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The embeddings of discrete series into some induced representations for an exceptional real simple Lie group of type G_2

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(Joint work with H.YAMASHITA in Hokkaido University)

The problem of determining embeddings of discrete series representations of a connected semisimple Lie group into its various kind of induced representations has interested us for twenty years or so. For embeddings into principal series representations of real rank one groups are well-studied (cf. [3]). For groups of higher real rank, H.Yamashita gave, in his paper [4], the complete description of the embeddings of discrete series into principal series or into Gel'fand-Graev representations in case of $SU(2,2)$. Using his method, we have computed in case of G_2 type groups, and the author is now computing Whittaker models for G_2 type groups. In this talk, the results in the joint work with H.Yamashita [5] are reported. Further the author intends to explain some results on Whittaker models.

1. Structure of a simple Lie algebra of type G_2

Let $\mathfrak{g}$ be a complex simple Lie algebra of type G_2, $\mathfrak{g}_0$ a normal real form of $\mathfrak{g}$, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$, and θ the corresponding Cartan involution. Here, $\mathfrak{k}_0 = \{X \in \mathfrak{g}_0 \,|\, \theta X = X\}$, $\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 \,|\, \theta X = -X\}$. We denote the complexifications of $\mathfrak{k}_0$, $\mathfrak{p}_0$ etc. by $\mathfrak{k}$, $\mathfrak{p}$ etc., omitting the subscript $_0$. Since $\mathfrak{k}_0 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\operatorname{rank} \mathfrak{g} = \operatorname{rank} \mathfrak{k} = 2$, and we can take a compact Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}_0$.

We denote the root system of $\mathfrak{g}$ relative to $\mathfrak{t}$ by Δ, and a positive system of Δ and the Weyl group of Δ are denoted by Δ^+ and W respectively. Then,

$$\Delta = \{\pm\alpha_1,\ \pm\alpha_2,\ \pm(\alpha_1+\alpha_2),\ \pm(2\alpha_1+\alpha_2),\ \pm(3\alpha_1+\alpha_2),\ \pm(3\alpha_1+2\alpha_2)\}.$$

Here, α_1 is the short simple root of Δ, and α_2 is the long simple root of Δ. They satisfy the following relations:

$$|\alpha_2|^2 = 3|\alpha_1|^2 = \frac{1}{4}, \quad <\alpha_1, \alpha_2^\vee> = -1, \quad <\alpha_2, \alpha_1^\vee> = -3.$$

The system of compact (resp. non-compact) roots is denoted by Δ_c (resp. Δ_n) and put $\Delta_c^+ = \Delta \cap \Delta_c$, $\Delta_n^+ = \Delta \cap \Delta_n$. We may assume that $\Delta_c^+ = \{\alpha_1,\ 3\alpha_1 + 2\alpha_2\}$.

Let $B(\cdot,\cdot)$ be the Killing form of $\mathfrak{g}$ and $\bar{X}$ the complex conjugate of $X \in \mathfrak{g}$ relative to the compact real form $\mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$ of $\mathfrak{g}$. We equip $\mathfrak{g}$ with an inner product $(\cdot,\cdot)$ defined

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by $(X,Y) = -B(X,\bar{Y})$. Consider the root space decomposition $\mathfrak{g} = \mathfrak{t} + \sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$, then there exists an element E_α of $\mathfrak{g}_\alpha$ for each root α such that

$$B(E_\alpha, E_{-\alpha}) = \frac{2}{|\alpha|^2}, \quad E_{-\alpha} = -\bar{E}_\alpha. \tag{1}$$

Moreover, we can take E_α's in the following way:

$$[E_{10}, E_{01}] = E_{11}, \tag{2}$$
$$[E_{10}, E_{11}] = 2E_{21}, \tag{3}$$
$$[E_{10}, E_{21}] = 3E_{31}, \tag{4}$$
$$[E_{32}, E_{-3,-1}] = E_{01}. \tag{5}$$

Here, E_{ij} stands for $E_{i\alpha_1+j\alpha_2}$, and E_{ij}'s are uniquely determined under conditions (1)-(5) above when E_{10} and E_{01} are given. From now on, E_{ij}'s are assumed to satisfy relations (1)-(5), and define H_{ij}, $\tilde{H}_1$, $\tilde{H}_2$ and $\mathfrak{a}_0$ by

$$H_{ij} = [E_{ij}, E_{-i,-j}],$$
$$\tilde{H}_1 = E_{01} + E_{0,-1},$$
$$\tilde{H}_2 = E_{21} + E_{-2,-1},$$
$$\mathfrak{a}_0 = \mathbf{R}\tilde{H}_1 + \mathbf{R}\tilde{H}_2.$$

