表現論シンポジウム講演集, 1995
pp.31-47

SPHERICAL FUNCTIONS OF THE PRINCIPAL SERIES REPRESENTATIONS OF $Sp(2,\mathbb{R})$ AND HYPERGEOMETRIC FUNCTIONS

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§ 0. Introduction.

The Whittaker model of the principal series representation of $Sp(2,\mathbb{R})$ is obtained in [MO1]. The system of differential equations for the Whittaker model is given by the Casimir operator and the shift operator which is defined by the Schmid operator.

We will give the explicit formula of the system of differential operators satisfied by spherical functions of the principal series representation of $Sp(2,\mathbb{R})$ using the method of [MO1]. Moreover, we will obtain series expansions and integral formulas of spherical functions. In the end, we will have some results about hypergeometric functions of two variables.

§ 1. Parabolic subgroups of Sp(2,$\mathbb{R}$).

For $G = Sp(2,\mathbb{R})$, let K be a maximal compact subgroup of G, which is isomorphic to $U(2)$. We define a minimal parabolic subgroup $P = MAN$ and the Jacobi parabolic subgroup $P_J = M_J A_J N_J$ as follows.

We put $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{k} = \mathrm{Lie}(K)$. For the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we define the maximal abelian subspace $\mathfrak{a} = \{\mathrm{diag}(x, y, -x, -y) | x, y > 0\} \subset \mathfrak{p}$. Above A and M are defined by $A = \exp \mathfrak{a}$, $M = Z_K(\mathfrak{a}) = \{\pm I_2, \pm\gamma_2\}$. Here $\gamma_2 = \mathrm{diag}(1, -1, 1, -1)$.

We choose the basis $\{H_1 = \mathrm{diag}(1, 0, -1, 0), H_2 = \mathrm{diag}(0, 1, 0, -1)\}$ of $\mathfrak{a}$, and its dual basis $\{e_1, e_2\}$ of $\mathfrak{a}^*$. Then we have the restricted root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm 2e_1, \pm 2e_2, \pm(e_1 \pm e_2)\}$ and we choose the positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{a}) = \{2e_1, 2e_2, e_1 \pm e_2\}$. The above N is defined by $\exp \mathfrak{n}$ for $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}(\alpha, \mathfrak{a})$.

For the subspace $\mathfrak{a}_J = \mathbf{R}H_1$ of $\mathfrak{a}$, we set $A_J = \exp \mathfrak{a}_J$, $M_J = M \cdot \exp \mathbf{R}H_2 \cdot \exp(\mathfrak{g}(2e_2, \mathfrak{a}) \oplus \mathfrak{g}(-2e_2, \mathfrak{a})) \simeq \{\pm 1\} \times SL(2, \mathbf{R})$, $\mathfrak{n}_J = \sum_{\alpha \in \Delta^+ \setminus \{2e_2\}} \mathfrak{g}(\alpha, \mathfrak{a})$ and $N_J = \exp \mathfrak{n}_J$.

§ 2. Representations of Sp(2,R).

We will define two types of representations of $Sp(2, \mathbf{R})$.

Definition 2.1. *Let σ be a irreducible unitary representation of M and we set $\mu \in \mathfrak{a}_{\mathbf{C}}^*$, $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = 2e_1 + e_2$.*

We call the representation (π, H_π) of G induced from the representation $\sigma \otimes a^{\mu+\rho} \otimes 1_N$ of P the principal series representation of G. This means

$$\begin{aligned} H_\pi &= \operatorname{Ind}_P^G(\sigma \otimes a^{\mu+\rho} \otimes 1_N) \\ &= \{f \in C^\infty(G) \mid f(mang) = \sigma(m) a^{\mu+\rho} f(g), \\ &\qquad\qquad \text{for } {}^\forall g \in G, {}^\forall m \in M, {}^\forall a \in A, {}^\forall n \in N\}. \end{aligned}$$

and the action of G is defined by $\pi(g)f(x) = f(xg)$ for ${}^\forall f \in H_\pi$ and ${}^\forall g \in G$.

Definition 2.2. *Let $\sigma_J = (\varepsilon, \xi)$ be a discrete series representation of $M_J \simeq \{\pm 1\} \times SL(2, \mathbf{R})$ and we set $\nu_1 \in \mathfrak{a}_{J,\mathbf{C}}^*$, $\rho_J = \frac{1}{2}\{(e_1 - e_2) + 2e_1 + (e_1 + e_2)\} = 2e_1$. Here, $\varepsilon \in \widehat{\{\pm 1\}}$ and ξ is a discrete series representation of $SL(2, \mathbf{R})$.*

We call the representation (π_J, H_{π_J}) of G induced from the representation $\sigma_J \otimes a_J^{\nu_1+\rho_J} \otimes 1_{N_J}$ of P_J the generalized principal series representation of G. This means

$$\begin{aligned} H_{\pi_J} &= \operatorname{Ind}_{P_J}^G(\sigma_J \otimes a_J^{\nu_1+\rho_J} \otimes 1_{N_J}) \\ &= \{f : G \to V_{\sigma_J} \mid f \text{ is of } C^\infty\text{-class}, \ f(m_J a_J n_J g) = \sigma_J(m_J) a_J^{\nu_1+\rho_J} f(g) \\ &\qquad\qquad \text{for } {}^\forall g \in G, \ {}^\forall m_J \in M_J, \ {}^\forall a_J \in A_J, \ {}^\forall n_J \in N_J\}, \end{aligned}$$

and the action of G is defined by $\pi_J(g)f(x) = f(xg)$ for ${}^\forall f \in H_{\pi_J}$ and ${}^\forall g \in G$.

Definition 2.3. *Irreducible representations of K are parametrized by the following set.*

$$\{\lambda = (l_1, l_2) \in \mathbf{Z} \oplus \mathbf{Z} \mid l_1 \geq l_2\}.$$

The irreducible representation $(\tau_{(l_1,l_2)}, V_{(l_1,l_2)})$ of K is $(l_1 - l_2 + 1)$-dimensional representation. We choose a basis $\{v_k \mid 0 \leq k \leq d = l_1 - l_2\}$ of $V_{(l_1,l_2)}$ such that each v_k is a weight vector and v_0 is the lowest weight vector, v_d is the highest weight vector.

We denote K-finite vectors of H_π and H_{π_J} by $H_{\pi,K}, H_{\pi_J,K}$ respectively. Then we have the following propositions about the multiplicity of K-types ([MO1], [MO2]).

