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# CAPELLI IDENTITIES, DEGENERATE SERIES AND HYPERGEOMETRIC FUNCTIONS*

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ABSTRACT. The Capelli identity is extended to the case of minors. Related differential operators in the generalized identities give the annihilator of the degenerate principal series for $GL(n)$ and characterizes the image of the Poisson transform of the hyperfunctions on several boundaries of $GL(n)$. Hypergeometric functions are defined through realizations of some special sections of the degenerate principal series and the realizations on boundaries of $GL(n)$ generalize Gelfand's hypergeometric functions. Related Radon transforms for Grassmannians are examined.

## 1. Generalized Capelli identities and related operators

The classical Capelli identity can be considered as a *quantization* of the formula $\det {}^tAB = \det A \cdot \det B$ in the linear algebra. We *quantize* more general identities

$$(1.1)\quad \det\Big(\sum_{\nu=1}^{n} x_{\nu i}y_{\nu j}\Big)_{\substack{1\le i\le m\\ 1\le j\le m}} = \sum_{1\le\nu_1<\cdots<\nu_m\le n} \det\Big(x_{\nu_i j}\Big)_{\substack{1\le i\le m\\ 1\le j\le m}} \cdot \det\Big(y_{\nu_i j}\Big)_{\substack{1\le i\le m\\ 1\le j\le m}}$$

for $2mn$ commutative variables $x_{\nu i}$ and $y_{\nu i}$ with $1 \le i \le m$ and $1 \le \nu \le n$ (cf. [Si, II §5 Theorem 9]), where the left hand side of (1.1) is zero if $m > n$, and we get

**Theorem 1.1.** (Generalized Capelli identities) *Let* $I = \{i_k\}_{1\le k\le m}$ *and* $J = \{j_\ell\}_{1\le \ell\le m}$ *be sequences of positive integers. Then*

$$(1.2)\quad \det\Big(\sum_{\nu=1}^{n} x_{\nu i_k}\frac{\partial}{\partial x_{\nu j_\ell}} + (m-\ell)\delta_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\ 1\le \ell\le m}}$$
$$= \begin{cases} \displaystyle\sum_{1\le\nu_1<\cdots<\nu_m\le n} \det\Big(x_{\nu_p i_q}\Big)_{\substack{1\le p\le m\\ 1\le q\le m}} \cdot \det\Big(\frac{\partial}{\partial x_{\nu_p j_q}}\Big)_{\substack{1\le p\le m\\ 1\le q\le m}} & \text{if } m\le n,\\ 0 & \text{if } m>n. \end{cases}$$

Here $\delta_{ij}$ is the Kronecker symbol and for a matrix $A = (A_{ij}) = \Big(A_{ij}\Big)_{\substack{1\le i\le m\\ 1\le j\le m}}$ with $(i,j)$-components $A_{ij}$ in an associative algebra, we define

$$(1.3)\qquad \det A = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}(\sigma)A_{\sigma(1)1}A_{\sigma(2)2}\cdots A_{\sigma(m)m},$$

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where $\mathfrak{S}_m$ is the $m$-th symmetric group.

Using the Laplace expansion of the determinant with respect to the first column, the theorem is proved by induction because (1.2) is a formula for arbitrary minors.

*Remark 1.2.* If $m = n$, Theorem 1.1 is reduced to the Capelli identity

$$\det\Big(\sum_{\nu=1}^{n} x_{\nu i}\frac{\partial}{\partial x_{\nu j}} + (m-j)\delta_{ij}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}} = \det\Big(x_{ij}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}}\cdot\det\Big(\frac{\partial}{\partial x_{ij}}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}}.$$

Put $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C}) = \mathrm{Lie}(GL(n,\mathbb{C}))$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{g}_{\mathbb{C}}$ is naturally identified with the $n^2$-dimensional vector space of $n\times n$-matrices whose components are complex numbers. Let $E_{kl}$ denote the element of $\mathfrak{g}_{\mathbb{C}}$ whose $(i,j)$-component equals $\delta_{ki}\delta_{lj}$. We put $E_i = E_{ii}$ for simplicity. Put $\mathfrak{a} = \sum_{i=1}^{n}\mathbb{C}E_i$, $\mathfrak{n} = \sum_{1\le i<j\le n}\mathbb{C}E_{ji}$ and $\bar{\mathfrak{n}} = \sum_{1\le i<j\le n}\mathbb{C}E_{ij}$.

**Definition 1.3.** Let $I = \{i_1,\dots,i_m\}$ and $J = \{j_1,\dots,j_m\}$ be sequences of positive integers with $1\le i_\mu\le n$ and $1\le j_\nu\le n$. Then for $\lambda\in\mathbb{C}$ we put

$$D_{IJ}(\lambda) = \det\Big(E_{i_\mu j_\nu} + (\lambda+m-\nu)\delta_{i_\mu j_\nu}\Big)_{\substack{1\le\mu\le m\\ 1\le\nu\le m}}. \tag{1.4}$$

We naturally identify $E_{ij}$ with the left invariant vector filed on $G = GL(n,\mathbb{C})$. Then $E_{ij}$ is of the form

$$E_{ij} = \sum_{\nu=1}^{n} x_{\nu i}\frac{\partial}{\partial x_{\nu j}} \tag{1.5}$$

under the coordinates $\Big(x_{ij}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}}\in G$ and the left hand side of (1.2) is identified with $D_{IJ}(0)$ and morover it becomes $D_{IJ}(\lambda)$ under the automorphism of $U(\mathfrak{g})$ satisfying $E_{ij}\mapsto E_{ij}+\lambda\delta_{ij}$. Hence we call $D_{IJ}(\lambda)$ *generalized Capelli operators.*

We quote some fundamental properties of $D_{IJ}(\lambda)$ from [O4, §2]:

**Theorem 1.4.** i) $D_{\sigma(I)\sigma'(J)}(\lambda) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma')D_{IJ}(\lambda)$ *for* $\sigma$, $\sigma'\in\mathfrak{S}_m$.

ii) *For an ordered set of indices* $I = \{i_1,\dots,i_m\}$ *and for* $k\in I$, *we define* $\iota_\ell^k(I) = \{i'_1,\dots,i'_m\}$ *by*

$$i'_p = \begin{cases} \ell & \text{if } i_p = k,\\ i_p & \text{if } i_p \ne k,\end{cases}$$

*for* $p = 1,\dots,m$. *Moreover we put* $\iota_\ell^k(I) = \emptyset$ *if* $k\notin I$. *Then*

$$[E_{k\ell}, D_{IJ}(\lambda)] = D_{\iota_k^\ell(I)J}(\lambda) - D_{I\iota_\ell^k(J)}(\lambda).$$

*Here we define* $D_{IJ}(\lambda) = 0$ *if* $\#I\ne\#J$.

iii) *Suppose* $I = \{i_1<\cdots<i_m\}$ *and* $J = \{j_1<\cdots<j_m\}$. *Then*

$$D_{IJ}(\lambda) = \begin{cases} 0 \mod \bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n} & \text{if } I\ne J,\\ \displaystyle\prod_{\nu=1}^{m}(E_{i_\nu i_\nu}+\lambda+\nu-1) \mod U(\mathfrak{g})\mathfrak{n} & \text{if } I = J.\end{cases}$$

The following lemma says that the power of matrices of $\big(D_{IJ}\big)_{IJ}$ also have the same property as in Theorem 1.4 i) and ii).

**Lemma 1.5.** *Fix a positive number $m$ with $1 \leq m \leq n$. Let $F(i)_{IJ}$ be elements of $U(\mathfrak{g})$ which satisfy*

$$
(1.6) \qquad \begin{aligned} F(i)_{\sigma(I)\sigma'(J)} &= \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')F(i)_{IJ} \quad \text{for } \sigma, \sigma' \in \mathfrak{S}_m, \\ [E_{k\ell}, F(i)_{IJ}] &= F(i)_{\iota_k^\ell(I)J} - F(i)_{I\iota_\ell^k(J)} \end{aligned}
$$

*for $i = 1$ and $2$. Here $I$ and $J$ are ordered sets of $m$ positive integers being smaller or equal to $n$ and we put $F(i)_{IJ} = 0$ if $\#I \neq \#J$. Then $F(3)_{IJ} = \sum_V F(1)_{IV}F(2)_{VJ}$ also satisfies (1.6) with $i = 3$. Here the sum runs over all sets $V = \{\nu_1, \nu_2, \ldots, \nu_m\}$ of indices with $1 \leq \nu_1 < \nu_2 < \cdots < \nu_m \leq n$.*

Viewing the above lemma, it is interesting to study the power of $\big(D_{IJ}(\lambda)\big)_{IJ}$.