Then we see that $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$. We equip $\mathfrak{a}_0^*$ with the lexicographic order with respect to the ordered basis $(\tilde{H}_1, \tilde{H}_2)$ of $\mathfrak{a}_0$. We denote the system of restricted roots of $\mathfrak{g}_0$ relative to $\mathfrak{a}_0$ by Ψ and the positive system of Ψ is denoted by Ψ^+. Then,

$$\Psi^+ = \{\nu_1, \nu_2, \nu_1+\nu_2, 2\nu_1+\nu_2, 3\nu_1+\nu_2, 3\nu_1+2\nu_2\}.$$

Here, ν_1 and ν_2 are linear forms on $\mathfrak{a}$ defined through the conditions:

$$\nu_1(\tilde{H}_1) = 0,\ \nu_1(\tilde{H}_2) = 2,\ \nu_2(\tilde{H}_1) = 1,\ \nu_2(\tilde{H}_2) = -3.$$

Using this Ψ^+, we define a subspace $\mathfrak{n}_0$ of $\mathfrak{g}_0$ by $\mathfrak{n}_0 = \sum_{\lambda\in\Psi^+}(\mathfrak{g}_\lambda)_0$, where $(\mathfrak{g}_\lambda)_0 = \{X \in \mathfrak{g}_0 \mid [H,X] = \lambda(H)X\ (\forall H \in \mathfrak{a}_0)\}$. Then we have an Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ of $\mathfrak{g}_0$.

Let $G_{\mathbf{C}}$ be a connected, simply connected simple Lie group with Lie algebra $\mathfrak{g}$, G the analytic subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{g}_0$. In this talk, we are going to consider mainly the embeddings of discrete series of G into its principal series.

2. Structure of the group G

Let $G = KAN$ be the Iwasawa decomposition of G corresponding to the decomposition of $\mathfrak{g}_0$ given in the last section. Put

$$(\mathfrak{k}_1)_0 = \sqrt{-1}\mathbf{R}H_{10} + \mathbf{R}(E_{10} - E_{-1,0}) + \sqrt{-1}\mathbf{R}(E_{10} + E_{-1,0}),$$
$$(\mathfrak{k}_2)_0 = \sqrt{-1}\mathbf{R}H_{32} + \mathbf{R}(E_{32} - E_{-3,-2}) + \sqrt{-1}\mathbf{R}(E_{32} + E_{-3,-2}),$$

then $\mathfrak{k}_0 = (\mathfrak{k}_1)_0 \oplus (\mathfrak{k}_2)_0$. For the structures of $(\mathfrak{k}_j)_0$, $j = 1, 2$, they are isomorphic to $\mathfrak{su}(2)$, and the isomorphims $\zeta_j : (\mathfrak{k}_j)_0 \longrightarrow \mathfrak{su}(2)$, $j = 1, 2$, are given as follows:

$$\begin{array}{ccll} \mathfrak{su}(2) & \longrightarrow & (\mathfrak{k}_1)_0 & (\text{resp.}\,(\mathfrak{k}_2)_0) \\ \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} & \longrightarrow & \sqrt{-1}H_{10} & (\text{resp.}\,\sqrt{-1}H_{32}) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \longrightarrow & E_{10} - E_{-1,0} & (\text{resp.}\,E_{32} - E_{-3,-2}) \\ \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} & \longrightarrow & \sqrt{-1}(E_{10} + E_{-1,0}) & (\text{resp.}\,\sqrt{-1}(E_{32} + E_{-3,-2}))\,. \end{array}$$

Then there exists a covering homomorphism σ of $SU(2) \times SU(2)$ onto K whose differential is $\zeta_1 \oplus \zeta_2$. For an element g of $SU(2) \times SU(2)$, the image of g under σ is denoted by $g^\ddagger$. Now, by comparing the unit lattice $\{H \in \mathfrak{t}_0 \mid \exp H = 1\}$ of K with that of $SU(2) \times SU(2)$, we can see that $K \simeq (SU(2) \times SU(2))/D$ with $D = \{1, (-1_2, -1_2)\}$. Here 1_2 is the unit matrix of degree 2. Put $M = \{m \in K \mid \mathrm{Ad}(m)|_{\mathfrak{a}_0} = \mathrm{id}_{\mathfrak{a}_0}\}$, then we obtain, by a straightforward calculation, that $M = \{1, m_1, m_2, m_1m_2\}$ with m_1 and $m_2 \in K$ given by

$$\begin{aligned} m_1 &= \left(\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \right)^\ddagger \\ m_2 &= \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)^\ddagger \end{aligned}$$

Therefore M is generated by two elements m_1 and m_2 with $m_1^2 = m_2^2 = 1$, $m_1m_2 = m_2m_1$, and $M \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Next, define a character $\sigma_{\epsilon_1,\epsilon_2}$ of M through $\sigma_{\epsilon_1,\epsilon_2}(m_i) = \epsilon_i$ for $i = 1, 2$. Then, $\hat{M} = \{\sigma_{\epsilon_1,\epsilon_2} \mid \epsilon_i = \pm 1\ (i = 1, 2)\}$.