Proposition 2.4. *For $\sigma \in \widehat{M}$, write $\varepsilon_1 = \sigma(-\gamma_2)$ and $\varepsilon_2 = \sigma(\gamma_2)$. The following hold.*

(i) *If $\varepsilon_1 = \varepsilon_2$, then the representation $\tau_{(l,l)}$ occurs in $H_{\pi,K}$ with multiplicity one for any $l \in \mathbb{Z}$ such that $(-1)^l = \varepsilon_1 = \varepsilon_2$ and other 1-dimensional representations do not occur. We call this case the even case.*

(ii) *If $\varepsilon_1 = -\varepsilon_2$, then for any integer l the K-type $\tau_{(l,l-1)}$ occurs in $H_{\pi,K}$ with multiplicity one. We call this case the odd case.*

Proposition 2.5. *For $\sigma_1 = (\varepsilon, \xi_l) \in \widehat{M_1}(l \geq 2)$ and $\nu_1 \in \mathfrak{a}^*_{1,\mathbb{C}}$, the following hold. Here ξ_l is the discrete series representation of $SL(2,\mathbb{R})$ with the Blattner parameter l.*

(i) *If $\varepsilon(-\gamma_2) = (-1)^l$, the representation $\tau_{(k,k)}(k \in \mathbb{Z}, k \equiv l \mod 2, k \geq l)$ and $\tau_{(l,k)}(k \in \mathbb{Z}, k \equiv l \mod 2, k \leq l)$ occurs in $H_{\pi_J,K}$ with multiplicity one and other 1-dimensional representations do not occur. We call this case the even case.*

(ii) *If $\varepsilon(-\gamma_2) = (-1)^{l+1}$, the representation $\tau_{(k,k-1)}(k \in \mathbb{Z}, k \geq l)$ and $\tau_{(l,k-1)}$ $(k \in \mathbb{Z}, k \equiv l, k \leq l)$ occurs in $H_{\pi_J,K}$ with multiplicity one and no 1-dimensional representations occur. We call this case the odd case.*

§ 3. Spherical functions.

For $(\eta, V_\eta), (\tau, V_\tau) \in \widehat{K}$, we define the following function spaces.

$$\begin{aligned} C^\infty_\eta(K\backslash G) = \{f : G \to V_\eta \mid\ & f \text{ is of } C^\infty\text{-class}, \\ & f(kg) = \eta(k)f(g), {}^\forall k \in K, {}^\forall g \in G\}, \\ C^\infty_{\eta,\tau}(K\backslash G/K) = \{f : G \to V_\eta \otimes V_{\tau^*} \mid\ & f \text{ is of } C^\infty\text{-class}, \\ & f(k_1 g k_2) = \eta(k_1) \otimes \tau^*(k_2)^{-1} f(g), {}^\forall k_1, k_2 \in K, {}^\forall g \in G\} \end{aligned}$$

Here V_{τ^*} is the contragredient representation of V_τ.

Let H be an admissible $(\mathfrak{g}, K)$-module and assume there is a non-trivial K homomorphism $i : V_\tau \hookrightarrow H$. Consider a $(\mathfrak{g}, K)$-module

$$\psi : H \longrightarrow C^\infty_\eta(K\backslash G).$$

Then,

$$\psi_H = \psi \circ i \in \mathrm{Hom}_K(V_\tau, C^\infty_\eta(K\backslash G)) \simeq C^\infty_\eta(K\backslash G) \otimes_K V_{\tau^*} \simeq C^\infty_{\eta,\tau}(K\backslash G/K).$$

We call this function the spherical function attached to H.

We can write ψ_H explicitly as follows.

Let H^* be the contragredient representation of H. If we assume there is a non-trivial K-homomorphism

$$j : V_\eta \hookrightarrow H,$$

then there exist non-trivial K-homomorphisms

$$i^* : V_{\tau^*} \hookrightarrow H^*,$$
$$j^* : V_{\eta^*} \hookrightarrow H^*.$$

Choose bases $\{v_n^\tau | 0 \le n \le d_\tau\}$, $\{v_n^\eta | 0 \le n \le d_\eta\}$ of V_τ and V_η and let $\{w_n^{\tau^*} | 0 \le n \le d_{\tau^*}\}$, $\{w_n^{\eta^*} | 0 \le n \le d_{\eta^*}\}$ be dual base of V_{τ^*} and V_{η^*} respectively. Then

$$\psi_H(g) = \sum_{n,m} \langle \pi(g) i(v_n^\tau), j^*(w_m^{\eta^*}) \rangle v_m^\eta \otimes w_n^{\tau^*}.$$

We will study spherical functions attached to $H_{\pi,K}$ and $H_{\pi_J,K}$.

$\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$ is determined by its restriction on A by the Cartan decomposition $G = KAK$. Since $A \simeq \mathfrak{a} = \mathbb{R}H_1 + \mathbb{R}H_2$, we consider ϕ as a function on $\mathbb{R}^2$ by $\phi(x, y) = \phi(xH_1 + yH_2)$.

For a differential operator D on G, there is the differential operator $R(D)$ on A such that $(D\phi)|_A = R(D)(\phi|A)$ for any $\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$. We call the operator $R(A)$ the radial part of D.

From the action of M and the Weyl group on the spherical function, the following propositions are easily proved.

Proposition 3.1(the even case). *Let $\eta = (k, k), \tau = (l, l) \in \widehat{K}$ (This means $\dim V_\eta = \dim V_\tau = 1$ and $\tau^* = (-l, -l)$). For $\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$, we have the following.*

(i) *If $k - l \equiv 1 \mod 2$, then $\phi = 0$.*

(ii) $\phi(y, x) = \phi(x, y)$.

(iii) $\phi(x, -y) = \begin{cases} \phi(x, y) & \text{If } k \equiv l \mod 4, \\ -\phi(x, y) & \text{otherwise.} \end{cases}$

Lemma 3.2(2-dimensional case). *Let $\eta = (k, k-1), \tau = (l, l-1) \in \widehat{K}$ (This means $\dim V_\eta = \dim V_\tau = 2$ and $\tau^* = (-l+1, -l)$). For $\phi = \sum_{0 \le i,j \le 1} \phi_{ij} v_i^\eta \otimes v_j^{\tau^*} \in C^\infty_{\eta,\tau}(K\backslash G/K)$, we have the following.*

(i) $\begin{cases} \phi_{00} = \phi_{11} \equiv 0 & \text{If } k - l \equiv 0 \mod 2, \\ \phi_{01} = \phi_{10} \equiv 0 & \text{If } k - l \equiv 1 \mod 2. \end{cases}$

(ii) $\begin{cases} \phi_{10}(x,y) = -\phi_{01}(y,x) & \textit{If } k-l \equiv 0 \mod 2, \\ \phi_{11}(x,y) = \phi_{00}(y,x) & \textit{If } k-l \equiv 1 \mod 2. \end{cases}$

(iii) $\phi(x,-y) = \begin{cases} \phi(x,y) & \textit{If } k-l \equiv 0,3 \mod 4, \\ -\phi(x,y) & \textit{If } k-l \equiv 1,2 \mod 4. \end{cases}$

§ 4 Casimir operator.

Define $H_{2e_i} = H_i$ $(i=1,2)$, $H_{e_1 \pm e_2} = H_1 \pm H_2$ and choose the basis E_α of $\mathfrak{g}(\mathfrak{a},\alpha)$ for $\alpha \in \Delta^+$ such that $[E_\alpha, {}^tE_\alpha] = H_\alpha$ holds. Then, the Casimir operator L of $Sp(2,\mathbb{R})$ is given by $L = H_1^2 + H_2^2 - 4H_1 - 2H_2 + 2E_{e_1-e_2}E_{-e_1+e_2} + 4E_{2e_1}E_{-2e_1} + 2E_{e_1+e_2}E_{-e_1-e_2} + 4E_{2e_2}E_{-2e_2}$.

