put $\nu = \lambda|_{\mathfrak{a}_E}$. We define

$$\begin{aligned}\mathcal{B}(G/B_E; L_{\nu,l}) = \{f \in \mathcal{B}(G)\,;\, f(gman) = \tau_l(m)^{-1} e^{(\nu-\rho_E)(\log a)} f(g),\\ g \in G,\, m \in K \cap M_E,\, a \in A_E,\, n \in N_E^+\}.\end{aligned}$$

The algebra $\mathbb{D}_l(M_E/K \cap M_E)$ acts from the right on $\mathcal{B}(G/B_E; L_{\nu,l})$. Let $\mathcal{B}(E; \lambda, l)$ denote the subspace of $\mathcal{B}(G/B_E; L_{\nu,l})$ consisting of solutions of the system

$$\mathcal{M}_{\lambda|\mathfrak{a}(E),l} \,:\, Df = \gamma_l^E(D)(\lambda|\mathfrak{a}(E))f, \quad D \in \mathbb{D}_l(M_E/K \cap M_E).$$

For $f \in \mathcal{B}(G/P; L_{\lambda,l})$ we define

$$\mathcal{P}^E f(x) = \int_{K \cap M_E} f(xk)\tau_l(k)dk.$$

The image of $\mathcal{P}^E$ is contained in $\mathcal{B}(E; \lambda, l)$. For $f \in \mathcal{B}(E; \lambda, l)$ we define

$$\mathcal{P}_E f(x) = \int_K f(xk)\tau_l(k)dk.$$

We have $\mathcal{P}_{\lambda,l} = \mathcal{P}_E \circ \mathcal{P}^E$.

We put $s_{w,i} = \langle \rho - w\lambda, H_i \rangle$ and $s_{\bar{w}} = (s_{w,i})_{\alpha_i \notin E}$, where $\bar{w} \in W_E \backslash W$ and w is a representative of $\bar{w}$ in W. Under the assumption

$$\frac{1}{2}(s_{\bar{1}} - s_{\bar{w}}) \notin \{0, 1, 2, \dots\}^k \text{ for all } \bar{w} \in W_E \backslash W \text{ with } \bar{w} \neq \bar{1}, \tag{2.3}$$

we can define boundary value map $\beta_{E,\bar{1},l}$ from $\mathcal{A}(G/K, \mathcal{M}_{\lambda,l})$ to $\mathcal{B}(E;\lambda,l)$, which satisfies

$$\beta_{E,\bar{1},l} \circ \mathcal{P}_E = \mathbf{c}^E(\lambda,l)\mathrm{id}. \tag{2.4}$$

Here $\mathbf{c}^E(\lambda,l)$ is the partial c-function given by

$$\mathbf{c}^E(\lambda,l) = \prod_{\alpha\in\Sigma^+\setminus\langle E\rangle} \mathbf{c}_\alpha(\lambda,l), \tag{2.5}$$

where

$$\mathbf{c}_{e_i\pm e_j}(\lambda,l) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}(\lambda_i\pm\lambda_j)\right)}{\Gamma\left(\frac{1}{2}(1+\lambda_i\pm\lambda_j)\right)}, \quad (1\le i<j\le n)$$

and

$$\mathbf{c}_{2e_j}(\lambda,l) = \frac{2^{1-\lambda_j}\Gamma(\lambda_j)}{\Gamma\left(\frac{1}{2}(1+\lambda_j+l)\right)\Gamma\left(\frac{1}{2}(1+\lambda_j-l)\right)} \quad (1\le j\le n).$$

Theorem 2.2 *If $\lambda \in \mathfrak{a}_c^*$ satisfies* (2.3) *and*

$$\mathbf{c}^E(\lambda,l) \neq 0, \tag{2.6}$$

then $\mathcal{P}_E$ is a bijection of $\mathcal{B}(E;\lambda,l)$ onto $\mathcal{A}(G/K,\mathcal{M}_{\lambda,l})$.