Put $P_m = MAN$ for M above, then we have a minimal parabolic subgroup of G and consider principal series $\mathrm{Ind}_{P_m}^G(\sigma_{\epsilon_1,\epsilon_2} \otimes e^\mu \otimes 1_N)$ for this P_m.

3. Structure of an irreducible K-module

Since $\mathfrak{k}_0 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{k} \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, every finite-dimensional irreducible $\mathfrak{k}$-module is an exterior tensor product of two finite-dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules. So we first explain some facts about irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules.

Let X, Y and H be elements of $\mathfrak{sl}(2, \mathbb{C})$ satisfying

$$[H, X] = 2X\,, \quad [H, Y] = -2Y\,, \quad [X, Y] = H\,.$$

For $(d+1)$-dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$-module (say V_d), there exists a basis $\{e_p^{(d)} \mid p = -d, -d+2, \ldots, d\}$ of V_d such that

$$\begin{cases} H \cdot e_p^{(d)} = p e_p^{(d)} \\ X \cdot e_p^{(d)} = x_p^{(d)} e_{p+2}^{(d)} \\ Y \cdot e_p^{(d)} = x_{p-2}^{(d)} e_{p-2}^{(d)} \end{cases} \qquad (p = -d, -d+2, \ldots, d)$$

with $x_p^{(d)} = \frac{1}{2}\sqrt{(d-p)(d+p+2)}$, where we regard $e_p^{(d)}$ as 0 if $p \notin \{-d, -d+2, \ldots, d\}$. Denote by $(\cdot,\cdot)$ the inner product on V_d for which $\{e_p^{(d)} \mid p = -d, -d+2, \ldots, d\}$ gives an orthonormal basis of V_d. Then $(\cdot,\cdot)$ is invariant under $\mathfrak{su}(2) \simeq \sqrt{-1}\mathbf{R}H + \mathbf{R}(X-Y) + \sqrt{-1}\mathbf{R}(X+Y)$. Now, $\mathfrak{p} \simeq V_3 \hat{\otimes} V_1$ as $\mathfrak{k}$-modules, where $\hat{\otimes}$ means an exterior tensor product. For the action of $\mathfrak{k}_1$ or $\mathfrak{k}_2$ on $\mathfrak{p}$, irreducible $\mathfrak{k}_j$-submodules of $\mathfrak{p}$ are given as follows:

- irreducible $\mathfrak{k}_1$-submodules of $\mathfrak{p}$:
  $\mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_1+\alpha_2} + \mathfrak{g}_{2\alpha_1+\alpha_2} + \mathfrak{g}_{3\alpha_1+\alpha_2} \simeq \mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_1+\alpha_2} + \mathfrak{g}_{2\alpha_1+\alpha_2} + \mathfrak{g}_{3\alpha_1+\alpha_2}$
- irreducible $\mathfrak{k}_2$-submodules of $\mathfrak{p}$:
  $\mathfrak{g}_{-3\alpha_1-\alpha_2} + \mathfrak{g}_{\alpha_2} \simeq \mathfrak{g}_{-2\alpha_1-\alpha_2} + \mathfrak{g}_{\alpha_1+\alpha_2} \simeq \mathfrak{g}_{-1\alpha_1-\alpha_2} + \mathfrak{g}_{2\alpha_1+\alpha_2} \simeq \mathfrak{g}_{-\alpha_2} + \mathfrak{g}_{3\alpha_1+\alpha_2}$

Put $\Delta_c^+ = \{\alpha_1, 3\alpha_1 + 2\alpha_2\}$, and let λ be a Δ_c^+-dominant, integral linear form on $\mathfrak{t}$ and $(\tau_\lambda, V_\lambda)$ a finite-dimensional irreducible representation of $\mathfrak{k}$ with highest weight λ. Then,

$$\begin{aligned} V_\lambda &\simeq V_r \hat{\otimes} V_s \quad \text{for} \quad r = \lambda(H_{10}),\ s = \lambda(H_{32}), \\ V_\lambda \otimes \mathfrak{p} &\simeq (V_r \otimes V_3) \hat{\otimes} (V_s \otimes V_1). \end{aligned}$$

Now, we are going to give an irreducible decomposition of a tensor product of two finite-dimensional irreducible $\mathfrak{sl}(2,\mathbf{C})$-modules. For two nonnegative integers m and n with $m \geq n$, $V_m \otimes V_n$ is decomposed as follows:

$$V_m \otimes V_n \simeq V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n}. \tag{6}$$

Let $P^{(m,n;j)}$ be the projection of $V_m \otimes V_n$ onto V_{m+n-2j} in (6), and we denote $e_p^{(m)} \otimes e_q^{(n)}$ by $e_{pq}^{(mn)}$.