We set $A' = \{a \in A \mid a^{2\alpha} \neq 1 \text{ for any } \alpha \in \Delta^+\}$.

Lemma 4.1. *For $a \in A'$ and $\alpha \in \Delta^+$,*

$$E_\alpha E_{-\alpha} = \frac{1}{(a^\alpha - a^{-\alpha})^2}(\mathrm{Ad}(a^{-1})X_\alpha)^2 - \frac{a^\alpha + a^{-\alpha}}{(a^\alpha - a^{-\alpha})^2}(\mathrm{Ad}(a^{-1})X_\alpha)X_\alpha + \frac{a^\alpha}{a^\alpha - a^{-\alpha}}H_\alpha + \frac{1}{(a^\alpha - a^{-\alpha})^2}X_\alpha^2$$

holds. Here, $X_\alpha = E_\alpha - {}^tE_\alpha \in \mathfrak{k}$.

For $X, Y \in U(\mathfrak{k}), H \in U(\mathfrak{a}), a \in A'$ and $\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$, we have

$$\begin{aligned}
&(\mathrm{Ad}(a^{-1})X \cdot H \cdot Y\phi)(a) \\
&= \frac{\partial^3}{\partial s \partial t \partial u}\bigg|_{s=t=u=0} \phi(a \cdot \exp s\,\mathrm{Ad}(a^{-1})X)\exp tH \exp uY) \\
&= \frac{\partial^3}{\partial s \partial t \partial u}\bigg|_{s=t=u=0} \phi(\exp sX \cdot a \cdot \exp tH \exp uY) \\
&= \frac{\partial^3}{\partial s \partial t \partial u}\bigg|_{s=t=u=0} \eta(\exp sX) \otimes \tau^*(\exp uY)^{-1}\phi(a \cdot \exp tH) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} \eta(X) \otimes (-\tau^*(Y))\phi(a \cdot \exp tH).
\end{aligned}$$

Therefore we can calculate the radial part of L by Lemma 4.1.

Proposition 4.2(the even case). *For $\eta = (k,k), \tau = (l,l) \in \widehat{K}$ and $\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$, the radial part $R(L)$ of L is given by*

$$\begin{aligned}
R(L)\phi = \{&L_0 - (k^2+l^2)(\mathrm{sh}^{-2}2x + \mathrm{sh}^{-2}2y) \\
&+ 2kl(\mathrm{ch}\,2x \cdot \mathrm{sh}^{-2}2x + \mathrm{ch}\,2y \cdot \mathrm{sh}^{-2}2y)\}\phi.
\end{aligned}$$

Here,

$$L_0 = \partial_x^2 + \partial_y^2 + \{2\coth 2x + \coth(x+y) + \coth(x-y)\}\partial_x$$
$$+ \{2\coth 2y + \coth(x+y) - \coth(x-y)\}\partial_y$$

and $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}$.

Proposition 4.3(the odd case). *For* $\eta = (k, k-1), \tau = (l, l-1) \in \widehat{K}$ *with* $k \equiv l \mod 2$ *and* $\phi = \phi_{01} v_0^\eta \otimes v_1^{\tau^*} + \phi_{10} v_1^\eta \otimes v_0^{\tau^*} \in C_{\eta,\tau}^\infty(K\backslash G/K)$, *the radial part* $R(L)$ *of* L *is given by*

$$\begin{aligned}
R(L)\phi = [\{L_0 &- \mathrm{sh}^{-2}(x+y) - \mathrm{sh}^{-2}(x-y) - ((k-1)^2 + (l-1)^2)\,\mathrm{sh}^{-2}2x \\
&-(k^2+l^2)\,\mathrm{sh}^{-2}2y + 2(k-1)(l-1)\,\mathrm{ch}\,2x\cdot\mathrm{sh}^{-2}2x \\
&+ 2kl\,\mathrm{ch}\,2y\cdot\mathrm{sh}^{-2}2y\}\phi_{01}(x,y) \\
&-\{\mathrm{ch}(x+y)\cdot\mathrm{sh}^{-2}(x+y) + \mathrm{ch}(x-y)\cdot\mathrm{sh}^{-2}(x-y)\}\phi_{10}(x,y)]v_0^\eta \otimes v_1^{\tau^*} \\
&+[\{L_0 - \mathrm{sh}^{-2}(x+y) - \mathrm{sh}^{-2}(x-y) - (k^2+l^2)\,\mathrm{sh}^{-2}2x \\
&-((k-1)^2 + (l-1)^2)\,\mathrm{sh}^{-2}2y + 2kl\,\mathrm{ch}\,2x\cdot\mathrm{sh}^{-2}2x \\
&+ 2(k-1)(l-1)\,\mathrm{ch}\,2y\cdot\mathrm{sh}^{-2}2y\}\phi_{10}(x,y) \\
&-\{\mathrm{ch}(x+y)\cdot\mathrm{sh}^{-2}(x+y) + \mathrm{ch}(x-y)\cdot\mathrm{sh}^{-2}(x-y)\}\phi_{01}(x,y)]v_1^\eta \otimes v_0^{\tau^*}.
\end{aligned}$$

§ 5 Shift operators.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the compact Cartan subalgebra of $\mathfrak{g}$. We define subspaces $\mathfrak{p}_\pm$ of $\mathfrak{p}_\mathbf{C}$ by $\mathfrak{p}_\pm = \sum_{\beta\in\Sigma_n^+} \mathfrak{g}_{\mathbf{C},\pm\beta}$ for the set of noncompact positive root

$$\Sigma_n^+ = \{\beta_1+\beta_2, 2\beta_1, 2\beta_2\} \subset \Sigma(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C})^+ = \{\beta_1 \pm \beta_2, 2\beta_1, 2\beta_2\}$$

with a certain basis $\{\beta_1, \beta_2\}$ of $\mathfrak{h}^*$. Then $\mathfrak{p}_\mathbf{C}$ is decomposed into the sum of two irreducible components $\mathfrak{p}_+ \simeq V_{(2,0)}$ and $\mathfrak{p}_- \simeq V_{(0,-2)}$ with respect to the adjoint action of K. We denote these K-modules by $\mathrm{Ad}_{\mathfrak{p}_\pm}$. Note that $\mathrm{Ad}_{\mathfrak{p}_-} \simeq \mathrm{Ad}^*_{\mathfrak{p}_+}$.

Definition 5.1(the Schmid operator). *Let* $\{X_i\}$ *be an orthonormal basis of* $\mathfrak{p}$ *with respect to the Killing form. We define the Schmid operator* ∇_τ *by*

$$\nabla_\tau : C_\tau^\infty(G/K) \ni \phi \to \sum_i R_{X_i}\phi \otimes X_i \in C^\infty_{\tau\otimes\mathrm{Ad}_\mathfrak{p}}(G/K).$$

Here, $\phi \in C_\tau^\infty(G/K)$ *and* $R_X\phi(x) = \frac{d}{dt}|_{t=0} f(x \exp tX)$. *Then* ∇_τ *is independent of the choice of an orthonormal basis of* $\mathfrak{p}$.

