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On Hardy-Littlewood-Schwartz space on Riemannian symmetric spaces

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1 Introduction

For a real number q ($2 \leq q < \infty$) and a Borel function f on $\mathbb{R}^n$ we put

$$\|f\|_{(q)} = \left(\int_{\mathbb{R}^n} |f(x)|^q |x|^{n(q-2)} dx \right)^{1/q}$$

and $J^q(\mathbb{R}^n)$ be the Banach space of all Borel functions f on $\mathbb{R}^n$ satisfying $\|f\|_{(q)} < \infty$. Hardy-Littlewood theorem([H-L]) says that if $f \in J^q(\mathbb{R}^n)$, then the Fourier transform $\tilde{f}$ of f is well-defined and there exists a constant $C_q > 0$ such that

$$\|\tilde{f}\|_q \leq C_q \|f\|_{(q)}.$$

On the other hand, if $1 \leq p \leq 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then the Fourier transform $\tilde{f}$ of $f \in L^p(\mathbb{R}^n)$ is well-defined and there exists a constant $B_p > 0$ such that

$$\|\tilde{f}\|_q \leq B_p \|f\|_p.$$

This is the Hausdorff-Young theorem. These two theorems suggest the resemblance between $L^p(\mathbb{R}^n)$ and $J^q(\mathbb{R}^n)$. In fact, if we put $f_\alpha(x) = 0(|x| \leq 1), = |x|^\alpha(|x| > 1)$ and $g_\beta(x) = |x|^\beta(|x| \leq 1), = 0(|x| > 1)$, then

$$f_\alpha \in L^p(\mathbb{R}^n) \Longleftrightarrow \alpha < -\frac{n}{p} \Longleftrightarrow f_\alpha \in J^q(\mathbb{R}^n)$$

and

$$g_\beta \in L^p(\mathbb{R}^n) \Longleftrightarrow \beta > -\frac{n}{p} \Longleftrightarrow g_\beta \in J^q(\mathbb{R}^n).$$

We have proved Hardy-Littlewood theorem and Hausdorff-Young theorem ([E-K1],[E-K2]) for spherical Fourier transform on Riemannian symmetric spaces G/K of noncompact type. The Euclidean space $\mathbb{R}^n$ is the symmetric space of the Euclidean motion group by the rotation group and is of rank one. The factor $|x|^n$ is a product of (distance from the origin)$^{\text{rank}}$ and the Jacobian with respect to

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX the polar decomposition. For a noncompact type symmetric space $X = G/K$ we denote by $\sigma(x)$ the distance of from the origin to x, by ℓ the rank of X and by $\Omega(x)$ the Jacobian with respect to the polar decomposition. Then there exists a constant $C_q > 0$ such that

$$\|\tilde{f}\|_q \leq C_q \left(\int_X |f(x)|^q \sigma(x)^{\ell(q-2)} \Omega(x)^{q-2} d\mu(x) \right)^{1/q},$$

for any K-biinvariant measurable function f on G whose value of integration in the right hand side is finite(Hardy-Littlewood theorem). There exists a constant $B_p > 0$ such that

$$\|\tilde{f}\|_q \leq B_q \|f\|_p,$$

for any K-biinvariant L^p function f on G(Hausdorff-Young theorem). We define $J^q(G)$ as the Banach space of all K-biinvariant measurable functions f on G satisfying $\|f\|_{(q)} < \infty$, where $\|f\|_{(q)}$ is defined by the right hand side of the Hardy-Littlewood inequality(see §3). Let $I^p(G) = L^p(K\backslash G/K)$ be the Banach space of K-biinvariant L^p-functions on G. If $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then it can be proved that the both spherical Fourier transforms of functions in $I^p(G)$ and $J^q(G)$ can be extended holomorphically to a certain tube domain([E-K1, Theorem 2],[E-K2, Theorem 2]).

The purpose of the present present paper is to point out more similarity between $I^p(G)$ and $J^q(G)$. There is an important dense subset of $I^p(G)$ which plays important roles in harmonic analysis. That is the Schwartz space $\mathcal{I}^p(G)$ of L^p type([T-V]). We define the Schwartz space of J^q type $\mathcal{J}^q(G)$(Hardy-Littlewood-Schwarts space) and investigte some properties of $\mathcal{J}^q(G)$. This is a Fréchet space and dense in $J^q(G)$. Furthermore, $\mathcal{I}^q(G)$ is contained in $\mathcal{J}^q(G)$. If $2 \leq q < 4$, then we can prove that $\mathcal{J}^q(G) = \mathcal{I}^p(G)$. And we prove that $\mathcal{J}^q(G) = \mathcal{I}^p(G)$ for all $q \geq 2$ if the rank of G/K is one.