4. Gradient type differential operators

The K-module $V_\lambda \otimes \mathfrak{p}$ decomposes as $V_\lambda \otimes \mathfrak{p} \simeq \bigoplus_{\beta \in \Delta_n} m(\beta) \cdot V_{\lambda+\beta}$, with multiplicity $m(\beta) = 0, 1$ for $\beta \in \Delta_n$. Take a positive system Δ^+ of Δ containing Δ_c^+, and put $V^- = \bigoplus_{\beta \in \Delta_n^+} m(-\beta) \cdot V_{\lambda-\beta}$. Let P_λ be the orthogonal projection of $V_\lambda \otimes \mathfrak{p}$ onto V^-. For a representation (τ, V) of K and a continuous representation (η, F) of a closed subgroup P of G on a Fréchet space F, define three function spaces $C_\tau^\infty(G)$, $C^\infty(G;\eta)$ and $C_\tau^\infty(G;\eta)$ by

$$\begin{aligned} C_\tau^\infty(G) &= \{f : G \overset{C^\infty}{\to} V \mid f(kg) = \tau(k) f(g) \quad (\forall (k,g) \in K \times G)\}, \\ C^\infty(G;\eta) &= \{f : G \overset{C^\infty}{\to} F \mid f(gp) = \delta_P(p)^{\frac{1}{2}} \eta(p)^{-1} f(g) \quad (\forall (g,p) \in G \times P)\}, \\ C_\tau^\infty(G;\eta) &= \{f : G \overset{C^\infty}{\to} V \otimes F \mid f(kgp) = \delta_P(p)^{\frac{1}{2}} (\tau(k) \otimes \eta(p)^{-1}) f(g) \\ &\qquad (\forall (k,g,p) \in K \times G \times P)\}, \end{aligned}$$

where δ_P is the modular function of P with respect to the left Haar measure of P.

Equip these spaces with the topology of uniform convergence of functions and their derivatives on each compact subset of G. Then G acts on the space $C^\infty(G;\eta)$ by left translation and we obtain a smooth representation of G on $C^\infty(G;\eta)$, which is denoted by $\mathrm{Ind}_P^G(\eta)$. By differentiating the action of G on $C^\infty(G;\eta)$, we see that $C^\infty(G;\eta)$ has a $(\mathfrak{g}, K)$-module structure. We denote this $(\mathfrak{g}, K)$-module also by $\mathrm{Ind}_P^G(\eta)$. We define a gradient-type differential operator $\mathcal{D}_\lambda$ on $C^\infty_{\tau_\lambda}(G)$ by

$$(\nabla f)(g) = \sum_j L_{X_j} f(g) \otimes \bar{X}_j\,,$$
$$(\mathcal{D}_\lambda f)(g) = P_\lambda(\nabla f(g))\,,$$

where L_X is the differentiation with respect to the right invariant vector field on G defined by an element X in $\mathfrak{g}$ and $\{X_j\}$ is an orthonormal basis of $\mathfrak{p}$ relative to the inner product $(\cdot,\cdot)$. Since $C^\infty_{\tau_\lambda}(G;\eta)$ is canonically embedded into $C^\infty_{\tau_\lambda}(G) \otimes F$, a differential operator $\mathcal{D}_{\lambda,\eta} : C^\infty_{\tau_\lambda}(G;\eta) \longrightarrow C^\infty_{\tau_\lambda^-}(G;\eta)$ can be defined by $\mathcal{D}_{\lambda,\eta} = (\mathcal{D}_\lambda \otimes \mathrm{id}_F)|_{C^\infty_{\tau_\lambda}(G;\eta)}$.

5. Parametrization of discrete series representations

Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$ as above, and T the maximal torus of K corresponding to $\mathfrak{t}_0$.

Definition 1. For a linear form λ on $\mathfrak{t}$, we say that λ is *K-integral* if the following assignment gives a unitary character of T:

$$T \ni \exp H \longmapsto e^{\lambda(H)} \in \mathbf{C}^\times \quad (H \in \mathfrak{t}_0)\,.$$

Let Ξ_c be the totality of Δ_c^+-dominant, regular, integral linear forms Λ on $\mathfrak{t}$. For each $\Lambda \in \Xi_c$, Δ^+ denotes the positive system of Δ for which Λ is Δ^+-dominant. By Harish-Chandra [1, Theorem 16], discrete series representations of G is parametrized by Ξ_c and we denote the discrete series of G with Harish-Chandra parameter Λ by π_Λ. Let Δ_J^+ $(J = I, II, III)$ be positive systems of Δ with simple roots listed below:

J	I	II	III
simple roots	$\alpha_1,\ \alpha_2$	$\alpha_1+\alpha_2,\ -\alpha_2$	$-\alpha_1-\alpha_2,\ 3\alpha_1+2\alpha_2$

For a discrete series π_Λ of G, the corresponding positive system $\Delta^+ = \{\alpha \in \Delta \,|\, (\alpha, \Lambda) > 0\} \subset \Delta$ is one of the above Δ_J^+'s. Define three subsets Ξ_J $(J = I, II, III)$ of Ξ_c by $\Xi_J = \{\Lambda \in \Xi_c \,|\, \Delta^+ = \Delta_J^+\}$. Put $\rho_c = \frac{1}{2}\sum_{\alpha\in\Delta_c^+}\alpha$, $\rho_n = \frac{1}{2}\sum_{\alpha\in\Delta_n^+}\alpha$ and $\lambda = \Lambda - \rho_c + \rho_n$. The discrete series π_Λ has the lowest K-type τ_λ and λ is called the *Blattner parameter* of π_Λ.