We choose $X_\beta \in \mathfrak{g}_{\beta,\mathbf{C}}$ $(\beta \in \Sigma_n^+)$ and set $X_{-\beta} = \overline{X}_\beta$ such that

$$\{C|\beta|(X_\beta + X_{-\beta}), \frac{C|\beta|}{\sqrt{-1}}(X_\beta - X_{-\beta}) \mid \beta \in \Sigma_n^+\}$$

is an orthonormal basis of $\mathfrak{p}$ for some constant $C > 0$. If we set

$$\nabla_\tau^\pm : C_\tau^\infty(G/K) \ni \phi \to \frac{1}{4} \sum_{\beta \in \Sigma_n^+} |\beta|^2 R_{X_{\pm\beta}} \phi \otimes X_{\mp\beta} \in C^\infty_{\tau\otimes \mathrm{Ad}_{\mathfrak{p}_\pm}}(G/K),$$

we can write $\nabla_\tau = 8C(\nabla_\tau^+ + \nabla_\tau^-)$ with respect to the above basis.

Definition 5.2 (Shift operators for the even case). *For $\tau = (l,l)$, $\tau \otimes \mathrm{Ad}_{\mathfrak{p}_\pm} \otimes \mathrm{Ad}_{\mathfrak{p}_\pm}$ has $\tau_\pm = (l \pm 2, l \pm 2)$ as a irreducible component with multiplicity 1 respectively. We define $\mathrm{pr}_e^\pm$ by*

$$\mathrm{pr}_e^\pm : C^\infty_{\tau\otimes \mathrm{Ad}_{\mathfrak{p}_\pm}\otimes \mathrm{Ad}_{\mathfrak{p}_\pm}}(G/K) \to C^\infty_{\tau_\pm}(G/K) \quad : \quad \textit{projection},$$

and we call the following differential operators of order 2 shift operators.

$$\begin{cases} D^+ = \mathrm{pr}_e^+ \circ \nabla^+_{\tau\otimes\mathrm{Ad}_{\mathfrak{p}_+}} \circ \nabla_\tau^+ : C^\infty_{(l,l)}(G/K) \to C^\infty_{(l+2,l+2)}(G/K) \\ D^- = \mathrm{pr}_e^- \circ \nabla^-_{\tau\otimes\mathrm{Ad}_{\mathfrak{p}_-}} \circ \nabla_\tau^- : C^\infty_{(l,l)}(G/K) \to C^\infty_{(l-2,l-2)}(G/K) \end{cases}$$

Shift operators are elements of the universal enveloping algebra $U(\mathfrak{g})$ and they are of the form

$$D^+ = X_{2\beta_1}X_{2\beta_2} + X_{2\beta_2}X_{2\beta_1} - \frac{1}{2}X^2_{\beta_1+\beta_2},$$
$$D^- = X_{-2\beta_1}X_{-2\beta_2} + X_{-2\beta_2}X_{-2\beta_1} - \frac{1}{2}X^2_{-\beta_1-\beta_2}.$$

We denote $D_l^\pm$ for $D^\pm|_{C^\infty_{(l,l)}(G/K)}$.

Remark 5.3. *$D^+_{l-2} \circ D^-_l$ is a map from $C^\infty_{(l,l)}(G/K)$ to $C^\infty_{(l,l)}(G/K)$. Especially for any $\eta \in \widehat{K}$ and $\tau = (l,l)$, this is a map from $C^\infty_{\eta,\tau}(K\backslash G/K)$ to $C^\infty_{\eta,\tau}(K\backslash G/K)$.*

Definition 5.4(Shift operators for the odd case). *For $\tau = (l, l-1)$, $\tau \otimes \mathrm{Ad}_{\mathfrak{p}_\pm}$ has $\tau_+ = (l+1,l), \tau_- = (l-1,l-2)$ as an irreducible component with multiplicity 1 respectively. We define $\mathrm{pr}_o^\pm$ by*

$$\mathrm{pr}_o^\pm : C^\infty_{\tau\otimes \mathrm{Ad}_{\mathfrak{p}_\pm}}(G/K) \to C^\infty_{\tau_\pm}(G/K) \quad : \quad \textit{the projection}.$$

and we call the following matrices whose entries are differential operators of order 1 shift operators.

$$\begin{cases} E^+ = \mathrm{pr}_o^+ \circ \nabla_\tau^+ : C^\infty_{(l,l-1)}(G/K) \to C^\infty_{(l+1,l)}(G/K) \\ E^- = \mathrm{pr}_o^- \circ \nabla_\tau^- : C^\infty_{(l,l-1)}(G/K) \to C^\infty_{(l-1,l-2)}(G/K) \end{cases}$$

$E^{\pm}$ *are of the form*

$$\begin{pmatrix}\varphi_0^+\\ \varphi_1^+\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}X_{\beta_1+\beta_2} & -X_{2\beta_1}\\ X_{2\beta_2} & \frac{1}{2}X_{\beta_1+\beta_2}\end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix},$$

$$\begin{pmatrix}\varphi_0^-\\ \varphi_1^-\end{pmatrix} = \begin{pmatrix}\frac{1}{2}X_{-\beta_1-\beta_2} & -X_{-2\beta_2}\\ X_{-2\beta_1} & -\frac{1}{2}X_{-\beta_1-\beta_2}\end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}$$

for $\varphi_0 v_0^{\tau^*} + \varphi_1 v_1^{\tau^*} \in C_\tau^\infty(G/K)$, $\varphi_0^{\pm} v_0^{\tau_\pm^*} + \varphi_1^{\pm} v_1^{\tau_\pm^*} \in C_{\tau_\pm}^\infty(G/K)$.

We denote $E_l^{\pm}$ *for* $E^{\pm}|_{C^\infty_{(l,l-1)}(G/K)}$.

Remark 5.5. $E_{l-1}^+ \circ E_l^-$ *is a map from* $C^\infty_{(l,l-1)}(G/K)$ *to* $C^\infty_{(l,l-1)}(G/K)$. *Especially for any* $\eta \in \widehat{K}$ *and* $\tau = (l, l-1)$, *this is a map from* $C^\infty_{\eta,\tau}(K\backslash G/K)$ *to* $C^\infty_{\eta,\tau}(K\backslash G/K)$.

We obtain radial parts of shift operators as below.