**Definition 1.6.** We define

$$
\begin{gathered}
F^1_{ij}(\lambda) = E_{ij} + \lambda\delta_{ij}, \\
\Big(F^k_{ij}(\lambda_1, \ldots, \lambda_k)\Big)_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} = \Big(F^1_{ij}(\lambda_1)\Big)_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} \cdots \Big(F^1_{ij}(\lambda_k)\Big)_{\substack{1\leq i\leq n \\ 1\leq j\leq n}}.
\end{gathered}
$$

**Theorem 1.7.** i) *$F^k_{ij}(\lambda_1, \ldots, \lambda_k)$ is a symmetric function of $\lambda_1, \ldots, \lambda_k$.*

ii) *The action of $\mathfrak{g}_{\mathbf{C}}$:*

$$
[E_{\mu\nu}, F^k_{ij}] = F^k_{\mu j}\delta_{\nu i} - F^k_{i\nu}\delta_{\mu j}.
$$

iii) *A formula related to the Bruhat decomposition:*

$$
\begin{aligned}
F^k_{ij}(\lambda_1, \ldots, \lambda_k) &\equiv 0 \mod U(\mathfrak{g})\mathfrak{n} \quad \text{if } i > j, \\
F^k_{ij}(\lambda_1, \ldots, \lambda_k) &\equiv 0 \mod \bar{\mathfrak{n}}U(\mathfrak{g}) \quad \text{if } i < j, \\
F^k_{ii}(\lambda_1, \ldots, \lambda_k) &\equiv F^{k-1}_{ii}(\lambda_1, \ldots, \lambda_{k-1}) \cdot (E_i + \lambda_k + i - 1) \\
&\qquad - \sum_{\nu=1}^{i-1} F^{k-1}_{\nu\nu}(\lambda_1, \ldots, \lambda_{k-1}) \mod U(\mathfrak{g})\mathfrak{n}, \\
F^k_{ii+1}(\lambda_1, \ldots, \lambda_k) &\equiv F^{k-1}_{ii+1}(\lambda_1, \ldots, \lambda_{k-1})(E_{i+1} + \lambda_k + i) \\
&\qquad + E_{ii+1}F^{k-1}_{ii}(\lambda_1, \ldots, \lambda_{k-1}) \mod U(\mathfrak{g})\mathfrak{n}.
\end{aligned}
$$

iv) *Let $\Theta = \{0 < n_1 < n_2 < \cdots < n_L = n\}$ be a sequence of positive integers. Put $n_0 = 0$ and define*

$$
\mathcal{J}_\mu = U(\mathfrak{g})\mathfrak{n} + \sum_{\nu=1}^{L} \sum_{j=n_{\nu-1}+1}^{n_\nu} U(\mathfrak{g})(E_j + \lambda_\nu + n_{\nu-1})
$$

*Then*

$$
\begin{aligned}
&F^k_{ii}(\lambda_1,\dots,\lambda_k)\\
&\equiv \begin{cases} 0 & \bmod \mathcal{J}_\mu \quad \text{if } i \le n_k,\\ \displaystyle\prod_{\nu=1}^{k}(\lambda_\nu - \lambda_{k+1} + n_{\nu-1} - n_\nu) & \bmod \mathcal{J}_\mu \quad \text{if } n_k < i \le n_{k+1},\end{cases}\\
&F^k_{ii+1}(\lambda_1,\dots,\lambda_k)\\
&\equiv \prod_{\nu=1}^{\ell-1}(\lambda_\nu - \lambda_\ell + n_{\nu-1} - n_\nu - n_{\ell-1} + i) \prod_{\nu=\ell+1}^{k}(\lambda_\nu - \lambda_\ell - n_{\ell-1} + i)E_{ii+1}\\
&\quad \bmod \mathcal{J}_\mu \quad \text{if} \quad n_{\ell-1} < i < n_\ell \ \text{ and } \ k \ge \ell.
\end{aligned}
$$

## 2. Degenerate principal series of $GL(n)$ (global version)

Let $G$ be $GL(n,\mathbb{R})$ or $U(p,q)$ with $p \ge q$ or $GL(m,\mathbb{H})$ or $GL(n,\mathbb{C})$, where $n = p+q$ and $n = 2m$. Our argument in the sequel is valid for the direct product of these groups but for simplicity we assume this. We will mainly consider the case when $G \ne GL(n,\mathbb{C})$ but the results also hold for $GL(n,\mathbb{C})$ with some obvious modifications which we will give if necessary.

Then $G$ is a real form of (a direct product of some copies of) $GL(n,\mathbb{C})$ and its Lie algebra $\mathfrak{g}$ is defined by the following involutions:

$$
\text{(2.1)}\qquad
\begin{aligned}
&\mathfrak{gl}(n,\mathbb{C}) \ni X \mapsto \begin{cases} \bar{X} & \text{if } \mathfrak{g} = \mathfrak{gl}(n,\mathbb{R}),\\ -I_{pq}{}^t\bar{X}I_{pq} & \text{if } \mathfrak{g} = \mathfrak{u}(p,q),\\ -J_m\bar{X}J_m & \text{if } \mathfrak{g} = \mathfrak{gl}(m,\mathbb{H}),\end{cases}\\
&\mathfrak{gl}(n,\mathbb{C}) \oplus \mathfrak{gl}(n,\mathbb{C}) \ni (X,Y) \mapsto (\bar{Y},\bar{X}) \qquad \text{if } \mathfrak{g} = \mathfrak{gl}(n,\mathbb{C}),
\end{aligned}
$$

where $I_{pq}$ is the permutation matrix representing $\begin{pmatrix} 1 & \cdots & q & q+1 & \cdots & p & p+1 & \cdots & p+q \\ p+q & \cdots & p+1 & q+1 & \cdots & p & q & \cdots & 1 \end{pmatrix}$ and $J_m$ is the direct product of $m$-copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Moreover $\mathfrak{gl}(n,\mathbb{C})$ is identified with a subalgebra of $\mathfrak{gl}(n,\mathbb{C}) \oplus \mathfrak{gl}(n,\mathbb{C})$ by the map $\mathfrak{gl}(n,\mathbb{C}) \ni X \mapsto (X,\bar{X})$. Then the element of the quaternion number field $\mathbb{H}$ is identified with a $2\times 2$ complex matrix by

$$
\text{(2.2)}\qquad \mathbb{H} \ni a+bi+cj+dk \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \quad \text{for } a,b,c,d \in \mathbb{R}.
$$

We choose maximally split Cartan subalgebras $\mathfrak{j}$ by

$$
\text{(2.3)}\quad \mathfrak{j} = \begin{cases} \sum_{i=1}^n \mathbb{R}E_i & \text{if } G = GL(n,\mathbb{R}),\\ \sum_{i=1}^n \mathbb{R}E_i + \mathbb{R}\sqrt{-1}E_i & \text{if } G = GL(n,\mathbb{C}),\\ \sum_{i=1}^m \mathbb{R}(E_{2i-1}+E_{2i}) + \mathbb{R}\sqrt{-1}(E_{2i-1}-E_{2i}) & \text{if } G = GL(m,\mathbb{H}),\\ \sum_{i=1}^q \mathbb{R}(E_i - E_{n+1-i}) + \mathbb{R}\sqrt{-1}(E_i + E_{n+1-i}) & \\ \qquad + \sum_{i=q+1}^{2n-q} \mathbb{R}\sqrt{-1}E_i & \text{if } G = U(p,q)\end{cases}
$$

so that they are compatible with our arguments in §1. Then $K = G \cap U(n)$ is a maximal compact subgroup of $G$ and $\theta : G \ni g \mapsto {}^t\bar{g}^{-1}$ or $\mathfrak{g} \ni X \mapsto -{}^t\bar{X}$ is the Cartan involution.

Put $\mathfrak{a}_\mathfrak{p} = \mathfrak{j} \cap \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{t} = \mathfrak{j} \cap \sqrt{-1}\mathfrak{gl}(n, \mathbb{R})$. Since the complexification $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of $(\mathfrak{g}, \mathfrak{j})$ is $(\mathfrak{gl}(n, \mathbb{C}), \mathfrak{a})$ (or the direct sum of its two copies), we define a fundamental system as in §1 and a compatible fundamental system $\Phi(\mathfrak{a}_\mathfrak{p})$ for the restricted root system for $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. Then the restricted root systems are of type $A^1_{n-1}$, $A^2_{n-1}$, $A^4_{m-1}$, $BC_q^{2(p-q),2,1}$ and $C_q^{2,1}$, respectively, where the superfices represent the multiplicities of roots in the order according to the length of roots.

Let $P_\Xi$ be the compatible parabolic subgroup of $G$ with the Langlands decomposition $P_\Xi = M_\Xi A_\Xi N_\Xi$ and the Levi decomposition $P_\Xi = L_\Xi N_\Xi$ with $L_\Xi = M_\Xi A_\Xi$.

The identity component of $M_\Xi$ is denoted by $M_\Xi^o$ and we put $\bar{N}_\Xi = \theta(N_\Xi)$. The Lie algebras of $K$, $M_\Xi$, $A_\Xi$, $N_\Xi$, $\bar{N}_\Xi$ and $L_\Xi$ are denoted by $\mathfrak{k}$, $\mathfrak{m}_\Xi$, $\mathfrak{a}_\Xi$, $\mathfrak{n}_\Xi$, $\bar{\mathfrak{n}}_\Xi$ and $\mathfrak{l}_\Xi$, respectively and then $\mathfrak{a}_\Xi \subset \mathfrak{j} \subset \mathfrak{a}$, $\mathbb{C} \otimes_\mathbb{R} \mathfrak{j} = \mathfrak{a}$ and $\mathbb{C} \otimes_\mathbb{R} \mathrm{Lie}(N_\Xi) \subset \mathfrak{n}$. Note that $\mathfrak{a}$ and $\mathfrak{n}$ should be replaced by $\mathfrak{a} \oplus \mathfrak{a}$ and $\mathfrak{n} \oplus \mathfrak{n}$, respectively, if $G = GL(n, \mathbb{C})$.

