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Macdonald polynomial as a vector valued character of the quantized universal enveloping algebra $U_q(gl(N))$

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P.I.Etingof and A.A.Kirillov Jr. discovered in [1] that Macdonald's polynomials P_λ (of type A_{N-1}) can be realized as *vector valued characters* of the quantized universal enveloping algebra $U_q(gl(N))$. To identify Macdonald's polynomials with the vector valued characters, they used a variant of the orthogonality relations by Weyl. On the other hand, one of the definitions of Macdonald's polynomials is due to the q-difference equation:

$$\sum_{i=1}^{N} \prod_{\substack{1\le j\le N \\ j\neq i}} \frac{q^k z_i - z_j}{z_i - z_j} T_{q,z_i} P_\lambda(z;q,q^k) = \left(\sum_{i=1}^{N} q^{k(N-i)+\lambda_i}\right) P_\lambda(z;q,q^k) \quad ,$$

where $(T_{q,z_i}f)(z) = f(z_1,\ldots,qz_i,\ldots,z_N)$ and $\lambda = (\lambda_1,\ldots,\lambda_N)$. The aim of this talk is the direct derivation of this q-difference equation from the quantized universal enveloping algebra: We calculate the radial part of a central element of $U_q(gl(N))$. Such statements are also contained in [1]. The new thing here is a direct computation of the radial part, in the spirit of the done in [4]. We believe that a method of direct computation of the radial part should be helpful in future research , for example in generalization to the case of quantum affine algebras, where the arguments of [1] do not work. We refer the audience to [2] for the detail.

1 Notations and main Theorem

Set a R-matrix as

$$R = \sum_{1\le i,j\le N} q^{\delta_{ij}} e_{ii}\otimes e_{jj} + (q-q^{-1}) \sum_{1\le i<j\le N} e_{ji}\otimes e_{ij} \quad ,$$

where e_{ij} denotes a matrix unit and $0<q<1$. Then the quantized universal enveloping algebra $U_q(gl(N))$ is a $\mathbb{C}$-algebra generated by the symbols L_{ij}^+ and L_{ji}^- for $1\le i<j\le N$ with the relations

$$R_{12}L_2^\pm L_1^\pm = L_1^\pm L_2^\pm R_{12}, \quad R_{12}L_2^- L_1^+ = L_1^+ L_2^- R_{12}$$

and

$$L_{ii}^{+}L_{ii}^{-} = 1 = L_{ii}^{-}L_{ii}^{+} \quad ,$$

where $L_{\frac{1}{2}}^{\pm} = \sum e_{ss} \otimes e_{ij} \otimes L_{ij}^{\pm}$ etc. The Hopf algebra structure is given by

$$\Delta(L_{ij}^{\pm}) = \sum_{s} L_{is} \otimes L_{sj}, \quad \varepsilon(L_{ij}^{\pm}) = \delta_{ij} \quad \text{and} \ \sum_{s} L_{is}S(L_{sj}) = \delta_{ij}.$$

Then we have the relation

$$\begin{aligned} &q^{\delta_{i_1 j_2}} S(L_{i_2 j_2})L_{i_1 j_1} + (q-q^{-1})\delta_{i_1 j_2} \sum_{i_1 < \ell \leq N} S(L_{i_2 \ell})L_{\ell j_1} \\ &= q^{\delta_{i_2 j_1}} L_{i_1 j_1} S(L_{i_2 j_2}) + (q-q^{-1})\delta_{j_1 i_2} \sum_{1 \leq \ell < j_1} L_{i_1 \ell} S(L_{\ell j_2}) \quad , \end{aligned}$$

which will be used frequently in our argument. The element

$$C^{GL(N)} = \sum_{1 \leq i,j \leq N} q^{2(N-i)} L_{ij}^{+} S(L_{ji}^{-})$$

belongs to the center of $U_q(gl(N))$, which was introduced by Reshetikhin-Takhtajan-Faddeev in [5] and plays a crucial role in this talk.