3 The Poisson transforms and the Hua equations

In this section we consider the Poisson transform for representations on boundaries G/P_Ξ and G/P_{Θ_k}. The system of differential equations that characterize the image of the Poisson transform, which we call the Hua equations, come from coefficients of a homogeneous differential operators between homogeneous vector bundles over G/K.

3.1 Representations on the Shilov boundary

Let $s \in \mathbb{C}$ and $\varepsilon \in \{\pm 1\}$. Let σ_ε denote the one-dimensional representation of M_Ξ that corresponds to the character $g \mapsto (\mathrm{sgn}(\det g))^\varepsilon$ $(g \in GL(n,\mathbb{R}))$ by the isomorphism $M_\Xi \simeq GL(n,\mathbb{R})$. Notice that $\sigma_\varepsilon \equiv 1$ if n is even or $\varepsilon = 0$. Let ρ' be the linear form on $\mathfrak{a}_\Xi$ defined by $\rho'(X_0) = n$. We define

$$\begin{aligned}\mathcal{B}(G/P_\Xi;s,\varepsilon) = \{f \in \mathcal{B}(G)\,;\, f(gman) = \sigma_\varepsilon(m)^{-1}e^{(s-\frac{1}{2}(n+1))\rho'(\log a)}f(g),\\ g\in G,\ m\in M_\Xi,\ a\in A_\Xi,\ n\in N_\Xi^+\}.\end{aligned} \tag{3.1}$$

For $s \in \mathbb{C}$ we define $\lambda_s^{\Xi} \in \mathfrak{a}_c^*$ by

$$\lambda_s^{\Xi} = \left(s - \tfrac{1}{2}(n-1), s - \tfrac{1}{2}(n-3), \ldots, s + \tfrac{1}{2}(n-1)\right).$$

Let $l \in \mathbb{Z}$ be such that $l \equiv \varepsilon \mod 2$. Then we have

$$\mathcal{B}(G/P_{\Xi}; s, \varepsilon) \subset \mathcal{B}(G/P, L_{\lambda_s^{\Xi}, l}). \tag{3.2}$$

For $f \in \mathcal{B}(G/P_{\Xi}; s, \varepsilon)$ we define

$$\mathcal{P}_{\Xi,s,l} f(x) = \int_K f(xk)\tau_l(k)dk.$$

By (3.2), we have

$$\mathcal{P}_s(\mathcal{B}(G/P_{\Xi}; s, \varepsilon)) \subset \mathcal{A}(G/K, \mathcal{M}_{\lambda_s^{\Xi}, l}).$$

For a representation (τ, V_τ) of K let $C^\infty(G/K, \tau)$ denote the space of C^∞-functions $f : G \to V_\tau$ satisfying $f(gk) = \tau(k)^{-1} f(g)$ for all $g \in G$ and $k \in K$. Let $\{E_i\}$ be a basis of $\mathfrak{p}^+$ and $\{E_j^*\}$ be the dual basis of $\mathfrak{p}^-$ with respect to B. Let $U(\mathfrak{g}_c)$ denote the universal enveloping algebra of $\mathfrak{g}_c$. We consider the element of $U(\mathfrak{g}_c) \otimes \mathfrak{k}_c$ defined by

$$\mathcal{H}^{\Xi} = \sum_{i,j} E_i E_j^* \otimes [E_j, E_i^*].$$

Notice that $\mathcal{H}^{\Xi}$ defines a homogeneous differential operator from $C^\infty(G/K; \tau_l)$ to $C^\infty(G/K; \tau_l \otimes \mathrm{Ad}_K|\mathfrak{k}_c)$, which does not depend on the choice of basis. Let $\mathfrak{q}$ denote the orthogonal complement of $\mathfrak{k} \cap \mathfrak{m}_{\Xi}$ in $\mathfrak{k}$ and p denote the orthogonal projection of $\mathfrak{k}_c$ onto $\mathfrak{q}_c$. We put

$$\mathcal{H}_{\mathfrak{q}}^{\Xi} = \sum_{i,j} E_i E_j^* \otimes p([E_j, E_i^*]).$$