Fix an irreducible finite dimensional representation $(\delta, V_\delta)$ of $M_\Xi$. For any element $\xi$ of the complex dual $\mathfrak{a}_\Xi^*$ of $\mathfrak{a}_\Xi$, we have an irreducible representation $(\tau_{\delta,\xi}, V_\delta)$ of $P_\Xi$ by $P \ni man \mapsto \delta(m)a^\xi \in V_\delta$ for $m \in M_\Xi$, $a \in A_\Xi$ and $n \in N_\Xi$. Then we define the space of hyperfunction sections of (degenerate) principal series of $G$:

$$\mathcal{B}(G/P_\Xi; \delta, \xi) = \{f \in \mathcal{B}(G) \otimes V_\delta;\ f(gp) = \tau_{\delta,\xi}(p^{-1})f(g) \quad \text{for} \quad p \in P_\Xi\}. \tag{2.4}$$

Here $\mathcal{B}(G)$ is the space of hyperfunctions on $G$ and they are left $G$-modules by $(\pi_g f)(x) = f(g^{-1}x)$ for $g \in G$.

Let $d(\delta, \xi)$ denote the lowest weight of the representation $\tau_{\delta,\xi}$ of $L_\Xi$ with respect to the order defined above. Then $d(\delta, \xi) \in \mathfrak{a}^*$ and $d(\delta, \xi) = d\delta + \xi$, where $d\delta$ is the lowest weight of $\delta$ and we identified $\mathfrak{a}$ and $\mathfrak{a}^*$ by the Killing form of $\mathfrak{gl}(n, \mathbb{C})$. Put

$$d(\delta, \xi) = \sum_{i=1}^n d(\delta, \xi)_i e_i. \tag{2.5}$$

Here the right hand side is replaced by $\sum_{i=1}^n d(\delta, \xi)_i (e_i, 0) + \sum_{i=1}^n d(\delta, \xi)'_i (0, e_i)$ if $G = GL(n, \mathbb{C})$.

**Theorem 2.1.** *For an irreducible finite dimensional representation $\delta$ of $M_\Xi$ we define the sequence $\Theta = \{n_1 < \cdots < n_L = n\}$ of positive integers so that for any $i$ with $1 \le i \le n$*

$$\begin{aligned} &\text{there exists } k \text{ with } n_{k-1} < i < n_k \\ &\qquad\qquad \text{if and only if } d(\delta, \xi)_i = d(\delta, \xi)_{i+1} \text{ for any } \xi \in \mathfrak{a}_\Xi^*, \end{aligned} \tag{2.6}$$

*where $n_0 = 0$ as in §1. Then any $u \in \mathcal{B}(G/P_\Xi; \delta, \xi)$ satisfies*

$$\begin{aligned} &D_{IJ}(-\lambda_k - n_{k-1})u = 0 \\ &\quad \text{for } \#I = \#J = n - n_k + n_{k-1} + 1 \text{ and } k = 1, \ldots, L, \end{aligned} \tag{2.7}$$

$$F^L_{ij}(-\lambda_1 - n_0, \ldots, -\lambda_L - n_{L-1})u = 0 \quad \text{for } i, j = 1, \ldots, n \tag{2.8}$$

*by putting*

$$\lambda_\nu = d(\delta,\xi)_{n_\nu} \quad \text{for } \nu = 1,\dots,L. \tag{2.9}$$

*Remark 2.2.* We give examples in the case when $P_\Xi$ is maximal and $\delta$ is trivial. Then $M_\Xi$ is characterized by a unique positive simple root $\alpha_k$ whose root space is not contained in $\mathfrak{m}_\Xi$. Note that $\dim \mathfrak{a}_\Xi^* = 2$. Then

$$\Theta = \begin{cases} \{k,n\} & \text{if } G = GL(n,\mathbb{R}), \\ \{k,n\} & \text{if } G = GL(n,\mathbb{C}). \\ \{2k,2m\} & \text{if } G = GL(m,\mathbb{H}), \\ \{q,2q\} & \text{if } G = U(q,q) \text{ and } k = q, \\ \{k,p+q-k,p+q\} & \text{if } G = U(p,q) \text{ and } 2k < p+q, \end{cases} \tag{2.10}$$

Moreover $\lambda_2 = 0$ if $G = U(p,q)$ and $2k < p+q$. Note that in the case of the maximal parabolic subgroup defining Shilov boundary of the symmetric space $G/K$ with $G = U(p,q)$, the above number $k$ equals $q$.

We give another realization of $\mathcal{B}(G/P_\Xi;\delta,\xi)$. Put $Z(\mathfrak{a}_\mathfrak{p}) = K \cap \exp\sqrt{-1}\mathfrak{a}_\mathfrak{p}$ whose elements are, in our case, diagonal matrices with diagonal components $\pm1$. Then the Cartan subgroup $J$ of $G$ which is the centralizer of $\mathfrak{j}$ equals $Z(\mathfrak{a}_\mathfrak{p})\exp\mathfrak{j}$. Note that $J \simeq \mathbb{Z}_2^\ell \times \exp\mathfrak{j}$, where

$$\ell = \begin{cases} n & \text{if } G = GL(n,\mathbb{R}), \\ 0 & \text{if } G = GL(n,\mathbb{C}) \text{ or } GL(m,\mathbb{H}) \text{ or } U(p,q). \end{cases} \tag{2.11}$$

Since $M_\Xi$ is a linear group, the representation $\delta = (\tau_\delta, V_\delta)$ of $M_\Xi$ has a unique highest weight vector $v_\delta \in V_\delta$ up to constant multiple. Similarly the dual $\delta^* = (\tau_{\delta^*}, V_\delta^*)$ of $\delta$ has a unique highest weight vector $v_\delta^*$ with $\langle v_\delta^*, v_\delta\rangle = 1$. Then the map

$$\mathcal{B}(G/P_\Xi;\delta,\xi) \ni f \mapsto \langle v_\delta^*, f\rangle \in \mathcal{B}(G)$$

is an injective $G$-homomorphism whose image equals

$$\begin{aligned} \mathcal{B}(G/P_\Xi;\delta,\xi)_\mathfrak{n} = \{u \in \mathcal{B}(G);\, &Xu = 0 \text{ for } X \in \mathfrak{n}, \\ &Hu = \langle d\delta + \xi, H\rangle \text{ for } H \in \mathfrak{a},\ \dim U(\mathfrak{m}_\Xi)u < \infty, \\ &u(gz) = \epsilon_\delta(z)^{-1}u(g) \text{ for } z \in Z(\mathfrak{a}_\mathfrak{p})\} \end{aligned} \tag{2.12}$$

and its inverse is given by $\mathcal{B}(G/P_\Xi;\delta,\xi)_\mathfrak{n} \ni \phi \mapsto \phi \otimes v_\delta$. Here $\epsilon_\delta : Z(\mathfrak{a}_\mathfrak{p}) \to \{\pm1\}$ is defined so that $\tau_\delta(z)v_\delta = \epsilon_\delta(z)v_\delta$ and the last condition given by $\epsilon_\delta$ is necessary only when $M_\Xi$ is not connected, which means $G = GL(n,\mathbb{R})$.

**Theorem 2.4.** *Fix a function $f$ in $\mathcal{B}(G/P_\Xi;\delta,\xi)_\mathfrak{n}$. Let $\lambda = (\lambda_1,\dots,\lambda_L) \in \mathbb{C}^L$ and let $\Theta = \{n_1 < \cdots < n_L = n\}$ be a sequence of positive integers. Assume*

$$\lambda_k = d(\delta,\xi)_i \quad \text{if } n_{k-1} < i \le n_k, \tag{2.13}$$

(2.14)

$$(\lambda_i + n_i) - (\lambda_j + n_{j-1}) \notin \{1,\dots,\max\{n_i - n_{i-1}, n_j - n_{j-1}\} - 1\} \quad \text{for } 1 \le i < j \le L.$$

*If $f$ satisfies (2.7) or (2.8), then*

$$E_{ii+1}f = 0 \quad \text{for } n_{k-1} < i < n_k \text{ and } k = 1,\dots,L. \tag{2.15}$$

## 3. Representations on symmetric spaces

In this section we review on representations realized on symmetric spaces of non-compact type and their characterization by differential equations. We consider a general real connected semisimple Lie group $G$. The notation used here is in general restricted only in this section. The statements in this section are known results or at least a reformulation or an easy consequence of known facts (cf. [H3], [K– ], [Ko], [KR], [O1], [Sn1] etc.) and this section can be read without referring to other sections.

Let $K$ be a maximal compact subgroup of $G$ modulo the center of $G$, $\theta$ be the corresponding Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$. Fix a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$. Let $\Sigma(\mathfrak{a}_\mathfrak{p})$ be the set of the roots defined by the pair $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ and fix a positive system $\Sigma(\mathfrak{a}_\mathfrak{p})^+$. We denote its Weyl group by $W(\mathfrak{a}_\mathfrak{p})$ and the fundamental system by $\Phi(\mathfrak{a}_\mathfrak{p})$, and the half of the sum of the positive roots counting their multiplicities is denoted by $\rho$. Let $G = KAN$ be the Iwasawa decomposition of $G$ corresponding to $\Sigma(\mathfrak{a}_\mathfrak{p})^+$. Then $P = MAN$ is a minimal parabolic subgroup of $G$. Here $M$ is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $K$. We denote by $\mathfrak{k}$, $\mathfrak{m}$ and $\mathfrak{n}$ the Lie algebras of $K$, $M$ and $N$, respectively.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$, which we identify with the algebra of left invariant differential operators on $G$. In general, for a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we denote by $U(\mathfrak{l})$ the universal enveloping algebra of the complexification $\mathfrak{l}_\mathbb{C}$ of $\mathfrak{l}$. Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}_\mathbb{C}$. Then the map sym of symmetrization of $S(\mathfrak{g})$ to $U(\mathfrak{g})$ defines a $K$-linear bijection. By the Killing form on $\mathfrak{g}_\mathbb{C}$ we identify the space $\mathcal{O}(\mathfrak{p}_\mathbb{C})$ of polynomial functions on the complexification $\mathfrak{p}_\mathbb{C}$ of $\mathfrak{p}$ with the symmetric algebra of $\mathfrak{p}_\mathbb{C}$. Let $\mathcal{O}(\mathfrak{p})^K$ be the space of all $K$-invariant polynomials on $\mathfrak{p}_\mathbb{C}$ and $\mathcal{H}$ be the space of all $K$-harmonic polynomials on $\mathfrak{p}_\mathbb{C}$. Then we have the following $K$-linear bijection

$$(3.1)\qquad \begin{aligned} \mathcal{H} \otimes \mathcal{O}(\mathfrak{p})^K \otimes U(\mathfrak{k}) &\xrightarrow{\sim} U(\mathfrak{g}) \\ h \otimes p \otimes k &\mapsto \operatorname{sym}(h) \otimes \operatorname{sym}(p) \otimes k \end{aligned}$$

because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and the $K$-linear bijection $\mathcal{H} \otimes \mathcal{O}(\mathfrak{p})^K \ni h \otimes p \mapsto hp \in \mathcal{O}(\mathfrak{p})$ studied by [KR].