Proposition 5.6(the even case). *For* $\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$ *with* $\eta = (k,k), \tau = (l,l)$, *radial parts of* $D_l^{\pm}$ *are given as follows.*

(i)
$$\begin{aligned}R(D_l^+)\phi = [&2\partial_{x_1}\partial_{x_2}\\
&+\{-2l\coth 2x_2 + 2k\,\mathrm{sh}^{-1}2x_2 + \coth(x_1+x_2) - \coth(x_1-x_2)\}\partial_{x_1}\\
&+\{-2l\coth 2x_1 + 2k\,\mathrm{sh}^{-1}2x_1 + \coth(x_1+x_2) + \coth(x_1-x_2)\}\partial_{x_2}\\
&+2(l\coth 2x_1 - k\,\mathrm{sh}^{-1}2x_1)(l\coth 2x_2 - k\,\mathrm{sh}^{-1}2x_2)\\
&-(l\coth 2x_2 - k\,\mathrm{sh}^{-1}2x_2)(\coth(x_1+x_2)+\coth(x_1-x_2))\\
&-(l\coth 2x_1 - k\,\mathrm{sh}^{-1}2x_1)(\coth(x_1+x_2)-\coth(x_1-x_2))]\phi.\end{aligned}$$

(ii)
$$\begin{aligned}R(D_l^-)\phi = [&2\partial_{x_1}\partial_{x_2}\\
&+\{2l\coth 2x_2 - 2k\,\mathrm{sh}^{-1}2x_2 + \coth(x_1+x_2) - \coth(x_1-x_2)\}\partial_{x_1}\\
&+\{2l\coth 2x_1 - 2k\,\mathrm{sh}^{-1}2x_1 + \coth(x_1+x_2) + \coth(x_1-x_2)\}\partial_{x_2}\\
&+2(l\coth 2x_1 - k\,\mathrm{sh}^{-1}2x_1)(l\coth 2x_2 - k\,\mathrm{sh}^{-1}2x_2)\\
&+(l\coth 2x_2 - k\,\mathrm{sh}^{-1}2x_2)(\coth(x_1+x_2)+\coth(x_1-x_2))\\
&+(l\coth 2x_1 - k\,\mathrm{sh}^{-1}2x_1)(\coth(x_1+x_2)-\coth(x_1-x_2))]\phi.\end{aligned}$$

Proposition 5.7 (the odd case). *For* $\phi = \phi_{01} v_0^\eta \otimes v_1^{\tau^*} + \phi_{10} v_1^\eta \otimes v_0^{\tau^*} \in C^\infty_{\eta,\tau}(K\backslash G/K)$, *and* $\phi^+ = \phi_{00}^+ v_0^\eta \otimes v_0^{\tau_+^*} + \phi_{11}^+ v_1^\eta \otimes v_1^{\tau_+^*} \in C^\infty_{\eta,\tau_+}(K\backslash G/K)$ *with*

$\eta = (k, k-1), \tau = (l, l-1)$ *and* $\tau_+ = (l-1, l-2)(k \equiv l \mod 2)$ *, radial parts of* E^+_{l-1}, E^-_l *are given as follows.*

$$\begin{aligned}
\text{(i) } R(E^+_{l-1})\phi^- = &[\{\partial_{x_2} - (l-2)\coth 2x_2 + k\,\mathrm{sh}^{-1} 2x_2 \\
&+ \frac{1}{2}(\coth(x_1+x_2) - \coth(x_1-x_2))\}\phi^-_{00} \\
&+ \frac{1}{2}(\mathrm{sh}^{-1}(x_1+x_2) + \mathrm{sh}^{-1}(x_1-x_2))\phi^-_{11}]v^\eta_0 \otimes v^{\tau^*}_1 \\
&+[-\{\partial_{x_1} - (l-2)\coth 2x_1 + k\,\mathrm{sh}^{-1} 2x_1 \\
&+ \frac{1}{2}(\coth(x_1+x_2) + \coth(x_1-x_2))\}\phi^-_{11} \\
&- \frac{1}{2}(\mathrm{sh}^{-1}(x_1+x_2) - \mathrm{sh}^{-1}(x_1-x_2))\phi^-_{00}]v^\eta_1 \otimes v^{\tau^*}_0 .\\
\text{(ii) } R(E^-_l)\phi = &[-\{\partial_{x_2} + l\coth 2x_2 - k\,\mathrm{sh}^{-1} 2x_2 \\
&+ \frac{1}{2}(\coth(x_1+x_2) - \coth(x_1-x_2))\}\phi_{01} \\
&- \frac{1}{2}(\mathrm{sh}^{-1}(x_1+x_2) - \mathrm{sh}^{-1}(x_1-x_2))\phi_{10}]v^\eta_0 \otimes v^{\tau^*_-}_0 \\
&+[\{\partial_{x_1} + l\coth 2x_1 - k\,\mathrm{sh}^{-1} 2x_1 \\
&+ \frac{1}{2}(\coth(x_1+x_2) + \coth(x_1-x_2))\}\phi_{10} \\
&+ \frac{1}{2}(\mathrm{sh}^{-1}(x_1+x_2) + \mathrm{sh}^{-1}(x_1-x_2))\phi_{01}]v^\eta_1 \otimes v^{\tau^*_-}_1 .
\end{aligned}$$

§ 6 Differential equations satisfied by spherical functions.

In this section, we will obtain the system of differential equations satisfied by spherical functions.

The action of $X \in U(\mathfrak{g})$ on $\psi_{H_\pi}(g) = \sum_{n,m}\langle \pi(g)i(v^\tau_n), j^*(w^{\eta^*}_m)\rangle v^\eta_m \otimes w^{\tau^*}_n$ is defined by

$$X\psi_{H_\pi}(g) = \sum_{n,m}\langle \pi(g)\pi(X)i(v^\tau_n), j^*(w^{\eta^*}_m)\rangle v^\eta_m \otimes w^{\tau^*}_n .$$

Since the Casimir operator L acts on $H_\pi = \mathrm{Ind}^G_P(\sigma \otimes a^{\mu+\rho} \otimes 1_N)$ $(\mu = (\mu_1, \mu_2))$ as the scalar called the infinitesimal character $\mu_1^2 + \mu_2^2 - 5$, then $\pi(L)i(v^\tau_n) = (\mu_1^2 + \mu_2^2 - 5)i(v^\tau_n)$. From this and the above equation, we have

$$L\psi_{H_\pi} = (\mu_1^2 + \mu_2^2 - 5)\psi_{H_\pi}.$$

It can be shown that $D^{\pm}\phi_{H_\pi}$ and $E^{\pm}\phi_{H_\pi}$ are also spherical functions attached to H_π. So are $D^+_{l-2}\circ D^-_l\phi_{H_\pi}$ and $E^+_{l-1}\circ E^-_l\phi_{H_\pi}$. As for D^+, for following two K-maps

$$p : V_{\tau^*}\otimes \mathrm{Ad}_{\mathfrak{p}_-}\otimes \mathrm{Ad}_{\mathfrak{p}_-}\to V_{\tau^*_+} : \text{projection},$$
$$m : V_\tau\otimes \mathrm{Ad}_{\mathfrak{p}_+}\otimes \mathrm{Ad}_{\mathfrak{p}_+}\ni v\otimes X\otimes Y\to \pi(YX)i(v)\in H_\pi,$$

$m\circ {}^tp\in \mathrm{Hom}_K(V_{\tau_+}$ and

$$D^+\phi_{H_\pi}(g)=\sum_{n,m}\langle \pi(g)\pi(X)m\circ {}^tp(v_n^\tau), j^*(w_m^{\eta^*})\rangle v_m^\eta\otimes w_n^{\tau^*}$$

hold. From the fact that the multiplicity of τ in $H_\pi|_K$ is one in both even and odd cases, the dimension of spherical functions attached to H_π in $C^\infty_{\eta,\tau}(K\backslash G/K)$ is one. Therefore $D^+_{l-2}\circ D^-_l$ and $E^+_{l-1}\circ E^-_l$ act on ψ_{H_π} as scalars. These scalars can be calculated by the realization of V_τ in $C^\infty(K)$ and the explicit action of the shift operator on the space.