We give the main result for the Shilov boundary;

Theorem 3.1 *Let $\varepsilon \in \{\pm 1\}$ and $l \in \mathbb{Z}$ be such that $l \equiv \varepsilon \mod 2$. Assume λ_s^{Ξ} satisfies conditions* (2.1) *and* (2.2). *Then the Poisson transform is a G-isomorphism of $\mathcal{B}(G/P_{\Xi}; s, \varepsilon)$ onto the space of the functions $u \in \mathcal{A}(G/K; \tau_l)$ that satisfy*

$$\mathcal{H}_{\mathfrak{q}}^{\Xi} u = \frac{\left(s + \frac{1}{2}(n+1)\right)\left(s - \frac{1}{2}(n+1)\right)}{8(n+1)^2} u \otimes \frac{\sqrt{-1}}{2} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{3.3}$$

Remark 3.2 Equation (3.3) can be written in the coordinates of the Siegel's upper half plane (cf. [21]).

Remark 3.3 We give some historical remarks.

(1) For $l = 0$ and $\lambda_s^{\Xi} = \rho$ the above theorem was proved by Johnson and Korányi [11] and Lassalle [16] for general Hermitian symmetric spaces of tube type. Previously there are several works for Siegel's upper half planes[1].

(2) For $l = 0$ the above theorem was proved in [24] for general Hermitian symmetric spaces of tube type. Korányi[2] also obtained the result under assumption on s stronger than ours. The generalization to line bundle for all Hermitian symmetric spaces of tube type seems to be a tractable problem.

(3) For $l = 0$ the above theorem was proved by Sekiguchi [21]. He assumed that the image consists of an eigenspace of invariant differential operators in addition to (3.3) and conjectured that the assumption can be removed. The assumption can be removed by using the fact that the space of the radial solution of (3.3) is one-dimensional (cf. Section 4).

(4) Berline and Vergne [2] generalized the above theorem to the Shilov boundary of all Hermitian symmetric space for $l = 0$ and $\lambda_s^{\Xi} = \rho$.

3.2 Representations on G/P_{Θ_k}

Let $s \in (\mathfrak{a}_{\Theta_k})_c^* \simeq \mathbb{C}^{n-k}$ and $l \in \mathbb{Z}$. We put

$$\rho(l) = (n - l, n - 1 - l, \ldots, 1 - l),$$

$$\lambda_{s,l,\pm}^{\Theta_k} = (s_1, \ldots, s_{n-k}, \mp l + k, \mp l + k - 1, \ldots, \mp l + 1),$$

and $\lambda_s^{\Theta_k} = \lambda_{s,0,\pm}^{\Theta_k}$. Throughout this subsection we sometimes write $\lambda = \lambda_{s,l,\pm}^{\Theta_k}$ for simplicity.

Let $\mathfrak{m}_{\Theta_k} = \mathfrak{k}_{\Theta_k} + \mathfrak{p}_{\Theta_k}$ be the Cartan decomposition corresponding to $\theta|\mathfrak{m}_{\Theta_k}$ and put $\mathfrak{p}_{\Theta_k}^{\pm} = \mathfrak{p}^{\pm} \cap \mathfrak{p}_{\Theta_k,c}$. Each element of $\mathfrak{p}_{\Theta_k}^{\pm}$ acts on $\mathcal{B}(\Theta_k; \lambda_{s,l,\pm}^{\Theta_k}, l)$ from the right as a differential operator. We denote by $\mathcal{B}(G/P_{\Theta_k}; s, l, \pm)$ the space of functions f in $\mathcal{B}(\Theta_k; \lambda_{s,l,\pm}^{\Theta_k}, l)$ that satisfy $\mathfrak{p}_{\Theta_k}^{\pm} f = 0$. If $l > k$ (resp. $l < -k$), the space of functions f in $\mathcal{B}(M_{\Theta_k}/K \cap M_{\Theta_k}; \tau_l)$ that satisfy $\mathfrak{p}_{\Theta_k}^{+} f = 0$ (resp. $\mathfrak{p}_{\Theta_k}^{-} f = 0$) is the space of holomorphic (resp. antiholomorphic) discrete series representation of M_{Θ_k} with lowest K-type τ_l. Let