2 Notations and Preliminaries

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$ with Cartan involution θ, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, and Σ the corresponding set of restricted roots. Let M' and M be the normalizer and the centralizer of $\mathfrak{a}$ in K, respectively, and we denote by $W = M'/M$ be the Weyl group of G/K and $|W|$ its order. Fix a Weyl chamber $\mathfrak{a}^+$ and put $A^+ = \exp \mathfrak{a}^+$. Let Σ^+ be the corresponding set of positive restricted roots and $|\Sigma^+|$ be its order. For $\alpha \in \Sigma^+$, $\mathfrak{g}_\alpha$ denotes the root subspace and $m_\alpha = \dim \mathfrak{g}_\alpha$ the multiplicity of α. Let $\mathfrak{n} = \sum_{\Sigma^+} \mathfrak{g}_\alpha$ and $\rho = \dfrac{1}{2} \sum_{\Sigma^+} m_\alpha \alpha$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. We denote by $G = KAN$ the corresponding decomposition of G. For $x \in G$, $H(x) \in \mathfrak{a}$ denotes the element uniquely determined by $x \in K\exp(H(x))N$. For $a \in A$, we write $\log a$ for $H(a)$.

Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$ and $\mathfrak{a}_{\mathbb{C}}^*$ its complexification. We denote by $< \ , \ >$ the Killing form of $\mathfrak{g}$. For $\lambda \in \mathfrak{a}^*$, let $H_\lambda \in \mathfrak{a}$ be the unique element determined by $\lambda(H) = < H_\lambda, H >$ for all $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$, we put $< \lambda, \mu > = < H_\lambda, H_\mu >$ and $|\lambda| = < \lambda, \lambda >^{1/2}$. Let $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$ and $\overline{N}$ denote the corresponding analytic subgroup of G. For $\epsilon > 0$ we put $C_{\epsilon\rho} = [w(\epsilon\rho); w \in W]$, the convex hull of the set $\{w(\epsilon\rho); w \in W\}$. For $0 < p < 2$ we define the tube domain T_p by $T_p = \mathfrak{a}^* + \sqrt{-1}C_{(2/p-1)\rho}$.

We denote by $C_c^\infty(G)$ the space of all compactly supported C^∞-functions on G and $C_c^\infty(G/K)$ and $C_c^\infty(K\backslash G/K)$ be the subspaces of $C_c^\infty(G)$ of right K-invariant and K-biinvariant functions, respectively. The Killing form induces euclidean measures on A and $\mathfrak{a}^*$. We normalize them by multiplying the factor $(2\pi)^{-\ell/2}$ and denote them by da and $d\lambda$, respectively, where $\ell = \dim\mathfrak{a}$, the rank of G/K. Let dk be the normalized Haar measure on K so that total measure is one. The Haar measures on N and $\overline{N}$ are normalized so that

$$\theta(dn) = d\overline{n}, \qquad \int_{\overline{N}} e^{-2\rho(H(\overline{n}))} d\overline{n} = 1.$$

And we normalize the Haar measure dx on G so that

$$\int_G f(x)dx = \int_{KAN} f(kan)e^{2\rho(\log a)}dkdadn, \qquad f \in C_c^\infty(G).$$

We denote by $\mathrm{vol}(K/M)$ the volume of K/M with respect to the K-invariant measure $d\mu(b)$ induced from the restriction of $- < \ , \ >$ to $\mathfrak{k}$. Let dk_M be the K invariant measure on K/M defined by $dk_M = \mathrm{vol}(K/M)^{-1}d\mu(k_M)$.

