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On tamely ramified supercuspidal representations of a unitary group over a quaternion field

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Introduction.

Let F be a non-archimedean local field of residue characteristic $p > 2$. Associated to an anisotropic maximal F-torus T, which splits over a tamely ramified extension of F, of a symplectic group G, L. Morris defined, in [M1], a system Ψ, called a cuspidal datum, which consists of elements c_i of the Lie algebra $\mathcal{T}$ of T (accurately, the algebra of F-rational points of $\mathcal{T}$), linear characters ψ_i of the centralizer in G of c_i in G, their conductoral exponents f_i and their dual elements c'_i in $\mathcal{T}$, and an irreducible (cuspidal) representation of a certain open compact subgroup of G. When G is a general linear group, this system generalizes an admissible character, defined by R. Howe [H], of such a minisotropic maximal F-torus of G. Having used the methods of R. Howe [H] and A. Moy [Mo], associated to such a datum Ψ, L. Morris constructed an open compact subgroup K of the symplectic group G and its an irreducible representation ρ, and showed that the representation π of G compactly induced from ρ is irreducible and supercuspidal. Slightly modifying the definition of cuspidal data of [M1], K. Kariyama showed in [K] that for a certain unitary group G over F with an involution except a symplectic group, the representations given in [M1] associated to special cuspidal data Ψ with c_i's (and c'_i's) restricted are also irreducible and supercuspidal.

In this note, we report that for a unitary group G over the quaternion division algebra D of F, similarly, we can define a cuspidal datum Ψ. related to an unramified quadratic extension F' of F in D, and can construct an irreducible supercuspidal representation π of G associated to such a cuspidal datum Ψ. Simultaneously, we can get an irreducible supercuspidal representation π' of the centralizer $G' = Z_G(F')$ of F' in G, which is isomorphic to a unitary group over F/F'. This shows that the representation π of G corresponds to the representation π' of $G' = Z_G(F')$ via the cuspidal datum Ψ.

Notations. Let p be a prime integer ≥ 2. For a p-field F, commutative or not, we denote by $\mathcal{O}_F$, $\mathcal{P}_F$ and ϖ_F the maximal order of F, the maximal ideal of $\mathcal{O}_F$, and a prime element of F, respectively. For commutative (p-)fields $L \supset F$ of finite extension degree, we denote by $\mathrm{Tr}_{L/F}$ the trace map from L into F. We denote by $\mathbb{Z}$ the set of rational integers.

§1. Unitary groups.

Throughout this note, F will be a non-archimedean local field of residue characteristic $p > 2$, and D will be the quaternion division algebra of F with the canonical involution $x \mapsto \bar{x}$. Put $\varpi = \varpi_F$ for simplicity. We may put

$$\varpi_D^2 = \varpi.$$

Let r be an integer ≥ 2, and V a vector space of dimension $2r$ over D. Let $I = \{\pm 1, \cdots, \pm r\}$, and $\{e_i\}_{i \in I}$ be a basis of V. Let $\varepsilon = \pm 1$. We define the form $f: V \times V \to D$ by

$$f(\sum_{i\in I} e_i x_i, \sum_{i\in I} e_i y_i) = \sum_{i=1}^{r} (x_i \overline{y_{-i}} + \varepsilon x_{-i}\overline{y_i})$$

where x_i, $y_i \in D$ for all $i \in I$. Then f is obviously a non-degenerate ε-hermitian form, and $\{e_i\}_{i\in I}$ is the Witt basis for f.

Denote by $\mathrm{End}_D(V)$ the ring of all the D-endomorphisms of V, and write

$$A = \mathrm{End}_D(V) .$$

Denote by

$$G = \mathrm{U}(V,f)$$

the unitary group of the form f on V. Then

$$G = \{g \in A = \mathrm{End}_D(V) : f(gv, gw) = f(v,w) \quad \text{for all} \quad v,\ w \in V\}.$$