$$\mathcal{B}(G/P_{\Theta_k}; s) = \{f \in \mathcal{B}(G)\,;\, f(gman) = e^{(\lambda-\rho)(\log a)} f(g),$$
$$g \in G,\ m \in M_{\Theta_k},\ a \in A_{\Theta_k},\ n \in N_{\Theta_k}^{+}\}.$$

[1]We refer to the references of the papers mentioned above.
[2]unpublished result

Notice that

$$\mathcal{B}(G/P_{\Theta_k}; s) = \mathcal{B}(G/P_{\Theta_k}; s, 0, +) \cap \mathcal{B}(G/P_{\Theta_k}; s, 0, -).$$

Let p_+ denote the projection of $\bigotimes^k \mathfrak{p}^+$ onto the K-irreducible component with highest weight $\gamma_1 + \cdots \gamma_{n-k+1}$. Let V_μ denote the irreducible representation of K with highest weight μ. We define an element of $U(\mathfrak{g}_c) \otimes V_{\gamma_1+\cdots\gamma_{n-k+1}}$ by

$$\mathcal{H}_-^{\Theta_k} = \sum_{i_1,\ldots,i_{n-k+1}} E_{i_1}^* \ldots E_{i_{n-k+1}}^* \otimes p_+(E_{i_1} \otimes \cdots \otimes E_{i_{n-k+1}}),$$

which does not depend on the choice of basis. Notice that $\mathcal{H}_-^{\Theta_k}$ defines a homogeneous differential operator from $C^\infty(G/K; \tau_l)$ to $C^\infty(G/K; \tau_l \otimes \mathrm{Ad}_K | V_{\gamma_1+\cdots\gamma_{n-k+1}})$. Define $\mathcal{H}_+^{\Theta_k}$ in a similar way.

We give the main result for G/P_{Θ_k};

Theorem 3.4 *Fix* $1 \leq k \leq n-1$. *Let* $s \in (\mathfrak{a}_{\Theta_k})_c^*$ *and assume that* $\lambda_{s,l,\pm}^{\Theta_k}$ *satisfies conditions* (2.3) *and* (2.6). *Then the Poisson transform* $\mathcal{P}_{\Theta_k}$ *is a G-isomorphism of* $\mathcal{B}(G/P_{\Theta_k}; s, l, \pm)$ *onto the space of the functions* $u \in \mathcal{A}(G/K; \tau_l)$ *that satisfy*

$$\mathcal{H}_\pm^{\Theta_k} u = 0 \tag{3.4}$$

and

$$Du = \gamma_l(D)(\lambda_{s,l,\pm}^{\Theta_k})u, \quad D \in \mathbb{D}_l(G/K). \tag{3.5}$$

For $l = 0$, we have the following corollary for the Poisson transform on the degenerate series representations;

Corollary 3.5 *Assume that* $\lambda_s^{\Theta_k}$ *satisfies conditions* (2.3) *and* (2.6). *Then the Poisson transform* $\mathcal{P}_{\Theta_k}$ *is a G-isomorphism of* $\mathcal{B}(G/P_{\Theta_k}; s)$ *onto the space of analytic functions* u *on* G/K *that satisfy*

$$\mathcal{H}_+^{\Theta_k} u = 0,$$

$$\mathcal{H}_-^{\Theta_k} u = 0,$$

and

$$Du = \gamma(D)(\lambda_s^{\Theta_k})u, \quad D \in \mathbb{D}(G/K).$$

For the maximal parabolic subgroup $P_{\Theta_{n-1}}$, we have the following refinement of the above theorem and the corollary;