We denote by $\mathcal{A}(G)$ and $\mathcal{B}(G)$ the space of real analytic functions and that of hyperfunctions on $G$, respectively. Then they are left $G$-modules by $(\pi_g f)(x) = f(g^{-1}x)$ for $g \in G$ and the functions $f$ in the spaces. We write $\mathcal{A}(G)_K$ for the space of all the $K$-finite elements of $\mathcal{A}(G)$.

By the decomposition

$$(3.2)\qquad U(\mathfrak{g}) = \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}_\mathfrak{p}) \oplus U(\mathfrak{a}_\mathfrak{p}) \oplus U(\mathfrak{g})\mathfrak{k}$$

coming from the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$, we define $D_{\mathfrak{a}_\mathfrak{p}} \in U(\mathfrak{a}_\mathfrak{p})$ for $D \in U(\mathfrak{g})$ so that $D - D_{\mathfrak{a}_\mathfrak{p}} \in \mathfrak{n}U(\mathfrak{n} + \mathfrak{a}_\mathfrak{p}) + U(\mathfrak{g})\mathfrak{k}$ and put $\gamma(D) = e^{-\rho} \circ D_{\mathfrak{a}_\mathfrak{p}} \circ e^\rho$. Here $e^\rho$ is the function on $A$ defined by $e^\rho(a) = a^\rho$.

Note that the kernel of the restriction of $\gamma$ to the space of all the $K$-invariants $U(\mathfrak{g})^K$ of $U(\mathfrak{g})$ equals $U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$ and the restriction defines the Harish-Chandra isomorphism

$$(3.3)\qquad \bar{\gamma} : \mathbb{D}(G/K) \simeq U(\mathfrak{g})^K/(U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}) \to U(\mathfrak{a}_\mathfrak{p})^W$$

onto the space $U(\mathfrak{a}_\mathfrak{p})^W$ of all the $W(\mathfrak{a}_\mathfrak{p})$-invariants in $U(\mathfrak{a}_\mathfrak{p})$. Here $\mathbb{D}(G/K)$ is the ring of invariant differential operators on $G/K$. Note also that $\mathrm{sym}(O(\mathfrak{p})^K)$ is isomorphic to $\mathbb{D}(G/K)$ through $\gamma$ and $\bar{\gamma}$ as $K$-modules.

Identifying $U(\mathfrak{a}_\mathfrak{p})$ with the space of polynomial functions on the complex dual $\mathfrak{a}_\mathbb{C}^*$ of $\mathfrak{a}_\mathfrak{p}$, we put

$$\gamma_\lambda(D) = \gamma(D)(\lambda) \in \mathbb{C} \tag{3.4}$$

for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $D \in U(\mathfrak{g})$. Now we define

$$J_\lambda = U(\mathfrak{g})\mathfrak{k} + \sum_{p \in O(\mathfrak{p})^K} U(\mathfrak{g})(\mathrm{sym}(p) - \gamma_\lambda(\mathrm{sym}(p))) \tag{3.5}$$

and

$$\mathcal{A}(G/K; \mathcal{M}_\lambda) = \{u \in \mathcal{A}(G); Du = 0 \quad \text{for } D \in J_\lambda\} \tag{3.6}$$

and put $\mathcal{A}(G/K;\mathcal{M}_\lambda)_K = \mathcal{A}(G)_K \cap \mathcal{A}(G/K;\mathcal{M}_\lambda)$. Here $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ is naturally a subspace of the space $\mathcal{A}(G/K)$ of real analytic functions on $G/K$ because the function in $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ is right $K$-invariant. Now we can state our basic theorem.

**Theorem 3.1.** *Fix $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and define a bilinear form*

$$\begin{array}{rcl} \mathcal{H} \otimes \mathcal{A}(G/K;\mathcal{M}_\lambda) & \to & \mathbb{C} \\ (h,u) & \mapsto & \langle h,u\rangle = (\mathrm{sym}(h)u)(e) \end{array} \tag{3.7}$$

*and for a subspace $V$ of $\mathcal{A}(G/K;\mathcal{M}_\lambda)$, put*

$$H(V) = \{h \in \mathcal{H}; \langle h,u\rangle = 0 \quad \text{for } u \in V\}. \tag{3.8}$$

i) *The bilinear form $\langle\ ,\ \rangle$ is $K$-invariant and non-degenerate.*
ii) *If $V$ is a subspace of $\mathcal{A}(G/K;\mathcal{M}_\lambda)_K$, then*

$$V = \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda)_K; \langle h,u\rangle = 0 \quad \text{for } h \in H(V)\}.$$

iii) *There are natural bijections between the following sets of modules.*

$$\begin{aligned} \mathfrak{V}(\lambda) &= \{V \subset \mathcal{A}(G/K;\mathcal{M}_\lambda); V \text{ is a close subspace of } C^\infty(G) \text{ and } G\text{-invariant}\}, \\ \mathfrak{V}(\lambda)_K &= \{V_K \subset \mathcal{A}(G/K;\mathcal{M}_\lambda)_K; V_K \text{ is a } \mathfrak{g}\text{-invariant subspace}\}, \\ \mathfrak{J}(\lambda) &= \{J \supset J_\lambda; J \text{ is a left ideal of } U(\mathfrak{g})\}. \end{aligned}$$

*Here the bijections are given by*

$$\mathfrak{V}(\lambda) \ni V \mapsto V \cap \mathcal{A}(G)_K \in \mathfrak{V}(\lambda)_K, \tag{3.9}$$

$$\mathfrak{V}(\lambda)_K \ni V_K \mapsto J_\lambda + \sum_{p \in H(V_K)} U(\mathfrak{g})\,\mathrm{sym}(p) \in \mathfrak{J}(\lambda), \tag{3.10}$$

$$\mathfrak{J}(\lambda) \ni J \mapsto \{u \in \mathcal{A}(G); Du = 0 \text{ for } D \in J\} \in \mathfrak{V}(\lambda). \tag{3.11}$$

The $G$-module

(3.12)

$$\mathcal{B}(G/P;\mathcal{L}_\lambda)=\{f\in\mathcal{B}(G);f(gman)=a^{\lambda-\rho}f(g)\quad\text{for }(g,m,a,n)\in G\times M\times A\times N\}$$

is the space of hyperfunction sections of spherical principal series of $G$ parametrized by $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$. Put $\mathcal{A}(G/P;\mathcal{L}_\lambda)=\mathcal{B}(G/P;\mathcal{L}_\lambda)\cap\mathcal{A}(G)$. Define the $K$-fixed element $K\times A\times N\ni(k,a,n)\mapsto 1_\lambda(kan)=a^{\lambda-\rho}$ of $\mathcal{A}(G/P;\mathcal{L}_\lambda)$ and put $P_\lambda(g)=1_{-\lambda}(g^{-1})$. By the $G$-invariant bilinear form

$$\begin{aligned}\mathcal{B}(G/P;\mathcal{L}_\lambda)\times\mathcal{A}(G/P;\mathcal{L}_{-\lambda})&\to\mathbb{C}\\(\phi,f)&\mapsto\langle\phi,f\rangle_\lambda=\int_K\phi(k)f(k)dk\end{aligned}\tag{3.13}$$

with the normalized Haar measure $dk$ on $K$, we have the Poisson transform

$$\begin{aligned}\mathcal{P}_\lambda:\mathcal{B}(G/P;\mathcal{L}_\lambda)&\to\mathcal{B}(G)\\\phi&\mapsto\mathcal{P}_\lambda\phi(g)=\langle\pi_{g^{-1}}\phi,1_{-\lambda}\rangle_\lambda=\int_K\phi(gk)dk\\&\qquad=\int_K\phi(k)P_\lambda(k^{-1}g)dk.\end{aligned}\tag{3.14}$$

Then it is known that the image of $\mathcal{P}_\lambda$ is contained in $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ because $DP_\lambda=\gamma_\lambda(D)P_\lambda$ for $D\in U(\mathfrak{g})^K$.