By means of $\{e_i\}_{i\in I}$, each $g \in A = \mathrm{End}_D(V)$ is expressed as follows:

$$g(\sum_{i\in I} e_i x_i) = \sum_{i\in I} e_i (\sum_{i\in I} g_{ij} x_j)$$

with $g_{ij} \in D$ for all i, $j \in I$. Thus we may set $g = (g_{ij}) \in \mathrm{M}_{2r}(D)$, where $\mathrm{M}_{2r}(D)$ denotes the ring of all the $2r\times 2r$ matrices with coefficients in D. So we may identify $A = \mathrm{End}_D(V)$ with $\mathrm{M}_{2r}(D)$. From now on, we shall use this identification freely. Put

$$J = \begin{pmatrix} 0_r & 1_r \\ \varepsilon 1_r & 0_r \end{pmatrix}$$

where 0_r and 1_r denote the zero and the identity matrices of rank r, respectively. We define the involution σ on $A = \mathrm{M}_{2r}(D)$ by

$$\sigma X = J^{-1}\,{}^t\overline{X} J$$

where for $X = (x_{ij})$, we put ${}^t\overline{X} = (\overline{x}_{ji})$. By the identification $A = \mathrm{End}_D(V) = \mathrm{M}_{2r}(D)$, we have

$$G = \{g \in \mathrm{M}_{2r}(D) : \sigma g \cdot g = 1_{2r}\}$$

where 1_{2r} denotes the identity matrix of rank r.

Throughout this note, F' will be a maximal subfield of D which is unramified extension of F. We identify every element $x \in D$ with $x \cdot 1_{2r} \in A = M_{2r}(D)$. Then D is a subfield of $A = M_{2r}(D)$ and the involution σ on D coincides with the canonical involution. Thus F' is also a subfield of $A = M_{2r}(D)$, and by Lemma 11.3 of [S]

$$\sigma x = x \quad \text{for all} \quad x \in F'.$$

We define a F'-vector subspace V' by

$$V' = \bigoplus_{i \in I} e_i F'.$$

Denote by $Z_A(F')$ the centralizer of F' in $A = M_{2r}(D)$. Since it is well-known that the centralizer of F' in D is F' itself, we have

$$Z_A(F') = M_{2r}(F').$$

Put $A' = M_{2r}(F')$. By the identification $A = \mathrm{End}_D(V) = M_{2r}(D)$, we thus have

$$A' = \mathrm{End}_{F'}(V') = M_{2r}(F').$$

Denote by f' the restriction of f to V'. Then $f' : V' \times V' \to F'$ is obviously a non-degenerate hermitian form over the quadratic extension F'/F with the non-trivial F-automorphism as the involution. Put $Z_G(F') = Z_A(F') \cap G$. Then it follows easily that $Z_G(F')$ is equal to the unitary group $U(V', f')$ of the form f' on V'. We denote by G' this. Thus

$$G' = Z_G(F') = U(V', f').$$

$A' = M_{2r}(F')$ is a subring of A which is invariant under σ. So we denote by σ' the restriction of σ to A'. Then, similarly for G,

$$G' = \{g \in A' = M_{2r}(F') : \sigma' g \cdot g = 1_{2r}\}.$$

§2. Lattice chains.

A subset L of V is called an $\mathcal{O}_D$-lattice of V, if L is a finitely generted $\mathcal{O}_D$-submodule of V which spans V over D. A system $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$, consisting of $\mathcal{O}_D$-lattices L_i's of V, is called an $\mathcal{O}_D$-lattice chain in V, if it satisfies the following two conditions:

(i) $L_i \supsetneqq L_{i+1}$ for all $i \in \mathbb{Z}$,

(ii) there is an integer $s \geq 1$ such that $L_{i+s} = L_i\varpi_D$ for all $i \in \mathbb{Z}$.

(cf. [BT1], [B], [M2]). We call by the $\mathcal{O}_D$-period of $\mathcal{L}$ this integer s. We say that an $\mathcal{O}_D$-lattice chain $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ in V is generated by the Witt basis $\{e_i\}_{i\in I}$, if L_i, for all $i \in \mathbb{Z}$, is written in the following form:

$$L_i = \bigoplus_{i\in I} e_j P_D^{n(i,j)} \quad \text{with} \quad n(i,j) \in \mathbb{Z} \quad \text{for all} \quad j \in I$$

(cf. 1.1 of [BK]).