Theorem 3.6 *Assume that $\lambda_{s,l,\pm}^{\Theta_{n-1}}$ satisfies conditions* (2.3) *and* (2.6). *Then the Poisson transform $\mathcal{P}_{\Theta_{n-1}}$ is a G-isomorphism of $\mathcal{B}(G/P_{\Theta_{n-1}}; s, l, \pm)$ onto the space of the functions $u \in \mathcal{A}(G/K; \tau_l)$ that satisfy*

$$\mathcal{H}_{\pm}^{\Theta_{n-1}} u = 0$$

and

$$L_l u = (\langle \lambda_{s,l,\pm}^{\Theta_{n-1}}, \lambda_{s,l,\pm}^{\Theta_{n-1}} \rangle - \langle \rho, \rho \rangle) u,$$

where L_l is the Laplace-Beltrami operator on $\mathcal{A}(G/K; \tau_l)$.

Corollary 3.7 *Assume that $\lambda_s^{\Theta_{n-1}}$ satisfies conditions* (2.3) *and* (2.6). *Then the Poisson transform $\mathcal{P}_{\Theta_{n-1}}$ is a G-isomorphism of $\mathcal{B}(G/P_{\Theta_{n-1}}; s)$ onto the space of analytic functions u on G/K that satisfy*

$$\mathcal{H}_{+}^{\Theta_{n-1}} u = 0,$$

$$\mathcal{H}_{-}^{\Theta_{n-1}} u = 0,$$

and

$$L u = (\langle \lambda_s^{\Theta_{n-1}}, \lambda_s^{\Theta_{n-1}} \rangle - \langle \rho, \rho \rangle) u,$$

where L is the Laplace-Beltrami operator on G/K.

Remark 3.8 We have coordinate expressions of (3.4). A function u on $H_n \simeq G/K$ satisfies $\mathcal{H}_{+}^{\Theta_k} u = 0$ if and only if u is annihilated by all minors of order $n-k+1$ of the matrix $(\partial/\partial z_{ij})_{1 \leq i,j \leq n}$. For $\mathcal{H}_{-}^{\Theta_k} u = 0$, replace $\partial/\partial z_{ij}$ by $\partial/\partial \bar{z}_{ij}$.

Remark 3.9 All results of this subsection hold for *all* Hermitian symmetric spaces[3].

4 Radial parts of the Hua equations

For l, $m \in \mathbb{Z}$, (τ_{-m}, τ_{-l})-radial parts of the elements of $\mathbb{D}_l(G/K)$ gives the hypergeometric differential operators of Heckman and Opdam [7] (cf. [23]). Radial part of the system of the differential equations that characterizes the image of the Poisson transform gives a quotient of the system of hypergeometric differential equations of Heckman and Opdam as D-modules.

[3]The paper on this subject is in preparation.

4.1 Case of the Shilov boundary

Let $m,\ l \in \mathbb{Z}$. Let $h(t;\ m, l)$ be the function on $\mathbb{R}^n$ given by

$$h(t;\ m, l) = \prod_{j=1}^{n} (\cosh t_j)^{\frac{1}{2}(m+l)} (\sinh t_j)^{\frac{1}{2}(l-m)}.$$