6. Explicit description of the projection P_λ

Since $\{e^{(rs)}_{pq} \mid p = -r, \dots, r;\ q = -s, \dots, s\}$ is a basis of $V_r \hat{\otimes} V_s \simeq V_\lambda$, an element f of $C^\infty_{\tau_\lambda}(G)$ is expressed uniquely in the following form:

$$f(g) = \sum_{p,q} c_{pq}(g) e^{(rs)}_{pq}, \tag{7}$$

where the sum is taken for $p = -r, -r+2, \dots, r$; $q = -s, -s+2, \dots, s$ and the coefficients c_{pq} are smooth functions on G.

In §5, we gave the parametrization of the discrete series. Fix an element Λ of Ξ_c and take the unique positive system Δ^+ so that Λ is Δ^+-dominant. Set $r' = \Lambda(H_{10})$ and $s' = \Lambda(H_{32})$ and put $r = \lambda(H_{10})$, $s = \lambda(H_{32})$ as before. Through the isomorphism $V_\lambda \otimes \mathfrak{p} \simeq (V_r \otimes V_1) \hat{\otimes} (V_s \otimes V_3)$, we identify these two $\mathfrak{k}$-modules. In the following, we list up the condition for Λ being an element of Ξ_J ($J = I, II, III$), the Blattner parameter λ of π_Λ, the condition for λ being far from the walls, and the projection P_λ.

Case I. $\Delta^+ = \Delta^+_I$: For $\Lambda \in \mathfrak{t}^*$,

$\Lambda \in \Xi_I$

$\Longleftrightarrow$ r', s' are positive integers with $s' - r' \geq 2$ and $s' - r'$ is even

$\Longleftrightarrow$ r, s are nonnegative integers with $s - r \geq 4$ and $s - r$ is even (*1)

In this case, we have

$\rho_n = 3\alpha_1 + 2\alpha_2$, $\lambda = \Lambda + \alpha_1 + \alpha_2$.

λ is far from the walls $\Longleftrightarrow$ $r \geq 4$, $s \geq 4$ (with (*1))

$P_\lambda = I \otimes P^{(s,1;1)}$

Case II. $\Delta^+ = \Delta^+_{II}$: For $\Lambda \in \mathfrak{t}^*$,

$\Lambda \in \Xi_{II}$

$\Longleftrightarrow$ r', s' are integers with $r' - s' \geq 2$, $r' \leq 3s' - 2$ and $r' - s'$ is even

$\Longleftrightarrow$ r, s are nonnegative integers with $r - s \geq 4$, $r \leq 3s$ and $r - s$ is even (*2)

In this case, we have

$\rho_n = 3\alpha_1 + \alpha_2$, $\lambda = \Lambda + \alpha_1$.

λ is far from the walls $\Longleftrightarrow$ $r \geq 7$, $s \geq 3$ (with ($*$2))

$P_\lambda = P^{(r,3;1)} \otimes P^{(s,1;1)} + P^{(r,3;2)} \otimes P^{(s,1;1)} + P^{(r,3;3)} \otimes P^{(s,1;1)} + P^{(r,3;3)} \otimes P^{(s,1;0)}$

Case III. $\Delta^+ = \Delta^+_{III}$: For $\Lambda \in \mathfrak{t}^*$,

$\Lambda \in \Xi_{III}$

$\Longleftrightarrow$ r', s' are positive integers with $s' \geq 1$, $r' - 3s' \geq 2$ and $r' - s'$ is even

$\Longleftrightarrow$ r, s are nonnegative integers with $r - 3s \geq 8$ and $r - s$ is even (*3)

In this case, we have

$\rho_n = 2\alpha_1$, $\lambda = \Lambda - \alpha_2$.

λ is far from the walls $\Longleftrightarrow$ $r \geq 8$, $s \geq 2$ (with ($*$3))

$P_\lambda = P^{(r,3;2)} \otimes I + P^{(r,3;3)} \otimes I$

7. Method for the determination of embeddings

To determine the embeddings of discrete series of G into its principal series, the same method as in Yamashita's paper [4] is used. Before stating the method, we prepare some notions.

Definition 2. A linear form μ is said to be *far from the walls* if $\lambda - \sum_{\beta \in Q} \beta$ is Δ_c^+-dominant for any subset Q of Δ_n^+.

Definition 3. Let P be a closed subgroup of G. A continuous representaton (η, F) of P on a Fréchet space F, is said to be *weakly cyclic* if there exists a continuous linear functional T on F such that for an element v of F, $T(\eta(p)v) = 0$ $(\forall p \in P)$ implies $v = 0$.