Let $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ be an $\mathcal{O}_D$-lattice chain, generated by the Witt basis $\{e_i\}_{i\in I}$, in V with $\mathcal{O}_D$-period s. Let $\mathfrak{A}$ and $\mathfrak{P}$ be the hereditary order in $A = \mathrm{End}_D(V)$ and its Jacobson radical, respectively, which are defined by $\mathcal{L}$ (cf. [R], [B], [M2]). Then, since $\mathfrak{P}$ is an invertible fractional ideal of $\mathfrak{A}$ (cf. also [R]),

$$\mathfrak{P}^n = \{X \in A = \mathrm{End}_D(V) : XL_i \subseteqq L_{i+n} \quad \text{for all} \quad i \in \mathbb{Z}\}$$

for all $n \in \mathbb{Z}$. In particular, $\mathfrak{A} = \mathfrak{P}^0$ and $\mathfrak{P} = \mathfrak{P}^1$.

Put

$$P = G \cap \mathfrak{A}.$$

P is called a parahoric subgroup of G in the sense of Bruhat-Tits Theory (cf. Lemma 3.11 of [BT2]). For all integers $n \geq 1$, put

$$P_n = \{g \in G:\ g-1 \in \mathfrak{R}^n\}.$$

Then $\{P_n\}_{n\geq 1}$ is a filtration of P.

Let $V' = \bigoplus_{i\in I} e_i F'$ be as in section 1. For the $\mathcal{O}_D$-lattice chain $\mathfrak{L} = \{L_i\}_{i\in\mathbb{Z}}$ in V, put

$$\mathfrak{L}' = \{L'_i\}_{i\in\mathbb{Z}} \quad \text{with} \quad L'_i = L_i \cap V'.$$

Then L'_i, for all $i \in \mathbb{Z}$, is obviously an $\mathcal{O}_{F'}$-lattice chain in V'. But $\mathfrak{L}' = \{L'_i\}_{i\in\mathbb{Z}}$ is not always an $\mathcal{O}_{F'}$-lattice chain in V'. That is, the condition

(2.1) $$L'_i \supsetneqq L'_{i+1} \quad \text{for all} \quad i \in \mathbb{Z}$$

does not always hold. Assume now that (2.1) holds. Then $\mathfrak{L}'$ is indeed an $\mathcal{O}_{F'}$-lattice chain in V'. For, by the assumption on F' of section 1, $\varpi = \varpi_D^2$ is a uniformizer of F'. Thus, for all $i \in \mathbb{Z}$,

$$L'_i\varpi = (L_i \cap V')\varpi_D^2 = L_i\varpi_D^2 \cap V' = L_{i+2s} \cap V' = L'_{i+2s}.$$

Thus the $\mathcal{O}_{F'}$-lattice chain $\mathfrak{L}'$ defines the hereditary order in $A' = \mathrm{End}_{F'}(V')$ and its Jacobson radical. Denote by $\mathfrak{A}'$ and $\mathfrak{R}'$ them respectively.

PROPOSITION 2.1. *Let notations be as above. Assume that* $\mathfrak{L}$ *is generated by the Witt basis* $\{e_i\}_{i\in I}$ *and that* $\mathfrak{L}'$ *satisfies* (2.1). *Then* $\mathfrak{L}'$ *is an* $\mathcal{O}_{F'}$*-lattice chain in* V' *with* $\mathcal{O}_{F'}$*-period* $2s$, *and for all* $n \in \mathbb{Z}$

$$\mathfrak{R}^n \cap A' = \mathfrak{R}'^{\,n}.$$

§3. Duals of lattices.

For a subset L of V, put

$$L^{\#} = \{v \in V:\ f(V,L) \subset \mathcal{O}_D\}.$$

Then L^* is also an $\mathcal{O}_D$-lattice in V. We say that an $\mathcal{O}_D$-lattice chain $\mathcal{L}$ in V is self-dual (with respect to f), if $L \in \mathcal{L}$ implies $L^* \in \mathcal{L}$.

In this section, assume that $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ is a self-dual $\mathcal{O}_D$-lattice chain in V with $\mathcal{O}_D$-period s, and that $\mathfrak{A}$, $\mathfrak{K}$, P and $\{P_n\}_{n\geq 1}$ be as in section 2.