For $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ and $w\in W(\mathfrak{a}_\mathfrak{p})$, let $A(\lambda,w)$ denote the intertwining operator of $\mathcal{B}(G/P;\mathcal{L}_\lambda)$ to $\mathcal{B}(G/P;\mathcal{L}_{w\lambda})$, which meromorphically depends on $\lambda$ and satisfies $(A(\lambda,w)\phi)(g)=\int_{\bar N_w}\phi(g\bar w n_w)dn_w$ if $\phi$ is continuous and moreover if $\mathrm{Re}\langle\lambda,\alpha\rangle<0$ for all $\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+$. Here $\bar w\in K$ is a representative of $w$, $\bar N_w=\theta(N)\cap\bar w^{-1}N\bar w$ and $dn_w$ is the Haar measure on $\bar N_w$ normalized by $\int_{\bar N_w}1_{-2\rho}(n_w)dn_w=1$.

For $\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})$ and $w\in W(\mathfrak{a}_\mathfrak{p})$, we put

$$\begin{aligned}\Sigma(\mathfrak{a}_\mathfrak{p})_o^+&=\{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+;\tfrac{\alpha}{2}\notin\Sigma(\mathfrak{a}_\mathfrak{p})\},\\\Sigma(w)&=\{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})_o^+;-w\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})^+\},\\e_\alpha(\lambda)&=\left\{\Gamma\left(\tfrac{\lambda_\alpha}{4}+\tfrac{m_\alpha}{4}+\tfrac12\right)\Gamma\left(\tfrac{\lambda_\alpha}{4}+\tfrac{m_\alpha}{4}+\tfrac{m_{2\alpha}}{2}\right)\right\}^{-1},\\e(\lambda)&=\prod_{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})_o^+}e_\alpha(\lambda),\quad e_w(\lambda)=\prod_{\alpha\in\Sigma(w)}e_\alpha(\lambda),\\d_\alpha(\lambda)&=\frac{\Gamma(m_\alpha+m_{2\alpha})}{\Gamma(\frac{m_\alpha}{2}+\frac{m_{2\alpha}}{2})}2^{1-\frac{\lambda_\alpha}{2}-\frac{m_\alpha}{2}}\sqrt{\pi}\,\Gamma\left(\tfrac{\lambda_\alpha}{2}\right),\\d(\lambda)&=\prod_{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})_o^+}d_\alpha(\lambda),\quad d_w(\lambda)=\prod_{\alpha\in\Sigma(w)}d_\alpha(\lambda),\end{aligned}\tag{3.15}$$

where $m_\alpha$ is the multiplicity of the root $\alpha$ and $\lambda_\alpha=2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$.

It is known that the regularized intertwining operators

$$\tilde A(w,\lambda)=d_w(-\lambda)A(w,\lambda)\tag{3.16}$$

depend holomorphically on $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$. Note that $\tilde A(w,\lambda)$ may be zero for some $\lambda$.

The following theorem is the main result in [K– ].

**Theorem 3.2.** *Fix $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$. Let $w$ be an element of $W$ with the minimal length which satisfies*

$$\mathrm{Re}\langle w\lambda, \alpha\rangle \geq 0 \quad \text{for all } \alpha \in \Sigma(\mathfrak{a}_{\mathfrak{p}})^+. \tag{3.17}$$

i) *$\mathcal{P}_\lambda$ defines a topological $G$-isomorphism of $\mathcal{A}(G/P;\mathcal{L}_\lambda)$ onto $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ if and only if $e(\lambda) \neq 0$.*

ii) *$\mathrm{Im}\, A(\lambda, w)$ and $\mathrm{Im}\,\mathcal{P}_\lambda$ are $G$-isomorphic under the Poisson transform*

$$\mathcal{P}_{w\lambda} : \mathcal{B}(G/P;\mathcal{L}_{w\lambda}) \rightrightarrows \mathcal{A}(G/K;\mathcal{M}_\lambda). \tag{3.18}$$

*Remark 3.3.* i) Since the image of the intertwining operator $A(\lambda, w)$ is always closed (cf. [O4, Lemma 7.3]), Theorem 3.1 says that the image of the Poisson transform $\mathcal{P}_\lambda$ of $\mathcal{B}(G/P;\mathcal{L}_\lambda)$ is always characterized as the space of solutions of a system of differential operators corresponding to suitable elements of $U(\mathfrak{g})$.

ii) Suppose $e(\lambda) \neq 0$. Let $\mathcal{D}'(G)$ and $C^\infty(G)$ denote the space of distributions and that of $C^\infty$-functions on $G$, respectively. Then (cf. [OS])

$$\begin{aligned}
&\mathcal{P}_\lambda(\mathcal{B}(G/P;\mathcal{L}_\lambda) \cap D'(G)) \\
&\quad = \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda); \text{there exist } C \text{ and } k \text{ with } |u(g)| \leq C \exp k|g|\}, \\
&\mathcal{P}_\lambda(\mathcal{B}(G/P;\mathcal{L}_\lambda) \cap C^\infty(G)) \\
&\quad = \{u \in \mathcal{A}(G/K;\mathcal{M}_\lambda); \text{there exist } k \text{ such that for any } D \in U(\mathfrak{k}) \\
&\qquad\quad \text{we can choose } C_D > 0 \text{ with } |\pi_D u(g)| \leq C_D \exp k|g|\}.
\end{aligned}$$

Here $U(\mathfrak{k})$ is the universal enveloping algebra of the complexification of $\mathfrak{k}$ and $|g| = \langle H, H\rangle^{\frac{1}{2}}$ with the Killing form $\langle\ ,\ \rangle$ if $g \in K \exp HK$ with $H \in \mathfrak{a}_{\mathfrak{p}}$. Note that $U(\mathfrak{k})$ in the above may be replaced by $U(\mathfrak{g})$.

**Theorem 3.4.** i) *The map*

$$\mathcal{H} \ni h \mapsto \pi_{\mathrm{sym}(h)} 1_\lambda \in \mathcal{A}(G/P;\mathcal{L}_\lambda)_K \tag{3.19}$$

*is $K$-linear. It is bijective if and only if $e(-\lambda) \neq 0$.*

ii)

$$\gamma_\lambda(D) = \pi_D(1_{-\lambda})(e) \quad \text{for } D \in U(\mathfrak{g}). \tag{3.20}$$

iii) *Putting*

$$\mathcal{H}_\lambda = \{h \in \mathcal{H}; \gamma_\lambda(\mathrm{sym}(\mathrm{Ad}(k)h)) = 0 \quad \text{for all } k \in K\} \tag{3.21}$$

*and*

$$\bar{J}_\lambda = J_\lambda + \sum_{h \in \mathcal{H}_\lambda} U(\mathfrak{g})\,\mathrm{sym}(h), \tag{3.22}$$

we have

$$\operatorname{Im}\mathcal{P}_\lambda = \{u \in \mathcal{A}(G);\ Du = 0 \quad \textit{for } D \in \bar{J}_\lambda\}. \tag{3.23}$$

For a subset $\Xi$ of $\Phi(\mathfrak{a}_\mathfrak{p})$, let $W_\Xi$ be a subgroup of $W(\mathfrak{a}_\mathfrak{p})$ generated by reflections with respect to the elements of $\Xi$ and put $P_\Xi = PW_\Xi P$. Let $P_\Xi = M_\Xi A_\Xi N_\Xi$ be the Langlands decomposition of $P_\Xi$ with $A_\Xi \subset A$. For an element $\mu$ of the complex dual $\mathfrak{a}^*_{\Xi,\mathbb{C}}$ of the Lie algebra $\mathfrak{a}_\Xi$ of $A_\Xi$, the space of hyperfunction sections of spherical degenerate series is defined by

$$\begin{aligned}\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu}) = \{f \in \mathcal{B}(G); f(gman) = a^{\rho-\mu}f(g)\\ \text{for } (g,m,a,n) \in G \times M_\Xi \times A_\Xi \times N_\Xi\}.\end{aligned} \tag{3.24}$$

Then as in the case of the minimal parabolic subgroup, we can define Poisson transform

$$\begin{aligned}\mathcal{P}_{\Xi,\mu} : \mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu}) &\to \mathcal{B}(G)\\ \phi &\mapsto (\mathcal{P}_{\Xi,\mu}\phi)(g) = \langle \pi_{g^{-1}}\phi, 1_{\Xi,-\mu}\rangle_{\Xi,\mu}\\ &= \int_K \phi(gk)dk = \int_K \phi(k)P_{\Xi,\mu}(k^{-1}g)dk.\end{aligned} \tag{3.25}$$

Here $\langle\ ,\ \rangle_{\Xi,\mu}$ is the bilinear form of $\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu}) \times \mathcal{A}(G/P_\Xi;\mathcal{L}_{\Xi,-\mu})$ defined on the integral over $K$, $1_{\Xi,\mu}(kman) = a^{\rho-\mu}$ for $(k,m,a,n) \in K \times M_\Xi \times A_\Xi \times N_\Xi$ and $P_{\Xi,\mu}(g) = 1_{\Xi,\mu}(g^{-1})$.