Let $P = \oplus_{1 \leq i \leq N} \mathbb{Z}\epsilon_i$ be the weight lattice for $GL(N)$ with a symmetric bilinear form $<,>: P \times P \mapsto \mathbb{Z}$ such that $<\epsilon_i, \epsilon_j> = \delta_{ij} (1 \leq i,j \leq N)$, P^+ the set of dominant integral weights in P:

$$P^+ = \{\lambda = \sum_{1 \leq i \leq N} \lambda_i \epsilon_i \in P; \lambda_1 \geq \cdots \geq \lambda_N\}.$$

By the pairing, we identify P with its dual $P^* = Hom_{\mathbb{Z}}(P, \mathbb{Z})$ in what follows. It is noted that

$$q^h L_{ij}^{\pm} = q^{<h, \epsilon_j - \epsilon_i>} L_{ij}^{\pm} q^h \quad \text{for } h \in P^*$$

and

$$L_{ii}^{\pm} = q^{\pm \epsilon_i}.$$

We write by V_λ an irreducible highest weight module of highest weight $\lambda \in P$, $V_\lambda(\mu)$ its weight μ-space. Corresponding to the intertwiner

$$\Phi_\lambda^k : V_{\lambda+(k-1)\rho} \longrightarrow V_{\lambda+(k-1)\rho} \otimes V_{(1-k)(\epsilon_1+\cdots+\epsilon_N)+(k-1)N\epsilon_1} \quad ,$$

for λ, $\rho = \sum_{1 \leq i \leq N}(N-i)\epsilon_i \in P$ and $k \in \mathbb{Z}_{\geq 1}$, we consider the function

$$Tr_{V_{\lambda+(k-1)\rho}}(\Phi_\lambda^k q^h) \in V_{(1-k)(\epsilon_1+\cdots+\epsilon_N)+(k-1)N\epsilon_1} \quad ,$$

where $h = h_1\epsilon_1 + \cdots + h_N\epsilon_N \in P^*$. In what follows we set $q^{h_i} = z_i$.

Furthermore, to calculate the radial part of $C^{GL(N)}$ we realize an irreducible module

$$V_{(1-k)(\epsilon_1+\cdots+\epsilon_N)+(k-1)N\epsilon_1}$$

by the highest weight module W_k in

$$det_q^{1-k} \otimes \mathbb{C}[x_1, \ldots, x_N;\ qx_jx_i = x_ix_j\ (j > i)] :$$

which is studied well in [3]: W_k is generated by the highest weight vector $det_q^{1-k} \otimes x_1^{N(k-1)}$ with relations

$$\begin{aligned} &L_{ij}^+ x_s = \delta_{si} x_j (q - q^{-1}) \quad (i < j) \quad , \quad L_{ii}^+ x_s = x_s q^{\delta_{si}} \quad , \\ &L_{ij}^+ det_q = q\delta_{ij} det_q \end{aligned}$$

and

$$\begin{aligned} &S(L_{ji}^-) x_s = \delta_{sj} x_i (q - q^{-1}) \quad (i < j) \quad , \quad S(L_{ii}^-) x_s = x_s q^{\delta_{si}} \quad , \\ &S(L_{ji}^-) det_q = q\delta_{ij} det_q \quad . \end{aligned}$$

We also note that

$$L_{ij}^+(\phi_1\phi_2) = \sum_s (L_{is}^+(\phi_1))(L_{sj}^+(\phi_2))$$

and

$$S(L_{ji}^-)(\phi_1\phi_2) = \sum_s (S(L_{si}^-)(\phi_1))(S(L_{js}^-)(\phi_2))$$

for $\phi_1, \phi_2 \in W_k$. The corresponding N-variable function

$$\chi_\lambda(z) = Tr_{V_{\lambda+(k-1)\rho}}(\Phi_\lambda^k q^h)$$

taking the value on W_k is our object. When we denote the weight μ-space of W_k by $W_k[\mu] = \{\phi \in W_k;\ q^h.\phi = q^{<h,\mu>}\phi\}$, χ_λ takes the value only on $W_k[0]$. Then we state our theorem as follows:

Theorem 1. *When we put* $\varphi_\lambda^k = \chi_\lambda^k / \chi_0^k$,

$$\sum_{i=1}^N \prod_{\substack{1 \le j \le N \\ j \ne i}} \frac{q^{2k} z_i - z_j}{z_i - z_j} T_{q^2, z_i} \varphi_\lambda(z) = \left(\sum_{i=1}^N q^{2(k(N-i)+\lambda_i)}\right) \varphi_\lambda(z) \quad .$$

It should be remarked that the statement of Theorem, which implies that φ_λ^k is a constant multiple of Macdonald's polynomial, is not new and contained in [1] (See Theorem 4 and Theorem 5 in [1]). A new thing is a method of its derivation presented in the next section: Direct calculation of the radial part of the central element $C^{GL(N)}$.

References

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