LEMMA 3.1. *Let notations and assumptions be above. Then, $\mathfrak{K}^n$, for all $n \in \mathbb{Z}$, is stable under σ.*

For a subset M of $A = \mathrm{M}_{2r}(D)$ which is stable under σ, we define

$$M_- = \{X \in A = \mathrm{M}_{2r}(D) : \sigma X + X = 0\}.$$

In particular, it is well-known that A_- is the Lie algebra of G. By Lemma 3.1, we get $\mathfrak{K}^n_-$ for all $n \in \mathbb{Z}$.

From now on, τ_D will be the reduced trace in D, and tr will be the trace in $A = \mathrm{M}_{2r}(D)$. Then, by Corollary 2 to Proposition 1 of Ch.IX-2 of [W], $\tau_D \circ \mathrm{tr}$ is the reduced trace in $A = \mathrm{M}_{2r}(D)$. Write

$$\tau_A = \tau_D \circ \mathrm{tr}.$$

For a subset M in $A = \mathrm{M}_{2r}(D)$, put

$$M^\vee = \{X \in A = \mathrm{M}_{2r}(D) : \tau_A(XM) \subseteq \mathcal{O}_F\}.$$

PROPOSITION 3.2. *Let $2i \geq j \geq i \geq 1$ and denote by $(P_i/P_j)^\wedge$ the Pontrjagin dual of P_i/P_j. Then there is an isomorphism of abelian groups*

$$\mathfrak{K}_-^{1-2s-j}/\mathfrak{K}_-^{1-2s-i} \longrightarrow (P_i/P_j)^\wedge,$$

where we recall that s is the $\mathcal{O}_D$-period of $\mathcal{L}$.

Define the function $\lambda\colon \mathbb{Z} \to \mathbb{Z}$ by

$$\lambda(n) = 1 - n - 2s \quad \text{for all} \quad n \in \mathbb{Z}.$$

§4. Cuspidal data.

From now on, Ω will be an additive character of F of conductor $\mathcal{O}_F$, that is, $\Omega|_{\mathcal{O}_F} \equiv 1$ and $\Omega|_{\mathfrak{p}_F^{-1}} \not\equiv 1$, and E will be a tamely ramified extension (of degree $< 2r$) over F' such that it is a subfield of $A' = \mathrm{End}_{F'}(V')$ (and so of $A = \mathrm{End}_D(V)$), and that it is stable under σ'.

Let $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ be a self-dual $\mathcal{O}_D$-lattice chain, generated by the Witt basis $\{e_i\}_{i\in I}$, in V with $\mathcal{O}_D$-period s. Let $\mathfrak{A}$ and $\mathfrak{P}$ be the hereditary order in $A = \mathrm{End}_D(V)$ and its Jacobson radical, respectively, which are defined by this $\mathcal{L}$. From now on, assume

(i) $\mathcal{L}' = \{L'_i\}_{i\in\mathbb{Z}}$ satisfies (2.1),

(ii) $\mathcal{L}'$ is self-dual with respect to f' (see section 1),

(iii) $\mathcal{L}'$ is stable under the multiplication by elements of $E^\times$, and

(iv) the $\mathcal{O}_E$-period of $\mathcal{L}'$, as an $\mathcal{O}_E$-lattice chain in V' by (iii), is at most 2.

By the assumption (i) on $\mathcal{L}'$ and by Proposition 2.1, $\mathcal{L}'$ defines the hereditary order $\mathfrak{A}'$ in $A' = \mathrm{End}_{F'}(V')$ and its Jacobson radical $\mathfrak{P}'$ as in section 2.

LEMMA 4.1. *E is unramified over $E_0 = E^\sigma$, the fixed field of E under σ.*

Let P and $\{P_n\}_{n\geq 1}$ be the parahoric subgroup of G and the filtration of P, respectively, defined by $\mathfrak{A}$ and $\mathfrak{P}^n$ for $n \geq 1$ in section 2..