For a discrete series π_Λ of G, the contragredient representation π_Λ^* of π_Λ is also a discrete series. Therefore we may consider the embeddings of π_Λ^* instead of those of π_Λ, and they are described as in the following theorem.

Theorem 1 (cf. [4, Theorem 2.4]). *Let Λ be in Ξ and (η, F) a weakly cyclic representation of a closed subgroup P of G. Then there exists a natural isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathrm{Ind}_P^G(\eta)) \simeq \mathrm{Ker}\, \mathcal{D}_{\lambda,\eta}$$

as linear spaces, if λ is far from the walls.

When P is a parabolic subgroup of G, let $P = M_P A_P N_P$ be a Langlands decomposition of P and L_P the Levi part $M_P A_P$ of P. We denote the Lie algebra of L_P (resp. A_P, N_P) by $\mathfrak{l}_0$ (resp. $\mathfrak{a}_{P0}$, $\mathfrak{n}_{P0}$) and put $K_L = L_P \cap K$. The complexification of $\mathfrak{l}_0$ (resp. $\mathfrak{a}_{P0}$, $\mathfrak{n}_{P0}$) is denoted by $\mathfrak{l}$ (resp. $\mathfrak{a}_P$, $\mathfrak{n}_P$) as usual. The Levi part L_P acts on $\operatorname{Ker}\mathcal{D}_{\lambda,1_N}$ by right translation. Here, 1_N stands for the trivial character of N_P. As in the case of $\operatorname{Ind}_P^G(\eta)$, $\operatorname{Ker}\mathcal{D}_{\lambda,1_N}$ has an $(\mathfrak{l}, K_L)$-module structure.

For an irreducible admissible representation σ of M_P and a linear form μ on $\mathfrak{a}_P$, $\xi = \sigma \otimes e^{\mu}$ is an irreducible admissible representation of L_P. Put $\tilde{\xi} = \sigma \otimes e^{\mu+\rho_P}$. Here, $\rho_P(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} H|_{\mathfrak{n}_P})$ $(H \in \mathfrak{a}_P)$. Then, next theorem gives a kind of Frobenius reciprocity on the embeddings into principal series.

Theorem 2 (cf. [4, Theorem 3.5]). *If the Blattner parameter λ of π_Λ is far from the walls, then*

$$\operatorname{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \operatorname{Ind}_P^G(\xi \otimes 1_N)) \simeq \operatorname{Hom}_{(\mathfrak{l},K_L)}(\tilde{\xi}^*, \operatorname{Ker}\mathcal{D}_{\lambda,1_N}),$$

as linear spaces. Here π_Λ^* denotes the discrete series of G contragredient to π_Λ.

8. Complete descrition of embeddings

Define an automorphism u of $\mathfrak{g}$ by

$$u = \left(\exp\frac{\pi}{4}\operatorname{ad}(E_{01} - E_{0,-1})\right)\cdot\left(\exp\frac{\pi}{4}\operatorname{ad}(E_{21} - E_{-2,-1})\right).$$

Note that u maps $\mathfrak{t}$ onto $\mathfrak{a}$. We take a representative $\Lambda \in \Xi_I$ of w-orbit through an element of Ξ_c, and let Λ_J $(J = I, II, III)$ be the unique element in $\Delta_J^+ \cap W\cdot\Lambda$, where W is the Weyl group of Δ. Further let $\tilde{\Lambda}$ be the Ψ^+-dominant element in $\mathfrak{a}^*$ conjugate to $\Lambda \circ u^{-1}$ under the action of the Weyl group $W(\Psi)$ of Ψ. Define discrete series representations π_J $(J = I, II, III)$ by $\pi_J = \pi_{\Lambda_J}$. Then these three π_J's are the mutually inequivalent discrete series with the same infinitesimal character Λ. Put ν_j, $j = 1, 2$, as $\nu_1 \circ u = -(2\alpha_1 + \alpha_2)$ and $\nu_2 \circ u = 3\alpha_1 + \alpha_2$, then these ν_j's are simple roots of Ψ^+. The reflection relative to ν_j is denoted by s_j. The following theorem describes the embeddings of discrete series of G into its principal series.