The following integral formula corresponds to the Cartan decomposition $G = KAK$.

$$\int_G f(x)dx = \frac{(2\pi)^{\ell/2}\mathrm{vol}(K/M)}{|W|} \int_{\mathfrak{a}} \prod_{\alpha\in\Sigma^+} |\sinh\alpha(H)|^{m(\alpha)}dH$$
$$\times \iint_{K\times K} f(k_1\exp(H)k_2)dk_1dk_2, \qquad f \in C_c^\infty(G).$$

We put

$$\Omega(\exp H) = \frac{(2\pi)^{\ell/2}\mathrm{vol}(K/M)}{|W|}\Pi_{\alpha\in\Sigma^+}|\sinh\alpha(H)|^{m(\alpha)}, \qquad H \in \mathfrak{a}.$$

By the W-invariance of $\Omega(a)(a \in A)$ we can extend it to G by $\Omega(x) = \Omega(a)$ for $x = k_1ak_2$, $k_1, k_2 \in K$, $a \in A$.

We put $\sigma(x) = \sqrt{< X, X >}$ for $x = k \exp X$, $k \in K$, $X \in \mathfrak{p}$.

3 Schwartz Space of L^p type

Let $I^p(G)$ be the Banach space of all K-biinvariant measurable function f on G such that

$$\|f\|_p = \left(\int_G |f(x)|^p dx \right)^{1/p} < \infty.$$

Of course, we identify two functions which differ only on a set of measure zero. Let

$$\varphi_\lambda(x) = \int_K e^{(\sqrt{-1}\lambda - \rho)(H(xk))} dk, \qquad x \in G,$$

be the elementary spherical function. Then φ_λ is bounded if and only if $\lambda \in T_1$. We put $\Xi = \varphi_0$. We denote the Harish-Chandra c-function is defined by

$$c(\lambda) = \int_{\overline{N}} e^{(-\sqrt{-1}\lambda + \rho)(H(\overline{n}))} d\overline{n}.$$

We define the spherical Fourier transform $\tilde{f}$ of $f \in I^1(G)$ by

$$\tilde{f}(\lambda) = \int_G f(x) \varphi_{-\lambda}(x) dx, \qquad \lambda \in \mathfrak{a}^*.$$

Let $L^2(\mathfrak{a}^*, \frac{1}{|W||c(\lambda)|^2} d\lambda)^W$ be the Hilbert space of W-invariant square integrable functions on $\mathfrak{a}^*$ with respect to the measure $\frac{1}{|W||c(\lambda)|^2} d\lambda$. Then the Plancherel theorem can be stated as follows(e.g.[W], p.338).

Lemma 1. *For $f \in I^1(G) \cap I^2(G)$, we have*

$$\|f\|_2 = \left(\frac{1}{|W|} \int_{\mathfrak{a}^*} |\tilde{f}(\lambda)| \frac{1}{|c(\lambda)|^2} d\lambda \right)^{1/2}.$$

And the map $f \longmapsto \tilde{f}$ can be extended to an isometry of $I^2(G)$ onto $L^2(\mathfrak{a}^*, \frac{1}{|W||c(\lambda)|^2} d\lambda)^W$.

The following is the Hausdorff-Young theorem([E-K1]).

Lemma 2. *Let $1 \leq p < 2$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. The spherical Fourier transform can be defined for the functions of $I^p(G)$ and, for each $f \in I^p(G)$, the spherical Fourier transform $\tilde{f}$ can be extended to a holomorphic function on IntTp and, for any $\eta \in \mathrm{Int}C_{(2/p-1)\rho}$, there exists a constant $B_{p,\eta} > 0$ such that*

$$\left(\frac{1}{|W|}\int_{\mathfrak{a}^*}|\tilde{f}(\xi+\sqrt{-1}\eta)|^q|c(\xi)|^{-2}d\xi\right)^{1/q} \leq B_{p,\eta}\|f\|_p, \qquad f \in I^p(G).$$

Let $U(\mathfrak{g}_C)$ be the universal enveloping algebra of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. Let $p > 0$. We denote by $\mathcal{I}^p(G)$ the space of all $f \in C^\infty(K\backslash G/K)$ such that for any $u \in U(\mathfrak{g}_C)$ and any integer $m \geq 0$,

$$\mu^p_{u,m}(f) = \sup_{x \in G}(1+\sigma(x))^m|(uf)(x)|\Xi(x)^{-2/p} < \infty.$$

Then $\mathcal{I}^p(G)$ is a Frécht space by the system of seminorms $\{\mu^p_{u,m}\}$ and is dense in $I^p(G)$([T-V]).