Now we remark

**Lemma 3.5.** *We have naturally*

$$\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu}) \subset \mathcal{B}(G/P;\mathcal{L}_{\mu+\rho(\Xi)}), \tag{3.26}$$

$$\operatorname{Im}\mathcal{P}_{\Xi,\mu} = \operatorname{Im}\mathcal{P}_{\mu-\rho(\Xi)}. \tag{3.27}$$

*Here we identify $\mathfrak{a}_\mathfrak{p}$ with its dual by the Killing form and define $\rho_\Xi = \rho|_{\mathfrak{a}_\Xi}$ and $\rho(\Xi) = \rho - \rho_\Xi$.*

**Corollary 3.6.** i) *$\mathcal{P}_{\Xi,\mu}$ is injective if $e(\mu + \rho(\Xi)) \neq 0$. In particular, the Poisson transform $\mathcal{P}_{\rho_\Xi} : \mathcal{B}(G/P_\Xi) \to \mathcal{A}(G/K;\mathcal{M}_\rho)$ is injective.*

ii) *If $e(-\mu + \rho(\Xi))e(\mu + \rho(\Xi)) \neq 0$, then $\mathcal{B}(G/P_\Xi;\mathcal{L}_{\Xi,\mu})$ is irreducible.*

*Remark 3.7.* i) The calculation of $\mathcal{H}_\lambda$ in (3.21) is equivalent to the determnination of the kernel of $P^\gamma(\lambda)$ defined by [Ko, §4].

ii) Under the notation in Lemma 3.5

$$[\mathcal{H}_{\mu+\rho(\Xi)} : \delta] \geq [\operatorname{Ind}_M^K \mathbf{1} : \delta] - [\operatorname{Ind}_{M_\Xi(K)}^K \mathbf{1} : \delta] \quad \text{for } \delta \in \hat{K} \tag{3.28}$$

and the equality holds if and only if $\mathcal{P}_{\Xi,\mu}$ is injective.

Many statements in this section can be generalized to line bundles (cf. [Sn1]) or vector bundles over $G/K$ but we will omit here.

## 4. Poisson transforms[1]

In this section we study the realization of degenerate principal series on the Riemannian symmetric space $G/K$. For simplicity we mention a result here when $G = GL(n,\mathbb{R})$ but a similar result holds in other cases (cf. [OSn]) where $G = GL(n,\mathbb{C})$ and $GL(n,\mathbb{H})$ and also in the case of sections of a homogeneous line bundles over $U(p,q)/U(p)\times U(q)$.

Given a positive integer $L$ and a sequence of positive integers $\Theta = \{n_1,\dots,n_L\}$ with $0 = n_0 < n_1 < \cdots < n_l = n$, we define a parabolic subgroup

$$(4.1)\qquad P_\Theta = \left\{ p = \begin{pmatrix} g_1 & & & 0 \\ * & g_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & g_L \end{pmatrix} \in GL(n,\mathbb{R}); g_k \in GL(n_k - n_{k-1},\mathbb{R}) \right\}.$$

Note that the subgroup $P_{\{1,\dots,n\}} = \{(x_{ij}) \in G; x_{ij} = 0 \text{ if } i < j\}$ is a minimal parabolic subgroup of $G$, which will be simply denoted by $P$.

For $\mu = (\mu_1,\dots,\mu_L) \in \mathbb{C}^L$, we define a $G$-submodule

$$(4.2)\qquad \mathcal{B}(G/P_\Theta;\mu) = \{f \in \mathcal{B}(G); f(xp) = \tau_{\Theta,\mu}(p^{-1})f(x) \text{ for } p \in P_\Theta\}$$

belonging to degenerate spherical principal series of $G$, where

$$(4.3)\qquad \tau_{\Theta,\mu}(p) = |\det g_1|^{-\mu_1}\cdots|\det g_L|^{-\mu_L} \text{ for } p \in P_\Theta$$

is the character of $P_\Theta$ under the expression of $p$ in (4.1).

Now we introduce the Poisson kernel

$$(4.4)\qquad \begin{aligned} &\Phi_\Theta^\mu(x) = \Phi_{n_1}(x)^{\xi_1}\cdots\Phi_{n_L}(x)^{\xi_L}, \\ &\Phi_m(x) = \sum_{1\le\nu_1<\cdots<\nu_m\le n} \left|\det\left(x_{i\nu_j}\right)_{\substack{1\le i\le m\\1\le j\le m}}\right|^2, \\ &2\xi_k = (\mu_k + n_{k-1}) - (\mu_{k+1} + n_{k+1}) \quad \text{for } k = 1,\dots,L-1, \\ &2\xi_L = \mu_L + n_{L-1}. \end{aligned}$$

It is easy to see that

$$\Phi_\Theta^\mu(pxk) = \tau_{\Theta,-\mu-2\rho(\Theta)}(p)\Phi_\Theta^\mu(x) \quad \text{for } p \in P_\Theta \text{ and } k \in K$$

and therefore we have a $G$-homomorphism

$$(4.5)\qquad \begin{aligned} \mathcal{P}_\Theta^\mu : \mathcal{B}(G/P_\Theta;\mu) \ni f \mapsto (\mathcal{P}_\Theta^\mu f)(x) = \int_K f(xk)dk = \int_K f(k)\Phi_\Theta^\mu(k^{-1}x)dk \\ \in \mathcal{B}(G/K). \end{aligned}$$

Note that $\mathcal{B}(G/P_\Theta;\mu) \subset \mathcal{B}(G/P;\zeta)$ with $\zeta = (\zeta_1,\dots,\zeta_n)$ defined by $\zeta_p = \mu_q$ for $n_{q-1} < p \le n_q$. The map $\mathcal{P}^\zeta_{\{1,\dots,n\}}$ was studied in [K– ] (cf. §3) and for a generic $\zeta$ it is injective and the image is the totality of the real analytic functions $u$ on $G/K$ which satisfy

$$(4.6)\qquad D_k u = \chi_\zeta(D_k)u \quad \text{for } k = 1,\dots,n.$$

Here $\mathbb{D}(G/K) = \mathbb{C}[D_1,\dots,D_n]$ with $\operatorname{ord} D_k = k$.

[1] Most parts in this section are taken from [O4, §5].

**Theorem 4.1.** i) *Any* $u \in \operatorname{Im} \mathcal{P}_\Theta^\mu$ *is real analytic and satisfies*

$$(4.7) \quad D_{IJ}(-\mu_k - n_{k-1})u = 0 \qquad \text{for } \#I = \#J = n - n_k + n_{k-1} + 1 \text{ and } k = 1, \dots, L$$

*and*

$$(4.8) \quad \begin{aligned} F_{ij}^L(-\mu_1 - n_0, \dots, -\mu_L - n_{L-1})u = (D_k - \chi_\zeta(D_k))u = 0 \\ \text{for } i, j = 1, \dots, n \text{ and } k = 1, \dots, L-1. \end{aligned}$$

ii) *If the condition*

$$(4.9) \qquad (\mu_i + n_i) - (\mu_j + n_{j-1} + 1) \notin \{0, 1, 2, 3, 4, \dots\} \quad \text{for } 1 \le i < j \le L$$

*holds,* $\mathcal{P}_\Theta^\mu$ *is a topological* $G$*-isomorphism onto the subspace of* $C^\infty(G/K)$ *which is the totality of the solutions of the system of equations* (4.7) *or equivalently* (4.8).

**Example 4.2.** Suppose $G = SL(2, \mathbb{R})$ and $P = \left\{ \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \in G \right\}$. Then the natural action on $\mathbb{P}_{\mathbb{C}}^1 = (\mathbb{C}^2 - \{0\})/\mathbb{C}^\times \ni \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ is given by the left multiplication. The fixed point group with respect to $i = z_1/z_2$ equals $SO(2)$ and the symmetric space $G/SO(2)$ is identified with the upper half plane $H_+ = \left\{ \left[\begin{smallmatrix} x+yi \\ 1 \end{smallmatrix}\right] \in \mathbb{P}_{\mathbb{C}}^1; y > 0 \right\}$. Since

$$\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{y}i + \frac{x}{\sqrt{y}} \\ \frac{1}{\sqrt{y}} \end{bmatrix} = \begin{bmatrix} x + yi \\ 1 \end{bmatrix} \in H_+ \subset \mathbb{P}_{\mathbb{C}}^1,$$

we have $\Psi_1 = \left(\sqrt{y}\right)^2 + \left(\frac{x}{\sqrt{y}}\right)^2 = \frac{x^2+y^2}{y}$. Hence if $\mu_1 = \mu_2 = 0$, we get the usual Poisson kernel $\Phi_{\{1,2\}}^\mu = \frac{y}{x^2+y^2}$ for $H_+$ and Theorem 4.1 says the isomorphism of the space of harmonic functions on $H_+$ onto that of hyperfunctions on the circle $G/P \simeq \mathbb{R} \cup \{\infty\} \subset \mathbb{P}_{\mathbb{C}}^1$.

**Example 4.3.** Suppose $\Theta = \{n\}$. Then $P_\Theta = G$ and $\mathcal{B}(G/P_\Theta; \mu) = \mathbb{C}|\det x|^\mu$ for generic $\mu \in \mathbb{C}$ and we have the equality (cf. Theorem 1.4 iii)):

$$D_{\{1,\dots,n\}\{1,\dots,n\}}(\lambda)|\det x|^\mu = (\mu + \lambda)(\mu + \lambda + 1) \cdots (\mu + \lambda + n - 1)|\det x|^\mu.$$

If we put $\lambda = 0$ and $\mu = s + 1$, this corresponds to Cayley's formula

$$\det \left( \frac{\partial}{\partial x_{ij}} \right)_{\substack{1 \le i \le n \\ 1 \le j \le n}} (\det x)^{s+1} = (s+1)(s+2) \cdots (s+n)(\det x)^s$$

in view of the Capelli identity. Note that this equality defines the $b$-function of $\det x$ and hence $\mu$ is the meromorphic parameter of $|\det x|^\mu$ and its poles are contained in $\{-1, -2, -3, \dots\}$ (cf. [B], [Sm]).