Theorem 3. *For $\Lambda \in \Xi_I$, $J = I, II, III$, $\sigma_{\epsilon_1,\epsilon_2} \in \hat{M}$, $\mu \in \mathfrak{a}^*$,*

$$\dim \operatorname{Hom}_{(\mathfrak{g},K)}(\pi_J, \operatorname{Ind}_P^G(\sigma_{\epsilon_1,\epsilon_2} \otimes e^{\mu} \otimes 1_N)) \leq 1,$$

and the equality holds if and only if

$$\mu = s\cdot\tilde{\Lambda} \quad \textit{and} \quad (\epsilon_1, \epsilon_2) \in S_\Lambda(J, s) \quad \textit{with an} \quad s \in W(J),$$

where $W(J)$ and $S_\Lambda(J, s)$ are subsets of $W(\Psi)$ and $\{\pm 1\} \times \{\pm 1\}$ defined respectively as follows:

$$W(I) = \{s_1,\ s_2s_1\},$$
$$W(II) = \{1,\ s_1,\ s_2,\ s_1s_2,\ s_2s_1\},$$
$$W(III) = \{s_2,\ s_1s_2\},$$

and

$$S_\Lambda(II,1) = \{((-1)^{r'+1}, (-1)^{\frac{1}{2}(r'-s'+2)}), ((-1)^{\frac{1}{2}(r'+s'+2)}, \pm 1)\},$$
$$S_\Lambda(J,s_1) = \begin{cases} \{((-1)^{r'+1}, (-1)^{\frac{1}{2}(r'+s')})\} & \text{for } J = I \\ \{((-1)^{\frac{1}{2}(r'+s')}, (-1)^{r'+1}), ((-1)^{\frac{1}{2}(r'+s'+2)}, \pm 1)\} & \text{for } J = II, \end{cases}$$
$$S_\Lambda(J,s_2) = \begin{cases} \{((-1)^{r'}, (-1)^{\frac{1}{2}(r'-s'+2)}), ((-1)^{r'+1}, \pm 1)\} & \text{for } J = II \\ \{((-1)^{\frac{1}{2}(r'-s')}, (-1)^{r'+1}), ((-1)^{\frac{1}{2}(r'-s'+2)}, (-1)^{r'})\} & \text{for } J = III, \end{cases}$$
$$S_\Lambda(J,s_1s_2) = \{(\pm 1, (-1)^{\frac{1}{2}(r'+s'+2)})\} \quad \text{for} \quad J = II, III,$$
$$S_\Lambda(J,s_2s_1) = \{((-1)^{\frac{1}{2}(r'-s'+2)}, \pm 1)\} \quad \text{for} \quad J = I, II.$$

Here $r' = \Lambda(H_{10})$ *and* $s' = \Lambda(H_{32})$.

9. Sketch of the proof of Theorem 3

Here we illustrate the outline of the proof in case $J = I$. Put $r = \lambda(H_{10})$ and $s = \lambda(H_{32})$, then they are non-negative integers so that $s - r$ is even and is not less than 4. Let f be a function in $C^\infty_{\tau_\lambda}(G; 1_N)$, then f can be expressed uniquely in the form

$$f(g) = \sum_{p,q} c_{pq}(g) e^{(rs)}_{pq},$$

with smooth functions c_{pq} on G. Rewriting the condition $\mathcal{D}_{\lambda,1_N} f = 0$ in terms of c_{pq}'s, we obtain the following system of differential equations:

$$(2L_1 - 2s + p + q - 2)c_{p,q+2} = 0, \tag{8}$$
$$\sqrt{s-q}(2L_2 + p + 3q)c_{pq} + 2\sqrt{(r+2-p)(r+p)(s+2+q)}c_{p-2,q+2} = 0, \tag{9}$$
$$-\sqrt{s+2+q}(p + 3q + 6 - 2L_2)c_{p,q+2} + 2\sqrt{(r-p)(r+2+p)(s-q)}c_{p+2,q} = 0, \tag{10}$$
$$(2s + p + q + 4 - 2L_1)c_{pq} = 0, \tag{11}$$

for $p = -r, -r+2, \ldots, r$ and $q = -s, -s+2, \ldots, s-2$. Here $L_1 = L_{\tilde{H}_1}$ and $L_2 = L_{\tilde{H}_2}$.

Since c_{pq}'s are determined by their values on A, we consider the equations for c_{pq}'s such as (8)-(11) only on A, though c_{pq}'s are functions on G. By (8) and (11), we have $(p+q)c_{pq} = 0$ if $q \neq \pm s$. So $c_{pq} = 0$ if $q \neq \pm s$ and $p + q \neq 0$. For c_{pq}'s with $q = \pm s$, (9) and (10) and the previous fact show that $c_{ps} = 0$ if $p \neq r$ and that $c_{p,-s} = 0$ if $p \neq -r$.

To determine the form of the function c_{pq}, define smooth functions $\tilde{c}_{pq}$ on $\mathbf{R}^2$ by

$$\tilde{c}_{pq}(x_1, x_2) = c_{pq}(\exp(x_1\tilde{H}_1 + x_2\tilde{H}_2)),$$

for real numbers x_1, x_2. Then equations (8)-(11) give a system of partial differential equations for $\tilde{c}_{pq}$'s. For instance, we can find the following equations for $\tilde{c}_{p,-p}$'s:

$$-\left(\frac{\partial}{\partial x_1}+s+2\right)\tilde{c}_{p,-p}=0\,, \tag{12}$$

$$-\sqrt{s+p}\left(\frac{\partial}{\partial x_2}+p\right)\tilde{c}_{p,-p}+\sqrt{(r+2-p)(r+p)(s+2-p)}\,\tilde{c}_{p-2,-(p-2)}=0\,, \tag{13}$$

$$\sqrt{s-p}\left(-\frac{\partial}{\partial x_2}+p\right)\tilde{c}_{p,-p}+\sqrt{(r-p)(r+2+p)(s+2+p)}\,\tilde{c}_{p+2,-(p+2)}=0\,. \tag{14}$$