We will give (τ_{-m}, τ_{-l})-radial parts the system $\mathcal{H}^{\Xi}u = 0$. For $l = m = 0$ this was done by Lassalle [16] for all tube domains. If $u \in C^\infty(G)$ satisfies

$$u(k_1 g k_2) = \tau_m(k_1)^{-1} u(g) \tau_l(k_2)^{-1} \quad \text{for all } g \in G,\ k_1,\ k_2 \in K \tag{4.1}$$

and $\mathcal{H}^{\Xi}u = 0$, then the function

$$\phi(t) = h(t;\ m, l) u(\exp(\textstyle\sum_{j=1}^n t_j X_j))$$

satisfies

$$\Big(\partial_{t_i}^2 + (2m \coth t_i + 2(1 - l - m) \coth 2t_i)\partial_{t_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} (\coth(t_i + t_j)(\partial_{t_i} + \partial_{t_j}) + (\coth(t_i - t_j)(\partial_{t_i} - \partial_{t_j})\Big)\phi = \Big(s^2 - (\tfrac{1}{2}(n+1) - l)^2\Big)\phi \tag{4.2}$$

for $1 \le i \le n$.

The system of differential equations *(4.2)* for $1 \le i \le n$ was investigated by Korányi [15] and Yan [26]. By [26, Theorem 2.1], there is a unique solution up to constant multiple subject to condition that it is W-invariant and analytic at $t = 0$, which corresponds to the (τ_{-m}, τ_{-l})-spherical function on G.

4.2 Case of $G/P_{\Theta_{n-1}}$

We will give (τ_{-m}, τ_{-l})-radial parts of the system $\mathcal{H}_{\pm}^{\Theta_{n-1}}u = 0$. The calculation for general n reduces to the case of $n = 2$, which was given by Iida [9][4].

If $u \in C^\infty(G)$ satisfies (4.1) and $\mathcal{H}_{\pm}^{\Theta_{n-1}}u = 0$, then the function

$$\phi(t) = h(t;\ \pm m, \pm l) u(\exp(\sum_{j=1}^{n} t_j X_j)) \tag{4.3}$$

[4]See also his article in this volume.

satisfies

$$(2\partial_{t_j}\partial_{t_k} + (\coth(t_j + t_k) - \coth(t_j - t_k))\partial_{t_j} \\ + (\coth(t_j + t_k) + \coth(t_j - t_k)\partial_{t_k})\phi = 0 \tag{4.4}$$

for all $1 \leq j < k \leq n$. Moreover, if u is an eigenfunction of the Laplace-Beltrami operator with eigenvalue $\langle \lambda_{s,l,\pm}^{\Theta_{n-1}}, \lambda_{s,l,\pm}^{\Theta_{n-1}} \rangle - \langle \rho, \rho \rangle$, then by [22, Proposition 2.6], the function ϕ given by (4.3) is a solution of the differential equation

$$\Big(\sum_{i=1}^{n} \partial_{t_i}^2 + \sum_{1 \leq j < k \leq n} ((\coth(t_j + t_k) + \coth(t_j - t_k))\partial_{t_j} \\ + (\coth(t_j + t_k) - \coth(t_j - t_k))\partial_{t_k}) \\ + \sum_{i=1}^{n} (\pm 2m \coth t_i + 2(1 \mp l \mp m) \coth 2t_i)\partial_{t_i}\Big)\phi \\ = (s^2 - (n \mp l)^2)\phi. \tag{4.5}$$

By the change of variables $y_i = -\sinh^2 t_i$ $(1 \leq i \leq n)$, the system of differential equations *(4.4)* for $1 \leq j < k \leq n$ and *(4.5)* becomes a system of differential equation that was investigated by Debiard and Gaveau [5]. In particular, if $m = l$, there is a unique solution up to constant multiple subject to condition that it is W-invariant and analytic at $t = 0$, which corresponds to the (τ_{-l}, τ_{-l})-spherical function on G.

References

[1] L. Barchini, *Szegö kernels associated with Zuckerman modules*, J. Funct. Anal. **131** (1995), pp. 145–182.

[2] N. Berline and M. Vergne, *Equations de Hua et noyau de Poisson*, Lecture Notes in Mathematics, vol. 880, pp. 1–51. Springer Verlag, Berlin-Heiderberg-New York, 1981.

[3] M.G. Davidson, T.J. Enright and R.J. Stanke, *Differential operators and highest weight representations*, Memoirs Amer. Math. Soc. **455** (1991).