Definition (Cuspidal datum). Let notations and assumptions be as above. Let n be an integer ≥ 2. A system $\Psi = \{(f_1, \beta_1, c_1, \psi_1), \cdots, (f_{n-1}, \beta_{n-1}, c_{n-1}, \psi_{n-1}); \xi_0\}$ is called a cuspidal datum relative to E/F' (cf. [M1], [K]), if the following conditions (i)-(v) are satisfied:

(i) Let $f_1, f_2, \cdots, f_{n-1}$ be a sequence of positive integers satifying

$$f_1 > f_2 > \cdots > f_{n-1} > 1.$$

(ii) Let $\beta_1, \beta_2, \cdots, \beta_{n-1}$ be a sequence of elements of E_- such that the subfields $F_i = F[\beta_i]$, generated by β_i over F, satisfy

$$F' \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} = E.$$

(iii) Let $c_1, c_2, \cdots, c_{n-1}$ be a sequence of elements of E_- such that for all i, $1 \leq i \leq n-1$,

(a) $c_i \in F_i$,

(b) $c_i \in \mathfrak{P}^{\lambda(f_i)} - \mathfrak{P}^{\lambda(f_i)+1}$,

(c) c_i is very cuspidal over F_{i-1} in the sense of [C]. That is, let a be the order of c_i in F_i (cf. Ch.I-4 of [W]), e the order of ramification of F_i over F_{i-1}, and ϖ_{i-1} a prime element of F_{i-1}. Then the followin two conditions are satified.

(1) $(a, e) = 1$, and

(2) the canonical image of $\varpi_{i-1}^{-a} c_i^e$ generates $\bar{F}_i$ over $\bar{F}_{i-1}$, where $\bar{F}_i$ and $\bar{F}_{i-1}$ are the residue class fields of F_i and F_{i-1}, respectively.

Put

$$G_0 = G \quad \text{and} \quad G_i = Z_{G_{i-1}}(\beta_i) \quad \text{for} \quad 1 \leq i \leq n-1,$$

where $Z_{G_{i-1}}(\beta_i)$ denotes the centralizer in G_{i-1} of β_i, and further put

$K(j) = P \cap G_i$ and $K_m(j) = P_m \cap G_j$ for $j \geq 0$ and $m \geq 1$. Then, by the conditions (ii) and (iii)(c), we can easily see

$$G_i \subset \mathrm{End}_{F_i}(V')$$

by induction on $i \geq 1$.

(iv) Let $\phi_1, \phi_2, \cdots, \phi_{n-1}$ be a sequence of linear characters of $G_1, G_2, \cdots, G_{n-1}$, respectively, that is, each ϕ_i is a character of G_i which factors through the determinant in $\mathrm{End}_{F_i}(V')$, such that

$$\phi_i(x) = \Omega(\tau_A(c_i(x-1))) \quad \text{for all} \quad x \in K_{j'_i}(i)$$

where j'_i is the integer part of $(f_i + 1)/2$.

We can see that the quotient $\overline{K(n-1)} = K(n-1)/K_1(n-1)$ is an algebraic group defined over a finite field (cf. 2.13 of [M2]). Thus we denote by $\overline{K(n-1)}^0$ the identity component of $\overline{K(n-1)}$.

(v) Let ξ_0 be an irreducible cuspidal representation of $\overline{K(n-1)}^0$ (cf. [DL]) such that the representation of $\overline{K(n-1)}$ induced from ξ_0 is irreducible. We denote by ξ the representation of $\overline{K(n-1)}$ obtained by inflating the induced representation under the canonical homomorphism $K(n-1) \to \overline{K(n-1)}$.

Remark. In [K], we have not defined the cuspidal data in the case of $f_n > 1$ as in [M1] and [M3]. We do not also consider this case in this note. However we can define the cuspidal data, and construct irreducible supercuspidal representations of G, as in [M1] and [M3], in this case.

§5. Open compact subgroups and their representations.

In this section, assume that E/F', $\mathscr{L} = \{L_i\}_{i\in\mathbb{Z}}$ and $\mathscr{L}' = \{L'_i\}_{i\in\mathbb{Z}}$ be as in section 4. Let n be an integer ≥ 2, and $\Psi = \{(f_1,\beta_1,c_1,\phi_1), \cdots, (f_{n-1},\beta_{n-1},c_{n-1},\phi_{n-1});\xi_0\}$ be a cuspidal datum relative to E/F' in section 4.