Theorem 6.1 (the even case). *For $\eta=(k,k),\tau=(l,l)$ with $k\equiv l \bmod 2$, the spherical function $\psi_{H_\pi}\in C^\infty_{\eta,\tau}(K\backslash G/K)$ satisfies the following system of differential equations.*

(i) $L\psi_{H_\pi}=(\mu_1^2+\mu_2^2-5)\psi_{H_\pi}$,

(ii) $D^+_{l-2}\circ D^-_l\psi_{H_\pi}=4\{\mu_1^2-(l-1)^2\}\{\mu_2^2-(l-1)^2\}\psi_{H_\pi}$.

Theorem 6.2 (the odd case). *For $\eta=(k,k-1),\tau=(l,l-1)$ with $k\equiv l \bmod 2$, the spherical function $\psi_{H_\pi}\in C^\infty_{\eta,\tau}(K\backslash G/K)$ satisfies the following system of differential equations.*

(i) $L\psi_{H_\pi}=(\mu_1^2+\mu_2^2-5)\psi_{H_\pi}$,

(ii) $E^+_{l-1}\circ E^-_l\psi_{H_\pi}=\begin{cases}-\{\mu_1^2-(l-1)^2\}\psi_{H_\pi} & \text{if } l:\text{odd}\\ -\{\mu_2^2-(l-1)^2\}\psi_{H_\pi} & \text{if } l:\text{even}\end{cases}$

For the generalized principal series H_{π_J}, we can obtain the system of differential equations satisfied by $\psi_{H_{\pi_J}}$ similarly.

Since the Casimir operator L acts of $H_{\pi_J}=\mathrm{Ind}^G_{P_J}(\sigma_J\otimes a_J^{\nu_1+\rho_J}\otimes 1_{N_J})$ as the scalar called the infinitesimal character $\nu_1^2+(l-1)^2-5$,

$$L\psi_{H_{\pi_J}}=(\nu_1^2+(l-1)^2-5)\psi_{H_{\pi_J}}.$$

On the other hand, for $\tau=(l,l)$, $D^-_l\phi_{H_{\pi_J}}$ is the spherical function attached to in H_{π_J} included in $C^\infty_{\eta,(l-2,l-2)}(K\backslash G/K)$, which must be 0 from Lemma 2.5. Therefore $D^-_l\psi_{H_{\pi_J}}=0$.

Similarly, for $\tau = (l, l-1)$, $E_l^- \psi_{H_{\pi_J}} = 0$.

Thus we have the following.

Theorem 6.3 (the even case). *For $\eta = (k,k), \tau = (l,l)$ with $k \equiv l \mod 2$, the spherical function $\psi_{H_{\pi_J}} \in C^\infty_{\eta,\tau}(K\backslash G/K)$ satisfies the following system of differential equations.*

(i) $L\psi_{H_{\pi_J}} = \{\nu_1^2 + (l-1)^2 - 5\}\psi_{H_{\pi_J}}$,

(ii) $D_l^- \psi_{H_{\pi_J}} = 0$.

Theorem 6.4 (the odd case). *For $\eta = (k, k-1), \tau = (l, l-1)$ with $k \equiv l \mod 2$, the spherical function $\psi_{H_{\pi_J}} \in C^\infty_{\eta,\tau}(K\backslash G/K)$ satisfies the following system of differential equations.*

(i) $L\psi_{H_{\pi_J}} = \{\nu_1^2 + (l-1)^2 - 5\}\psi_{H_{\pi_J}}$,

(ii) $E_l^- \psi_{H_{\pi_J}} = 0$.

Remark 6.5.

(i) *In the case $k = l = 0$ and more than or equal two variables, systems of differential equations in* Theorem 6.1 *and* Theorem 6.3 *are defined in* [DG1], [DG2] *with more general parameters, which are generalizations of root multiplicities without using the geometry of G/K.*

The polynomial solutions of the system in Theorem 6.1 *with $k = l = 0$ are given in* [DG2] *and the general solution of the system in* Theorem 6.3 *with $k = l = 0$ are obtained in* [DG1].

(ii) *In the case $k = l = 0$, the system in* Theorem 6.1 *are defined as a member of the family of commuting differential operators invariant under the action of B_2-type Weyl group in* [OO] *and the system in* Theorem 6.3 *are defined as the reducible system of that. Those systems have more parameters than the systems defined in* [DG1], [DG2].

§ 7 Spherical functions of H_{π_J}.

We will obtain solutions of systems of differential equations in Theorem 6.3 and 6.4. At first, we take the conjugate of differential equations by the function $\delta(x,y;k,l) = (\mathrm{ch}\, x \cdot \mathrm{ch}\, y)^{\frac{k+l}{2}}(\mathrm{sh}\, x \cdot \mathrm{sh}\, y)^{-\frac{k-l}{2}}$. In the even case, this makes the shift operator independent from k and l and the Casimir operator become the similar form in the case $k = l = 0$. This is done in [H2] and [Sh] for the Casimir operator. Secondly, we change variables as $x_1 = -\mathrm{sh}^2 x, x_2 = -\mathrm{sh}^2 y$.

Proposition 7.1(the even case). *If $\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$ is the solution of the system in Theorem 6.3, then $\psi(x,y) = \delta(x,y;k,l)\phi(x,y)$ satisfies the following* *differential equations with* $x_1 = -\operatorname{sh}^2 x, x_2 = -\operatorname{sh}^2 y$. *Here* $\partial_{x_i} = \frac{\partial}{\partial x_i}$

$$
\begin{aligned}
&\text{(i)}\ \ [\sum_{i=1}^{2} y_i(y_i-1)\partial_{y_i}^2 + \{(2-l)y_1 - 1 - \frac{k-l}{2} + \frac{y_1(y_1-1)}{y_1-y_2}\}\partial_{y_1} \\
&\qquad + \{(2-l)y_2 - 1 - \frac{k-l}{2} - \frac{y_2(y_2-1)}{y_1-y_2}\}\partial_{y_2} - \frac{1}{4}\{\nu_1^2 - (l-2)^2\}]\psi = 0, \\
&\text{(ii)}\ \ [\partial_{y_1}\partial_{y_2} - \frac{1}{2}\frac{1}{y_1-y_2}\partial_{y_1} + \frac{1}{2}\frac{1}{y_1-y_2}\partial_{y_2}]\psi = 0.
\end{aligned}
$$