*Remark 4.4.* i) Theorem 4.1 in the case $\Theta = \{1, n\}$ is given in [O1], where it is conjectured that in general there exists a system of operators $D$ satisfying a suitable condition for $\gamma(D)$ and characterizing the image of the Poisson transform. When $G = GL(n, \mathbb{R})$, the conjecture corresponds to Theorem 4.1 and Corollary 2.1.

ii) If $\operatorname{Re} \mu_j < \operatorname{Re} \mu_{j+1} + 1$ for $j = 1, \dots, L-1$, then (4.9) is valid.

iii) In [J1] and [J2] some differential equations characterizing $\operatorname{Im} \mathcal{P}_\Theta^0$ are given, which are less explicit than ours.

iv) Let $w_\Theta \in \mathfrak{S}_n$ with $\operatorname{Ad}(w_\Theta)\bar{\mathfrak{n}} \cap \mathfrak{n} = \operatorname{Lie}(P_\Theta) \cap \mathfrak{n}$. As is mentioned in §3, $\operatorname{Im} \mathcal{P}_\Theta^\mu = \operatorname{Im} \mathcal{P}_{\{1,\dots,n\}}^{\zeta'}$ with $\zeta_j' = \zeta_{w_\Theta(j)} + w_\Theta(j) - j$ and hence for $\eta \in \mathbb{C}^n$, it is interesting to study the system of differential equations characterizing $\operatorname{Im} \mathcal{P}_{\{1,\dots,n\}}^\eta$.

## 5. Hypergeometric functions[2]

In general, suppose $G$ is a linear reductive Lie group and moreover suppose that we are given three closed subgroups $P_\Theta$, $Q_1$ and $Q_2$ of $G$ and their finite dimensional representations $(\tau_\Theta, V_\Theta)$, $(\tau_1, V_1)$ and $(\tau_2, V_2)$, respectively, which satisfy the conditions

(5.1) $P_\Theta$ is a parabolic subgroup of $G$,

(5.2) the double cosets $Q_j\backslash G/P_\Theta$ have open cosets for $j = 1$ and 2.

Define the degenerate principal series as in §2:

$$\mathcal{B}(G/P_\Theta;\tau_\Theta) = \{\phi(g) \in \mathcal{B}(G)\otimes V_\Theta; \phi(gp) = \tau_\Theta(p^{-1})\phi(g) \text{ for } p \in P_\Theta\}.$$

Let $P_\Theta = L_\Theta N_\Theta$ be a Levi decomposition of $P_\Theta$. Fix a Cartan involution $\theta$ of $G$ with $\theta(L_\Theta) = L_\Theta$. Then $K = \{g \in G; \theta(g) = g\}$ is a maximal compact subgroup of $G$. Let $V_\Theta^*$ be the dual space of $V_\Theta$ and let $\tau_\Theta^*$ be the representation of $P_\Theta$ on $V_\Theta^*$ such that

$$\mathcal{B}(G/P_\Theta;\tau_\Theta)\times\mathcal{A}(G/P_\Theta;\tau_\Theta^*) \ni (\phi,\psi) \mapsto \langle\phi,\psi\rangle_\Theta = \int_K \langle\phi(k),\psi(k)\rangle dk \in \mathbb{C} \tag{5.3}$$

defines a $G$-invariant bilinear form. Note that $(\tau_\Theta^*, V_\Theta^*)$ is the tensor product of the contragredient representation of $\tau_\Theta$ and the character $\det(\mathrm{Ad})$ of $P_\Theta$ on $\mathfrak{n}_\Theta = \mathrm{Lie}(N_\Theta)$, and that if $\langle\phi(k),\psi(k)\rangle$ is an integrable functions on $K$,

$$\langle\phi,\psi\rangle_\Theta = \int_{\bar N_\Theta} \langle\phi(n),\psi(n)\rangle dn \tag{5.4}$$

with a suitably normalized Haar measure $dn$ on $\bar N_\Theta = \theta(N_\Theta)$.

**Definition 5.1.** For given functions $\phi \in \mathcal{B}(G/P_\Theta;\tau_\Theta)\otimes V_1$ and $\psi \in \mathcal{B}(G/P_\Theta;\tau_\Theta^*)\otimes V_2$ satisfying

$$\begin{aligned}\phi(q_1x) &= \tau_1(q_1)\phi(x) \quad \text{for } q_1 \in Q_1,\\ \psi(q_2x) &= \tau_2(q_2)\psi(x) \quad \text{for } q_2 \in Q_2,\end{aligned} \tag{5.5}$$

we call a $V_1 \otimes V_2$-valued function

$$\Phi_{\phi,\psi}(x) = \int_K \langle\phi(xk),\psi(k)\rangle dk \tag{5.6}$$

on $G$ a *hypergeometric function.*

By the $G$-invariance of the bilinear form we have

$$\Phi_{\phi,\psi}(q_1xq_2) = \tau_1(q_1)\tau_2(q_2^{-1})f(x) \quad \text{for } (q_1,q_2) \in Q_1\times Q_2 \tag{5.7}$$

and for elements $D$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}_\mathbb{C} = \mathbb{C}\otimes\mathrm{Lie}(G)$

$$\pi_D(\Phi_{\phi,\psi}) = 0 \quad \text{if } \pi_D(\phi) = 0. \tag{5.8}$$

Note that the following map defines a $G$-homomorphism:

$$\mathcal{B}(G/P_\Theta;\tau_\Theta) \ni f \mapsto (\mathcal{T}_\psi f)(x) = \int_K \langle f(xk),\psi(k)\rangle dk \in \mathcal{B}(G/Q_2;\tau_2). \tag{5.9}$$

[2] This section is extracted from [O4, §6].

**Example 5.2.** Suppose $P_\Theta$ is a minimal parabolic subgroup. If $Q_1$ and $Q_2$ are a maximal compact subgroup, the integral representations of the corresponding hypergeometric functions are Eisenstein integrals. In particular, if $\tau_1$ and $\tau_2$ are trivial representations, we have the integral representations of zonal spherical functions.

Now we will consider the case when $G = GL(n, \mathbb{R})$ and furthermore $P_\Theta$ and $Q_2$ are parabolic subgroups. In particular, we examine the case when they are maximal, namely, $\Theta = \{\ell, n\}$ and $Q_2 = P_{\{k,n\}}$. Suppose $\ell < k < n$ and $n = m\ell$ with a positive integer $m$ satisfying $m > 1$. To assure the existence of the nontrivial intertwining operator (5.9), we assume $\tau_\Theta = \tau_{\mu,0}$ with $\mu = (k, 0) \in \mathbb{C}^2$. In this case, the intertwining operator $A(\lambda, w)$ in §3 with $w = \begin{pmatrix} 1 & 2 & \cdots & \ell & \ell+1 & \ell+2 & \cdots & k & k+1 & \cdots & n \\ k-\ell+1 & k-\ell+2 & \cdots & k & 1 & 2 & \cdots & k-\ell & k+1 & \cdots & n \end{pmatrix}$ converges for any continuous function $f \in \mathcal{B}(G/P_\Theta; \tau_\Theta)$. Note that the corresponding kernel function is a measure whose support is the compact double coset in $Q_2\backslash G/P_\Theta$ and $\bar{N}_w = \{(x_{ij}) \in G; x_{ij} = \delta_{ij} \text{ if } i \geq \ell \text{ or } j \leq \ell \text{ or } j > k\}$.

**Lemma 5.3.** *The intertwining operator $A(\lambda, w)$ defines a G-homomorphism*

$$\begin{aligned} \mathcal{T}_w : \mathcal{B}(G/P_{\{\ell,n\}}; (k, 0)) &\to \mathcal{B}(G/P_{\{k,n\}}; (\ell, 0)). \\ \phi &\mapsto \int_{O(k)\times I_{n-k}} \phi(gk)dk \end{aligned} \tag{5.10}$$

Through the anti-automorphism $G \ni g \mapsto g^{-1} \in G$, $\mathcal{B}(G/P_{\{\ell,n\}}; (k, 0))$ is canonically identified with the space of hyperfunctions $\phi$ on the $\ell n$-dimensional manifold

$$M(\ell, n) = \left\{ \left(t_{ij}\right) = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \cdots & \cdots & \cdots \\ t_{\ell 1} & \cdots & t_{\ell n} \end{pmatrix}; \ t_{ij} \in \mathbb{R} \ \text{ and } \ \operatorname{rank}\left(t_{ij}\right) = \ell \right\} \tag{5.11}$$

which satisfy

$$\phi(g(t_{ij})) = |\det g|^{-k}\phi((t_{ij})) \quad \text{for} \quad g \in GL(\ell, \mathbb{R}). \tag{5.12}$$

Similarly $\mathcal{B}(G/P_{\{k,n\}}; (\ell, 0))$ is canonically identified with the space of hyperfunctions $\Phi$ on the $kn$-dimensional manifold

$$M(k, n) = \left\{ \left(y_{ij}\right) = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots \\ y_{k1} & \cdots & y_{kn} \end{pmatrix}; \ y_{ij} \in \mathbb{R} \ \text{ and } \ \operatorname{rank}\left(y_{ij}\right) = k \right\} \tag{5.13}$$

which satisfy

$$\Phi(g(y_{ij})) = |\det g|^{-\ell}\Phi((y_{ij})) \quad \text{for} \quad g \in GL(k, \mathbb{R}). \tag{5.14}$$

Denoting $GL(\ell, \mathbb{R})_+ = \{g \in GL(\ell, \mathbb{R}); \det g > 0\}$, we choose $Q_1 = GL(\ell, \mathbb{R})_+ \times \cdots \times GL(\ell, \mathbb{R})_+ \subset GL(n, \mathbb{R})$ and define $\tau_1$ by

$$\tau_1(g) = (\det g_1)^{-\alpha_1} \cdots (\det g_m)^{-\alpha_m} \quad \text{for} \quad g = (g_1, \ldots, g_m) \in Q_1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$ with the condition

$$\alpha_1 + \cdots + \alpha_m + k = 0. \tag{5.15}$$

Let $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{1, -1\}^m$. Using a function

$$|s|_{\epsilon_p}^{\alpha_p} = \begin{cases} |s|^{\alpha_p} & \text{if } \epsilon_p s > 0, \\ 0 & \text{if } \epsilon_p s \le 0 \end{cases} \tag{5.16}$$

on $\mathbf{R}$, we give a function

$$\phi(t) = \frac{1}{2}\Big(\prod_{p=1}^{m} \Big|\det\Big(t_{ij}\Big)_{\substack{1\le i\le \ell \\ (p-1)\ell<j\le p\ell}}\Big|_{\epsilon_p}^{\alpha_p} + \prod_{p=1}^{m} \Big|\det\Big(t_{ij}\Big)_{\substack{1\le i\le \ell \\ (p-1)\ell<j\le p\ell}}\Big|_{-\epsilon_p}^{\alpha_p}\Big) \tag{5.17}$$

in Definition 5.1, which belongs to $\mathcal{B}(G/P_{\{\ell,n\}};(k,0))$.