Let $S(\mathfrak{a}^*_C)$ be the symmetric algebra over $\mathfrak{a}^*_C$ and for $s \in S(\mathfrak{a}^*_C)$ we denote by $\partial(s)$ the corresponding differential operator on $\mathfrak{a}^*_C$. Let $0 < p < 2$. We define the space $\overline{\mathcal{Z}}(T_p)$ to be the set of all W-invariant holomorphic function F on $\mathrm{Int}T_p$ such that for any $s \in S(\mathfrak{a}^*_C)$ and any integer $m \geq 0$,

$$\zeta^p_{s,m}(F) = \sup_{\lambda \in \mathrm{Int}T_p}(1+|\lambda|^2)^m|(\partial(s)F)(\lambda)| < \infty.$$

Then the following important theorem is due to Trombi-Varadarajan.

Lemma 3([T-V]). *Let $0 < p < 2$. Then, for $f \in \mathcal{I}^p(G)$, the integral $\tilde{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx$ converges absolutely for all $\lambda \in T_p$. The function $\tilde{f}$ lies in $\overline{\mathcal{Z}}(T_p)$ and the spherical Foutier transform $f \longmapsto \tilde{f}$ is a linear topological isomorphism of $\mathcal{I}^p(G)$ onto $\overline{\mathcal{Z}}(T_p)$.*

4 Hardy-Littlewood-Schwartz Space

For $q \geq 2$ we define the Banach space $J^q(G)$ of all K-biinvariant measurable function f on G such that

$$\|f\|_{(q)} = \left(\int_G |f(x)|^q\sigma(x)^{l(q-2)}\Omega^{q-2}dx\right)^{1/q} < \infty.$$

Then the next Hardy-Littlewood theorem holds([E-K2]).

Lemma 4. *Let $2 \leq q < \infty$. Then the spherical Fourier transform can be defined for $f \in J^q(G)$ and there exists a constant $C_q > 0$, independent of f, such that*

$$\left(\frac{1}{|W|}\int_{\mathfrak{a}^*} |\tilde{f}(\lambda)|^q |c(\lambda)|^{-2} d\lambda\right)^{1/q} \leq C_q \|f\|_{(q)}.$$

We denote by $\mathcal{J}^q(G)$ the set of all $f \in C^\infty(K\backslash G/K)$ such that for any $u \in U(\mathfrak{a}_C^*)$ and any integer $m \geq 0$,

$$\nu_{u,m}^q(f) = \sup_{x \in G}(1+\sigma(x))^m |(uf)(x)| \sigma(x)^{\ell(1-2/q)} \Omega(x)^{1-2/q}(x) \Xi(x)^{-2/q} < \infty.$$

Then $\mathcal{J}^q(G)$ is a Fréchet space by the system of the seminorms $\{\nu_{u,m}^q\}$. We call $\mathcal{J}^q(G)$ the Hardy-Littlewood-Schwartz space on G/K.

5 Some Inclusion Properties

The following estimate of (1) is an immediate consequence of the definition of $\Omega(x)$. The statement (2) is due to Harish-Chandra[Ha1, Theorem 3].

Lemma 5. *(1) We put $c_1 = 2^{-|\Sigma^+|}(2\pi)^{\ell/2} vol(K/M)|W|^{-1}$. Then*

$$\Omega(a) \leq c_1 e^{2\rho(\log a)} \quad a \in A^+,$$
$$\Omega(a) \sim c_1 e^{2\rho(\log a)} \quad a \in A^+ \quad \text{and} \quad a \to \infty.$$

(2) There exist constants $c_2 > 0$ and $d > 0$ such that

$$1 \leq e^{\rho(\log a)} \Xi(a) \leq c_2 (1 + |\log a|)^d \quad a \in A^+.$$

By using the estimtes in Lemma 5, we can prove the following theorem.

Theorem 1. *Let $1 < p < 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then we have*

$$\mathcal{I}^p(G) \subset \mathcal{J}^q(G) \subset I^p(G) \cap J^q(G)$$

and each inclusion map is continuous.

We can get the following lemma by the same way as that of Harish-Chandra ([Ha2]Lemma20).

Lemma 6. *The space $C_c^\infty(K\backslash G/K)$ is dense in $\mathcal{J}^q(G)$.*

If we use Lemma 5 again, we have the following theorem.

Theorem 2. *Let $q > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\frac{1}{p} - \frac{1}{q} < \frac{1}{r} \leq \frac{1}{p}$, then*

$$\mathcal{J}^q(G) \subset I^r(G).$$

And the inclusion map is continuous.