Proposition 7.2(the odd case). *Let* $\phi = \sum_{i,j=0}^{1} \phi_{ij} v_i^{\eta} \otimes v_j^{\tau^*} \in C_{\eta,\tau}^{\infty}(K\backslash G/K)$ *satisfy the system in Theorem 6.4. If we set*

$$
\begin{cases}
\psi_{01}(x,y) = \delta(x,y;k,l)\cdot(\operatorname{ch} x)^{-1}\phi_{01}(x,y), \\
\psi_{10}(x,y) = \delta(x,y;k,l)\cdot(\operatorname{ch} y)^{-1}\phi_{10}(x,y),
\end{cases}
$$

then ψ_{01} *satisfies the following differential equations with* $x_1 = -\operatorname{sh}^2 x, x_2 = -\operatorname{sh}^2 y$.

$$
\begin{aligned}
&\text{(i)}\ \ [\sum_{i=1}^{2} y_i(y_i-1)\partial_{y_i}^2 + \{(2-l)y_1 - 1 - \frac{k-l}{2} + \frac{y_1(y_1-1)}{y_1-y_2}\}\partial_{y_1} \\
&\qquad + \{-ly_2 - \frac{k-l}{2} - 3\frac{y_2(y_2-1)}{y_1-y_2}\}\partial_{y_2} - \frac{1}{4}\{\nu_1^2 - (l-1)^2 - 2\}]\psi_{01} = 0, \\
&\text{(ii)}\ \ [\partial_{y_1}\partial_{y_2} - \frac{1}{2}\frac{1}{y_1-y_2}\partial_{y_1} + \frac{3}{2}\frac{1}{y_1-y_2}\partial_{y_2}]\psi_{01} = 0.
\end{aligned}
$$

In order to obtain these equations, we use the relation $\phi_{10}(y,x) = -\phi_{01}(x,y)$.

In [DG1], the solution of the system in Theorem 7.1 is studied. We can obtain the solution of the system in Theorem 7.2 by the same method in [DG1]. Key lemmas are the following two lemmas. For $B_1 = B_2$, Lemma 7.3 is proved in [DG1] and Lemma 7.4 is proved in [DG2].

Lemma 7.3. *When* $\operatorname{Re} B_1, \operatorname{Re} B_2 > 0$, *the function* $f(y_1, y_2)$ *which is analytic around the origin and satisfies*

$$
[\partial_{y_1}\partial_{y_2} - B_2\frac{1}{y_1-y_2}\partial_{y_1} + B_1\frac{1}{y_1-y_2}\partial_{y_2}]f = 0
$$

has the following series expansion and integral representation.

(i) $f(y_1,y_2)=\sum_{m_i\geq 0}\frac{(B_1)_{m_1}(B_2)_{m_2}\xi(m_1+m_2)}{m_1!m_2!}y_1^{m_1}y_2^{m_2}$

Here we set

$$(\lambda)_k=\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$$

and ξ *is some function on* $\mathbb{N}$.

(ii) $f(y_1,y_2)=\int_0^1 F(ty_1+(1-t)y_2)t^{B_1-1}(1-t)^{B_2-1}dt$

Here F *is some analytic function around the origin.*

(iii) *If two equations in* (i) and (ii) coincide with each other, then

$$F(s)=\frac{\Gamma(B_1+B_2)}{\Gamma(B_1)\Gamma(B_2)}\sum_{n\geq 0}\frac{(B_1+B_2)_k\xi(k)}{k!}s^k$$

holds.

Lemma 7.4. *Let* $P=\sum_{i=1}^2 y_i(y_i-1)\partial_{y_i}^2+\{(A+B_1-B_2+1)y_1+B_2-C+2B_2\frac{y_1(y_1-1)}{y_1-y_2}\}\partial_{y_1}+\{(A-B_1+B_2+1)y_2+B_1-C-2B_1\frac{y_2(y_2-1)}{y_1-y_2}\}\partial_{y_2}-\lambda$, $L=z(z-1)\frac{d^2}{dz^2}-\{C-(A+B_1+B_2+1)z\}\frac{d}{dz}-\lambda$ *and let the linear operator* T_{B_1,B_2} *on the functions which are analytic around the origin be*

$$(T_{B_1,B_2}f)(y_1,y_2)=\int_0^1 f(ty_1+(1-t)y_2)t^{B_1-1}(1-t)^{B_2-1}dt$$

for $A,C,\lambda\in\mathbb{C}$ *and* $\mathrm{Re}\,B_1,\mathrm{Re}\,B_2>0$. *Then we have*

$$P\circ T_{B_1,B_2}=T_{B_1,B_2}\circ L.$$

Using these lemmas, we can show the following theorems.

Theorem 7.5(the even case).

(i) *The analytic solution of the system in Theorem 7.1 has the following series expansion up to scalar.*

$$\psi(y_1,y_2)=\sum_{m_i\geq 0}\frac{(\frac{1}{2})_{m_1}(\frac{1}{2})_{m_2}(-\mu_+)_{m_1+m_2}(-\mu_-)_{m_1+m_2}}{m_1!m_2!(1)_{m_1+m_2}(\frac{3+k-l}{2})_{m_1+m_2}}y_1^{m_1}y_2^{m_2}$$

Here we set $\mu_\pm=(l-2\pm\nu_1)/2$.

(ii) *The analytic solution of the system in Theorem 7.1 has the following integral representation up to scalar.*

$$\psi(y_1,y_2)=\int_0^1 {}_2F_1(-\mu_+,-\mu_-;\frac{3+k-l}{2};ty_1+(1-t)y_2)t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt$$

Here ${}_2F_1$ *is the classical* Gaussian *hypergeometric function which is analytic around the origin.*

Remark 7.6. *When* $\nu_1 = \pm l$ *(this means* $-\mu_{\mp} = 1$*), the solution given above becomes* Appell's *hypergeometric function*

$$F_1(-l+1, 1/2, 1/2, (3+k-l)/2; y_1, y_2).$$

Theorem 7.7(the odd case). (i) *The analytic solution of the system in Theorem 7.2 has the following series expansion up to scalar.*

$$\psi_{01}(y_1, y_2) = \sum_{m_i \geq 0} \frac{(\frac{3}{2})_{m_1}(\frac{1}{2})_{m_2}(-\mu_+)_{m_1+m_2}(-\mu_-)_{m_1+m_2}}{m_1!m_2!(2)_{m_1+m_2}(\frac{3+k-l}{2})_{m_1+m_2}} y_1^{m_1} y_2^{m_2}$$

Here we set $\mu_\pm = (l - 2 \pm \sqrt{\nu_1^2 - 2l + 1})/2$.