Put $x_{w(i)w(j)} = y_{ij}$ for $i = 1, \dots, \ell$ and $j = 1, \dots, n$. Then in $GL(n, \mathbf{R})$

$$\begin{pmatrix} t_{11} & & t_{1k} & & \\ \cdots & \cdots & \cdots & & \\ t_{\ell 1} & & t_{\ell k} & & \\ & 1 & & & \\ & & \ddots & & \\ & & & & 1 \end{pmatrix} w^{-1} \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots \\ y_{k1} & \cdots & y_{kn} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \sum_{\nu=1}^k t_{1\nu}x_{\nu 1} & \cdots & \sum_{\nu=1}^k t_{1\nu}x_{\nu n} \\ \cdots & \cdots & \cdots \\ \sum_{\nu=1}^k t_{\ell\nu}x_{\nu 1} & \cdots & \sum_{\nu=1}^k t_{\ell\nu}x_{\nu n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} w^{-1} \tag{5.18}$$

and we have our hypergeometric function

$$\Phi(\alpha, \epsilon; x) = \int_{\mathbf{R}^{\ell(k-\ell)}} \prod_{p=1}^{m} \Big|\det\Big(\sum_{\nu=1}^{k} t_{i\nu}x_{\nu j}\Big)_{\substack{1\le i\le \ell \\ (p-1)\ell<j\le p\ell}}\Big|_{\epsilon_p}^{\alpha_p} \prod_{1\le i\le \ell<j\le k} dt_{ij} \tag{5.19}$$

with the convention

$$t_{ij} = \delta_{ij} \quad \text{if} \quad 1 \le j \le \ell. \tag{5.20}$$

**Theorem 5.4.** *The hypergeometric function $\Phi(\alpha, \epsilon; x)$ on $M(k,n) = \{(x_{ij}); x_{ij} \in \mathbf{R},\ 1 \le i \le k,\ 1 \le j \le n,\ \mathrm{rank}(x_{ij}) = k\}$ satisfies the following equations.*

i) *A right $GL(\ell, \mathbf{R})_+ \times \cdots \times GL(\ell, \mathbf{R})_+$-invariance:*

$$\Phi(xg) = \Big(\prod_{p=1}^{m} (\det g_p)^{\alpha_p}\Big)\Phi(x) \quad \textit{for} \quad g = g_1 \otimes \cdots \otimes g_m \quad \textit{with} \quad g_p \in GL(\ell, \mathbf{R})_+.$$

ii) *A left $GL(k, \mathbf{R})$-invariance:*

$$\Phi(gx) = |\det g|^{-\ell}\Phi(x) \quad \textit{for} \quad g \in GL(k, \mathbf{R}).$$

iii) *Generalized Capelli operators:*

$$\det\Big(\frac{\partial}{\partial x_{i_\mu j_\nu}}\Big)_{\substack{1\le\mu\le\ell+1 \\ 1\le\nu\le\ell+1}} \Phi(x) = 0 \quad \textit{for} \quad 1 \le i_1 < \cdots < i_{\ell+1} \le k \textit{ and } 1 \le j_1 < \cdots < j_{\ell+1} \le n. \tag{5.21}$$

The two invariances in Theorem 5.4 are infinitesimally as follows:

(5.22)

$$\sum_{i=1}^{k} x_{i,\mu+\ell p} \frac{\partial \Phi}{\partial x_{i,\nu+\ell p}} = \alpha_p \delta_{\mu\nu} \Phi \quad \text{for } 1 \leq \mu \leq \ell,\ 1 \leq \nu \leq \ell,\ 0 \leq p < m,$$

$$\sum_{\nu=1}^{n} x_{i\nu} \frac{\partial \Phi}{\partial x_{j\nu}} = -\ell \delta_{ij} \Phi \quad \text{for } 1 \leq i \leq k,\ 1 \leq j \leq k. \tag{5.23}$$

*Remark 5.5.* i) Note that $GL(n, \mathbb{R})/P_{\{\ell,n\}} \simeq O(n)/O(\ell) \times O(n-\ell)$. Hence the definition of $\Phi(\alpha, \epsilon; x)$ is an integration of a function on $O(n)/O(\ell) \times O(n-\ell)$ over its submanifold $O(k)/O(\ell) \times O(k-\ell)$, which is a Radon transform on real Grassmannians (cf. [H4, Chap. 1]).

ii) Suppose $\ell = 1$. The integration can be rewritten as

$$\Phi(\alpha, \epsilon; x) = C \int_{t_1^2+\cdots+t_k^2=1} \prod_{j=1}^{n} \Big| \sum_{p=1}^{k} t_p x_{pj} \Big|_{\epsilon_p}^{\alpha_p} \hat{\omega} \tag{5.24}$$

with a suitable constant $C$ and

$$\hat{\omega} = \sum_{p=1}^{k} (-1)^{p+1} t_p dt_1 \wedge dt_2 \wedge \cdots \wedge dt_{p-1} \wedge dt_{p+1} \wedge \cdots \wedge dt_k. \tag{5.25}$$

This integral representation coincides with the one given in [G] and the corresponding equations in Theorem 5.4 with $\ell = 1$ are same as in [GG]. This hypergeometric function is also studied by [A].

For the analysis of hypergeometric functions in the case when $G = P_{\{\ell,n\}}$ and $Q_2 = P_{\{k,n\}}$, the following theorem is essential.

**Theorem 5.6.** *The intertwining operator in* Lemma 5.3 *is the topological $G$-isomorphism onto the solution space $V_{k,n}^{\ell}$ of the system* (5.21) *under our realization using* (5.13) *and* (5.14) *if*

$$0 < \ell < k < n \quad \text{and} \quad \ell + k < n. \tag{5.26}$$

**Corollary 5.7.** *Let $Q$ be a closed subgroup of $GL(n, \mathbb{R})$ and let $(\tau, V)$ be a finite dimensional representation of $Q$. If* (5.26) *holds, the integral transformation $\mathcal{R}$ : $\phi \mapsto \int_{SO(k)} \phi(kx)dk$ is a bijection between $S(\ell, n; \tau)$ and $S(k, n; \ell, \tau)$. Here $S(\ell, n; \tau)$ is the space of $V$-valued hyperfunctions $\phi$ on $M(\ell, n)$ satisfying* (5.12) *and*

$$\phi(tg) = \tau(g^{-1})\phi(t) \quad \text{for } g \in Q. \tag{5.27}$$

*Moreover $S(k, n; \ell, \tau)$ is the space of of $V$-valued hyperfunctions $\Phi(x)$ on $M(k, n)$ satisfying the equations given in* Theorem 5.4 ii) *and* iii) *and*

$$\Phi(xg) = \tau(g^{-1})\Phi(x) \quad \text{for } g \in Q. \tag{5.28}$$

*In this corollary 'hyperfunctions' can be replaced by 'Schwartz's distributions' or '$C^\infty$-functions' or 'real analytic functions', which is clear by our way of the proof.*

Lastly we give other examples of $Q_1$ for the same $P_\Theta$ and $Q_2$.

**Example 5.9.** For $A = (A_{ij}) \in \mathfrak{gl}(n, \mathbf{R})$ with $n = m\ell$ and for $\mu = 1, \dots, m$ and $\nu = 1, \dots, m$, put

$$\begin{aligned} \mathfrak{u}_1 &= \{A \in \mathfrak{gl}(n, \mathbf{R}); M_{\mu\nu}(A) = 0 \ \text{ if } \ \mu \neq \nu - 1\}, \\ \mathfrak{u}_2 &= \{A \in \mathfrak{gl}(n, \mathbf{R}); M_{\mu\nu}(A) = 0 \ \text{ if } \ \mu > \nu, \\ M_{\nu,\nu+i}(A) &= M_{\nu+1,\nu+i+1}(A) \ \text{ for } \ i = 1, \dots, m-2 \ \text{ and } \ \nu = 1, \dots, m-i-1\}, \\ M_{\mu\nu}(A) &= (A_{(\mu-1)\ell+i,(\nu-1)\ell+j})_{\substack{1 \le i \le \ell \\ 1 \le j \le \ell}}. \end{aligned}$$

If $Q_1$ equals the nilpotent group $\exp \mathfrak{u}_1$ or $\exp \mathfrak{u}_2$, condition (5.2) is valid and we can define hypergeometric functions with respect to the character of $Q_1$ which has $m-1$ continuous parameters. When $\ell = 1$, the hypergeometric functions correspond to those discussed in [GRS] and [KHT].

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