Associated to Ψ, we get the following subgroups of G', as in Definition 2 of 2.2 of [K], which are similar to those of G in section 4:

$$G'_0 = G' \quad \text{and} \quad G'_i = Z_{G'_{i-1}}(\beta_i) \quad \text{for} \quad 1 \leq i \leq n-1,$$

$$K'(j) = P' \cap G'_j \quad \text{and} \quad K'_m(j) = P'_m \cap G'_j \tag{5.1}$$

for $j \geq 0$ and $m \geq 1$, where $P' = G' \cap \mathscr{A}'$ and $P'_m = G' \cap (1+\mathscr{R}'^{\,n})$ (see section 4 for $\mathscr{A}'$ and $\mathscr{R}'$).

PROPOSITION 5.1. *Let notations and assumptions be as above. Then Ψ for G over D is as well a cuspidal datum relative to E for $G' = Z_G(F')$ over F'/F in Definition 2 of 2.2 of* [K].

By Proposition 5.1 and by [K], we can hence get an open compact subgroup K' of G' form the subgroups $K'_j(m)$ of (5.1), and an irreducible representation ρ' of K', associated to Ψ, as a cuspidal datum for G'.

Similarly, we can construct an open compact subgroup K of G and an irreducible representation ρ of K, associated to Ψ for G as follows: Put

$$j_k = [f_k/2], \quad j'_k = f_k - j'_k \quad \text{for} \quad 1 \leq k \leq n-1,$$

and

$$j_n = j'_n = 1,$$

where $[r]$ denotes the integer part of a rational number r. We can define subgroups of G as follows:

$$K_k = K_{j_n}(n-1) \cdots K_{j_{k+1}}(k)$$
$$L_k = K_{j_k}(k-1) \cdots K_{j_1}(0)$$
$$L_k^* = K_{j_k'}(k-1) K_{j_{k-1}}(k-2) \cdots K_{j_1}(0)$$
for $1 \leq k \leq n-1$. Put
$$K_0 = K_{j_n}(n-1) \cdots K_{j_1}(0) \quad \text{and} \quad K = K(n-1)K_0.$$
Then obviously $K_0 = K_k L_k$ for $1 \leq k \leq n-1$.

Now, fix k, $1 \leq k \leq n-1$. By definition, $K(n-1)K_k \subset G_k$. Thus we can restrict the character ψ_k in Ψ to $K(n-1)K_k$. We can further extend the restricted character to a one-dimensional representation $\tilde{\mu}_k$ of $K(n-1)K_k L_k^*$, by letting
$$\tilde{\mu}_k(1+x) = \Omega(\tau_A(c_k x)) \quad \text{for} \quad 1+x \in L_k^*.$$

(a) Let f_k be even, i.e., $f_k = 2j_k$. Then $L_k = L_k^*$ and $K = K(n-1)K_k L_k = K(n-1)K_k L_k^*$. Hence $\tilde{\mu}_k$ is a desired one-dimensional representation of K. We put $\eta_k = \tilde{\mu}_k$.

(b) Let f_k be odd, i.e., $f_k = 2j_k + 1$. Then $L_k \neq L_k^*$ and $K_k L_k^* \subsetneqq K_k L_k = K_0$. Denote by
$$\mu_k = \tilde{\mu}_k | K_k L_k^*$$
the restriction of $\tilde{\mu}_k$ to $K_k L_k^*$. We want to get an irreducible representation of K_0 which contains μ_k, and to extend it to $K = K(n-1)K_0$.

Put
$$\mathcal{H} = K_0/\mathrm{Ker}\,\mu_k \quad \text{and} \quad \mathcal{Z} = K_k L_k^*/\mathrm{Ker}\,\mu_k.$$
Then we can see that $\mathcal{Z}$ is a central subgroup of $\mathcal{H}$, and that the form $\langle\ ,\ \rangle$ on $K_0 \times K_0$ can be defined by
$$\langle g, h \rangle = \mu_k(ghg^{-1}h^{-1}) \quad \text{for} \quad g,\ h \in K_0,$$

and that it also induces the form on $\mathcal{H}/\mathcal{Z} \times \mathcal{H}/\mathcal{Z}$ (see Lemma 10 of section 4 of [K]). We still denote by $<\ ,\ >$ it.