If we choose $r = 2$, then we have the following corollary.

Corollary. *If $2 \leq q < 4$, then $\mathcal{J}^q(G) \subset I^2(G)$ and the inclusion map is continuous.*

The condition $\frac{1}{p} - \frac{1}{q} < \frac{1}{r}$ in Theorem 3 is necessary for the regularity of the function $\Omega(a)^{r(2/q-1)+1}$ on the walls of the Weyl chamber A^+ except for the origin. Hence if the rank $\ell = 1$, Theorem 3 holds for $r \geq p$ and Corollary does for $q \geq 2$. In fact, we have the following proposition.

Proposition. *We assume that the rank ℓ of G/K is one. Let $q \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have $\mathcal{J}^q(G) \subset I^r(G)$ for all $r \geq p$ and, especially, $\mathcal{J}^q(G) \subset I^2(G)$.*

6 Fourier Transforms of $\mathcal{J}^q(G)$

Lemma 7. *Let φ be a measurable function on G such that there exist a constant $C > 0$ and an integer $m \leq 0$ satisfying*

$$|\varphi(x)| \leq C\Xi(x)^{2/q}(1 + \sigma(x))^m \qquad (x \in G). \tag{4.1}$$

Then

$$L(f) = \int_G (uf)(x)\varphi(x)dx$$

converges absolutely for all $f \in \mathcal{J}^q(G)$ and $u \in U(\mathfrak{g}_C)$, and L is a continuous linear functional on $\mathcal{J}^q(G)$. If φ and $u^*\varphi$ satisfy the inequality of the same type of (4.1), then

$$\int_G uf \cdot \varphi dx = \int_G f \cdot u^*\varphi dx,$$

where u^* is the adjoint differential operator of u.

Let $Z_K(U(\mathfrak{g}_C))$ the centralizer of K in $U(\mathfrak{g}_C)$. For any $u \in U(\mathfrak{g}_C)$ we can find a unique element $a_u \in U(\mathfrak{a}_C)$ such that $u - a_u \in \mathfrak{k}U(\mathfrak{g}_C) + U(\mathfrak{g}_C)\mathfrak{n}$. For any $z \in Z_K(U(\mathfrak{g}_C))$, we put $\tau(z) = e^{\rho} \circ a_z \circ e^{-\rho}$. Then $\tau(z) \in U(\mathfrak{a}_C)$. The following lemma is due to Trombi-Varadarajan([T-V], Lemma 3.5.3)

Lemma 8. *Let $s \in S(\mathfrak{a}_{\mathbf{C}}^*)$ and $d_s = deg(s)$. Then, if $z \in Z_K(U(\mathfrak{g}_{\mathbf{C}}))$, $(z - \tau(z)(\lambda))^{d_s+1}\partial(s)(\varphi_\lambda(x)) = 0$ for all $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ and $x \in G$. Furthermore, given $u \in U(\mathfrak{g}_{\mathbf{C}})$, there exist constants $c_{u,s} > 0$ and $m_{u,s} \geq 0$ such that for all $x \in G$, $\lambda \in T_p$,*

$$|\partial(s)u\varphi_\lambda(x)| \leq c_{u,s}\{(1+|\lambda|)(1+\sigma(x)\}^{m_{u,s}}\Xi(x)^{2(1-1/p)}.$$

By Lemma 7 and 8 we can prove the following Lemma 3 using the method by Trombi and Varadarajan([T-V]Theorem 3.5.5).

Theorem 3. *Let $1 < p < 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $f \in \mathcal{J}^q(G)$, then the integral*

$$\tilde{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx$$

converges absolutely for any $\lambda \in T_p$. And the map $f \longmapsto \tilde{f}$ is a continuous map of $\mathcal{J}^q(G)$ to $\overline{\mathcal{Z}}(T_p)$.

7 Coincidence Theorem

By Lemma 1, Lemma 3, Corollary of Theorem 2, Theorem 3 and Proposition we can conclude the following statement.

Theorem 4. *If $2 \leq q < 4$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then $\mathcal{J}^q(G) = \mathcal{I}^p(G)$. If the rank of G/K is one, then $\mathcal{J}^q(G) = \mathcal{I}^p(G)$ for all $q \geq 2$.*

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