These three equations tell that $\tilde{c}_{p,-p}(x_1,x_2) = a_p\exp(-(s+2)x_1+rx_2)$ for a scalar constant a_p. In a similar way, c_{rs} and $c_{-r,-s}$ are determined up to scalar multiples. By using (13) and (14) again, we have the following inductive relation for constants a_p's,

$$a_{p-2}=\sqrt{\frac{(s+p)(r+p)}{(s+2-p)(r+2-p)}}\,a_p\,,$$

for $p=-r+2,\ldots,r$. This implies that $a_p=\alpha_p a_r$ with

$$\alpha_p=\sqrt{\frac{2r(2r-2)\cdots(r+p+2)}{(r-p)(r-p-2)\cdots 2}\cdot\frac{(s+r)(s+r-2)\cdots(s+p+2)}{(s-p)(s-p-2)\cdots(s-r+2)}}\,.$$

Define linear forms μ_1, μ_2 on $\mathfrak{a}$ through

$$\begin{aligned}\mu_1(\tilde{H}_1)&=-(s+2)\,, & \mu_1(\tilde{H}_2)&=r\,,\\ \mu_2(\tilde{H}_1)&=-(s-r+4)/2\,, & \mu_2(\tilde{H}_2)&=-(r+3s)/2\,.\end{aligned}$$

Argue as above for c_{rs} and $c_{-r,-s}$, we see that $\operatorname{Ker}\mathcal{D}_{\lambda,1_N}$ is contained in the linear span of the following three functions f_* ($*=0,+,-$):

$$\begin{aligned}f_0(a)&=\sum_p \alpha_p a^{\mu_1}e^{(rs)}_{p,-p}\\ f_+(a)&=a^{\mu_2}(e^{(rs)}_{rs}+e^{(rs)}_{-r,-s})\\ f_-(a)&=a^{\mu_2}(e^{(rs)}_{rs}-e^{(rs)}_{-r,-s})\qquad (a\in A)\,,\end{aligned}$$

where $a^\mu=\exp(\mu(\log a))$, and extend these f_*'s to G through $f_*(kan)=\tau_\lambda(k)f_*(a)$ for $k\in K$, $a\in A$, $n\in N$. It is easily seen that these f_*'s actually form a basis of $\operatorname{Ker}\mathcal{D}_{\lambda,1_N}$.

Since for minimal parabolic subgroup P_m, the pair $(\mathfrak{l}, K_L)$ in Theorem 2 is $(\mathfrak{a}, M)$, we study the MA-module structure of $\operatorname{Ker}\mathcal{D}_{\lambda,1_N}$. For this purpose, we decompose $\operatorname{Ker}\mathcal{D}_{\lambda,1_N}$ into irreducibles by seeking its suitable basis. In case $J=I$, the subspace $\mathbf{C}f_*$ for each of the above three f_*'s is an MA-invariant subspace of $\operatorname{Ker}\mathcal{D}_{\lambda,1_N}$ and as MA-modules

$$\begin{aligned}\mathbf{C}f_0&\simeq(\sigma_{(-1)^r,(-1)^{(r+s)/2}})\otimes e^{\mu_1}\,,\\ \mathbf{C}f_+&\simeq(\sigma_{(-1)^{(s-r)/2},1})\otimes e^{\mu_2}\,,\\ \mathbf{C}f_-&\simeq(\sigma_{(-1)^{(s-r)/2},-1})\otimes e^{\mu_2}\,.\end{aligned}$$

Applying Theorem 2 to this result for the MA-module structure of $\operatorname{Ker}\mathcal{D}_{\lambda,1_N}$ and rewriting the parameters, we can specialize parameters ϵ_1, ϵ_2 and μ satisfying

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathrm{Ind}_P^G(\sigma_{\epsilon_1,\epsilon_2} \otimes e^\mu \otimes 1_N)) \neq (0),$$

for discrete series π_Λ whose Blattner parameter is far from the walls. Keeping in mind the fact that each discrete series representation of G is self-contragredient and calculating $s\cdot\tilde{\Lambda}$ for each s in the Weyl group of Ψ, we can verify the assertion in the theorem if the Blattner parameter of π_Λ is far from the walls.

To get rid of the restriction that λ is far from the walls, Zuckerman's translation functors can be used. See [6, Corollary 5.5] and [2, Theorem B.1].

For the case $J = II$ or $J = III$, by similar but more complicated computations, we can derive the statement in the theorem.

The results on Whittaker models of G are to be given in the talk.

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