Put $W = \mathcal{H}/\mathcal{Z}$. Then W has a finite-dimensional vector space structure over the prime field $\mathbb{F}_p$ (cf. 4.9 of [M1]).

PROPOSITION 6.2. *Let notations and assumptions be as above. Then* $<\ ,\ >$ *is the non-degenerate alternating form on* $W \times W$.

It follows that $\mathcal{H}$ is a finite Heisenberg group with center $\mathcal{Z}$ associated with the non-degenerate alternating space $(W, <\ ,\ >)$ (cf. [G]). Further, we can show that the action of $K(n-1)$ on $\mathcal{H}/\mathcal{Z}$ via conjugation leaves $<\ ,\ >$ invariant (see also Lemma 10 of section 4 of [K]). Hence, by the well-knowen Heisenberg construction, we can get an irreducible representation η_k of $K = K(n-1)K_0$ from μ_k, regarded as a character of $\mathcal{Z}$ (see Theorem 2.4 of [G]), as we claimed above.

Varying k, we hence get the irreducible representations $\eta_1, \cdots, \eta_{n-1}$ of K. Let ξ be the irreducible representation of $K(n-1)$ defined in Definition (v) of section 4. We can extend it to an irreducible representation of $K = K(n-1)L_{n-1}$, also denoted by ξ, by setting ξ to be trivial on L_{n-1}.

Finally, put

$$\rho = \eta_1 \otimes \cdots \otimes \eta_{n-1} \otimes \xi.$$

This ρ is a desired irreducible representation of K.

§6. Supercupidal representations.

In this section, we keep notations and assumptions of section 5.

Let Ψ be a cuspidal datum relative to E/F'. Let ρ (resp. ρ') be the irreducible representation of the open compact subgroup K (resp. K'). of G (resp. G'), defined in section 5, associated to Ψ. Denote by

$$\pi = \text{c-Ind}(\rho:K,G)$$

the compactly induced representation of ρ from K to G, and similarly,

$$\pi' = \text{c-Ind}(\rho':K',G').$$

Then we get the following:

THEOREM 6.1 (Theorem in section 5 of [K]). *π' is an irreducible supercuspidal representation of G'.*

We also get a similar theorem for π which can be proved in a similar way.

Let H be an open subgroup of G, and ν be a smooth representation of H on a vector space U over $\mathbb{C}$, the set of complex numbers. We say that $g \in G$ intertwines ν, if $\mathrm{Hom}_{H\cap g^{-1}Hg}(\nu, \nu^g) \neq (0)$ holds, where $\nu^g(h) = \nu(ghg^{-1})$ for all $h \in H \cap g^{-1}Hg$.

LEMMA 6.2. *If $g \in G$ intertwines ρ, then $g \in KG_{n-1}K$.*

LEMMA 6.3. *Let ξ be the irreducible representation of K defined in section 5. Then, if $g \in G_{n-1}$ intertwines ρ, it intertwines $\xi|_{K(n-1)}$.*

By definitions in section 4, we have $K(n-1) = K'(n-1)$, $K_1(n-1) = K_1'(n-1)$ and $\overline{K(n-1)} = \overline{K'(n-1)}$. The structures of these groups have been studied in section 3 of [K]. By those results, we can prove the following:

LEMMA 6.4. *If* $g \in G_{n-1}$ *intertwines* $\xi|_{K(n-1)}$, *then* $g \in K(n-1)$.

It follows from Lemmas 6.1-6.4 that $g \in G$ intertwines ρ, then $g \in K$. Hence by Proposition 1.5 of [C], we get the following:

THEOREM 6.5. π *is an irreducible supercuspidal representation of* G.

Theorems 6.1 and 6.5 show that an irreducible supercuspidal representation π of G corresponds to an irreducible supercuspidal representation π' of $G' = Z_A(F')$ via a cuspidal datum Ψ relative to E/F' for G.

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