[4] M.G. Davidson and R.J. Stanke, *Szegö maps and highest weight representations*, Pacific J. Math. **158** (1993), pp. 67–91.

[5] A. Debiard and B. Gaveau, *Représentation intégrale de certaines séries de fonctions sphériques d'un système de racines BC*, J. Funct. Anal. **96** (1991), pp. 256–296.

[6] S. Helgason, *A duality for symmetric spaces with applications to group representations*, Adv. in Math. **5** (1970), pp. 1-154.

[7] G.J. Heckman and E.M. Opdam, *Root systems and hypergeometric functions* I, Comp. Math. **64** (1987), pp. 329–352.

[8] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in Classical Domains*, Amer. Math. Soc., Providence, 1963.

[9] M. Iida, *Spherical functions of the principal series representations of $Sp(2,\mathbb{R})$ as hypergeometric functions of C_2-type*, preprint, 1995, UTMS 95-24, Dept. of Mathematical Sciences, Univ. of Tokyo.

[10] S. Ishikawa, *The range characterizations of the totally geodesic Radon transform on the real hyperbolic space*, preprint, 1995, UTMS 95-47, Dept. of Mathematical Sciences, Univ. of Tokyo.

[11] K. Johnson and A. Korányi, *The Hua operators on bounded symmetric domains of tube type*, Ann. of Math. **111** (1980), pp. 589–608.

[12] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, T. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math. **107** (1978), pp. 1–39.

[13] A. Knapp, *Introduction to representations in analytic cohomology*, Contemporary Mathematics **154**, 1993.

[14] A. Knapp and N. Wallach, *Szegö kernels assòciated with discrete series*, Invent. Math. **34** (1976), pp. 163–200

[15] A. Korányi, *Hua-type integrals, hypergeometric functions and symmetric polynomials*, International symposium in memory of Hua Loo Keng, vol. II, Analysis, (S. Gong et al., eds.), Science Press, Beijing and Springer-Verlag, Berlin, 1991, pp. 169–180.

[16] M. Lassalle, *Les équations de Hua d'un domaine borné symétrique de type tube*, Invent. math. **77** (1984), pp. 129–161.

[17] T. Oshima, *Generalized Capelli identities and boundary value problems for $GL(n)$*, preprint, 1995, UTMS 95-44, Dept. of Mathematical Sciences, Univ. of Tokyo.

[18] ———, *Capelli identities, degenerate series and hypergeometric functions*, in this volume.

[19] W. Schmid, *On the realization of the discrete series of a semisimple Lie group*, Rice Univ. Studies **56** (1970), pp. 99–108.

[20] H. Sekiguchi, *The Penrose transform for certain non-compact homogeneous manifolds of $U(n,n)$*, Thesis, Tokyo University, 1995.

[21] J. Sekiguchi, *Invariant system of differential equations on Siegel's upper half-plane*, Seminar reports on unitary representations, vol. VII (1987), pp. 97–126.

[22] N. Shimeno, *Eigenspaces of invariant differential operators on a homogeneous line bundle on a Riemannian symmetric space*, J. Fac. Sci. Univ. Tokyo, Sect. 1A **37** (1990), pp. 201–234.

[23] ———, *The Plancherel formula for spherical functions with a one-dimensional K-type on a simply connected simple Lie group of Hermitian type*, J. Funct. Anal. **121** (1994), pp. 330–388.

[24] ———, *Boundary value problems for the Shilov boundary of a bounded symmetric domain of tube type*, J. Funct. Anal. (to appear).

[25] ———, *Poisson integrals and the Hua equations*, Proceedings of the 34-th Symposium on Real Analysis and Functional Analysis, (1995), (in Japanese).

[26] Z. Yan, *A class of generalized hypergeometric functions in several variables*, Canad. J. Math. **44** (1992), pp. 1317–1338.