(ii) *The analytic solution of the system in Theorem 7.2 has the following integral representation up to scalar.*

$$\psi_{01}(y_1, y_2) = \int_0^1 {}_2F_1(-\mu_+, -\mu_-; \frac{3+k-l}{2}; ty_1 + (1-t)y_2)t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt$$

Here ${}_2F_1$ *is the classical* Gaussian *hypergeometric function which is analytic around the origin.*

Remark 7.8. *When* $\nu_1 = \pm\sqrt{l^2 + 6l + 3}$ *(this means* $-\mu_- = 2$*), the solution given above is* Appell's *hypergeometric function*

$$F_1(-l, 3/2, 1/2, (3+k-l)/2; y_1, y_2).$$

Remark 7.9. *The hypergeometric series defined in above theorems is denoted by*

$$F_{10}\begin{pmatrix} a & b & c_1 & c_2 & ; y_1, & y_2 \\ d & e & & & & \end{pmatrix}$$
$$= \sum_{m_i \geq 0} \frac{(a)_{m_1+m_2}(b)_{m_1+m_2}(c_1)_{m_1}(c_2)_{m_2}}{m_1!m_2!(d)_{m_1+m_2}(e)_{m_1+m_2}} y_1^{m_1} y_2^{m_2}$$

in [T].

§ 8 Appell's hypergeometric functions.

Differential operators

$$\begin{aligned} R_1 &= w_1(1-w_1)\partial_{w_1}^2 - w_1 w_2 \partial_{w_1}\partial_{w_2} \\ &\qquad + \{\gamma - (\alpha+\beta+1)w_1\}\partial_{w_1} - \beta w_2 \partial_{w_2} - \alpha\beta, \\ R_2 &= w_2(1-w_2)\partial_{w_2}^2 - w_1 w_2 \partial_{w_1}\partial_{w_2} \\ &\qquad + \{\gamma' - (\alpha+\beta'+1)w_2\}\partial_{w_2} - \beta' w_1 \partial_{w_1} - \alpha\beta', \end{aligned}$$

are known as Appell's hypergeometric differential operators whose analytic kernel is usually written by $F_2(\alpha;\beta,\beta';\gamma,\gamma';w_1,w_2)$.

On the other hand, the system of differential equations in Theorem 7.1 and 7.2 can be written by the following P and Q in general.

$$\begin{aligned} P =& \sum_{i=1}^{2} y_i(y_i-1)\partial_{y_i}^2 \\ &+ \{(A+B_1-B_2+1)y_1 + B_2 - C + 2B_2\frac{y_1(y_1-1)}{y_1-y_2}\}\partial_{y_1} \\ &+ \{(A-B_1+B_2+1)y_2 + B_1 - C - 2B_1\frac{y_2(y_2-1)}{y_1-y_2}\}\partial_{y_2} - \lambda \\ Q =& \partial_{y_1}\partial_{y_2} - B_2\frac{1}{y_1-y_2}\partial_{y_1} + B_1\frac{1}{y_1-y_2}\partial_{y_2} \end{aligned}$$

For parameters A, B_1, B_2, λ, we denote $-\mu_\pm$ by solutions of the quadratic equation $x^2-(A+B_1+B_2)x-\lambda=0$. From easy calculation, we can see these operators are related as follows.

Lemma 8.1. *Let P, Q, R_1 and R_2 be as above. If we set $w_1 = 1 - y_1/y_2, w_2 = 1/y_2$, then we have*

$$\begin{aligned} (-w_2)^{-k}\circ P \circ (-w_2)^k &= \frac{2-2w_1-2w_2+w_1w_2}{w_1}R_1 + w_2 R_2 \\ (-w_2)^{-k}\circ Q \circ (-w_2)^k &= -\frac{w_2^2}{w_1}R_1, \end{aligned}$$

with

$$\begin{cases} \alpha = k, \\ \beta = B_1, \beta' = k+1-C, \\ \gamma = B_1+B_2, \gamma' = 2k+1-A-B_1-B_2, \\ k = -\mu_+ \ \text{or}\ k = -\mu_-. \end{cases}$$

From above lemma and Theorem 7.5 and 7.7, we expect that there is a relation between Appell's hypergeometric functions F_2 and hypergeometric function F_{10}. In fact, the following theorem holds.

Theorem 8.2. *Appell's hypergeometric function F_2 and hypergeometric function F_{10} have the following relation for $B, C, \mu_\pm, C+\mu_\pm, D+\mu_\pm, D+2\mu_\pm, C-D-\mu_\pm \notin \mathbb{Z}$. Here we set $B = B_1 + B_2$ and $D = A + B_1 + B_2$.*

$$
\begin{aligned}
&F_{10}\begin{pmatrix} -\mu_+ & -\mu_- & B_1 & B_2 & \\ B & C & & & ;\ y_1,\ y_2 \end{pmatrix} \\
=&\frac{\Gamma(\mu_+ - \mu_-)\Gamma(C)}{\Gamma(-\mu_-)\Gamma(\mu_+ + C)} \\
&\times (-y_2)^{\mu_+} F_2(-\mu_+; B_1, -\mu_+ - C + 1; B, \mu_- - \mu_+ 1; 1 - y_1/y_2, 1/y_2) \\
+&\frac{\Gamma(\mu_- - \mu_+)\Gamma(C)}{\Gamma(-\mu_+)\Gamma(\mu_- + C)} \\
&\times (-y_2)^{\mu_-} F_2(-\mu_-; B_1, -\mu_- - C + 1; B, \mu_+ - \mu_- + 1; 1 - y_1/y_2, 1/y_2).
\end{aligned}
$$

Here $(-y_2)^{\mu_+}$ and $(-y_2)^{\mu_-}$ are defined in $|\arg(-y_2)| < \pi$.

Remark 8.3.

(i) *The equation in* Theorem 8.2 *is reduced to the relation between Appell's hypergeometric functions F_1 and F_2:*

$$
\begin{aligned}
&F_1(A; B_1, B_2; C; y_1, y_2) \\
=&\frac{\Gamma(B-A)\Gamma(C)}{\Gamma(B)\Gamma(C-A)} \\
&\times (-y_2)^{-A} F_2(A; B_1, A - C + 1; B, A - B + 1; 1 - y_1/y_2, 1/y_2) \\
+&\frac{\Gamma(A-B)\Gamma(C)}{\Gamma(A)\Gamma(C-B)} \\
&\times (-y_2)^{-B} F_2(B; B_1, B - C + 1; B, -A + B + 1; 1 - y_1/y_2, 1/y_2),
\end{aligned}
$$

for $A, B, C, A-B, B-C, C-A \notin \mathbb{Z}$, when $\lambda = -AB$.

Moreover, this relation is reduced to the famous relation of the Gaussian hypergeometric function:

$$
\begin{aligned}
{}_2F_1(A, B; C; x) = &\frac{\Gamma(B-A)\Gamma(C)}{\Gamma(B)\Gamma(C-A)}(-x)^{-A}{}_2F_1(A, A-C+1; A-B+1; 1/x) \\
+&\frac{\Gamma(A-B)\Gamma(C)}{\Gamma(A)\Gamma(C-B)}(-x)^{-B}{}_2F_1(B, B-C+1; B-A+1; 1/x)
\end{aligned}
$$

by setting $y_1 = y_2 = x$.

(ii) *The equation in* Theorem 8.2 *can be also obtained by using connection formulas of F_2 given in* [T Proposition 2.1 (5)], *where the relation between F_1 and F_2 is not mentioned.*

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