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The orbit and θ correspondence for some dual pairs

by

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§ 0 Introduction. — Since the fundamental paper of A.Weil [W] on the construction of the θ representation for the two-fold covering of Sp_n (symplectic group), there have been several approaches to extend Weil's basic construction to a more general context. The issue at hand is which of the many important properties of the θ representation are to be generalized and in which direction the generalization will go. For instance one possibility is to find analogues of θ for higher metaplectic coverings of GL_n [K-P]. Here the point is to find an "automorphic module" which comes from the residual spectrum and to determine certain uniqueness properties about certain types of Fourier coefficients of elements in this module. Another possible direction of generalizing [W] is to determine for a reductive group (other than Sp_n) the automorphic modules which have "smallest" Fourier coefficients. In particular this means that at least one of the local components of the automorphic module has smallest Gelfand-Kirillov dimension. It is in this direction that we are concerned in this talk.

Let G be a semi-simple Lie group defined over some local field . If A and B are two subgroups of G we say that (A, B) is a dual pair if A is the commutant of B in G and B the commutant of A. A unitary unramified irreducible representation π of G is called minimal if it is associated to a coadjoint orbit of minimal dimension. If we restrict π to $A \times B$ we may ask whether we get a Howe type correspondance between suitable subsets of the admissible duals of A and B. For a finite prime v and for the class of unramified representations (those admitting a non zero fixed vector under the maximal compact subgroup) the Howe conjecture takes a more precise form. Specifically if $\mathcal{H}_{A_v}$ and $\mathcal{H}_{B_v}$ are the spherical Hecke algebras of A_v and B_v, we want a homomorphism

$$\psi_v : \mathcal{H}_{A_v} \longrightarrow \mathcal{H}_{B_v} \qquad \operatorname{rank}(A_v) \geq \operatorname{rank}(B_v)$$

with the property that for all $z_v \in \mathcal{H}_{A_v}$

$$\pi_{sm}(z_v) = \pi_{sm}(\psi_v(z_v))$$

(*) For the main part, this talk is a report on a joint work with S.Rallis where π_{sm} is the smooth version of π. For the infinite primes the corresponding conjecture is that there exists a homomorphism

$$\psi_\infty : \mathcal{Z}_{A_\infty} \longrightarrow \mathcal{Z}_{B_\infty}$$

between the centers of the enveloping algebras of A_v and B_v so that

$$\pi_{sm}(z_\infty) = \pi_{sm}(\psi_\infty(z_\infty))$$

for all $z_\infty \in \mathcal{Z}_{A_\infty}$. For the symplectic case this was proved, a long time ago, by R.Howe himself [Ho].

It is this very precise form of the Howe type conjecture that we investigate, in our joint work with S.Rallis [R-S] but only for the infinite primes. We expect that if there is a construction of the collapsing ψ_∞ above then a corresponding construction of ψ_v exists for finite v. The evidence comes from several dual pairs analyzed in [G-R-S] and from [K].

§ **1 The minimal representation.** — Let us briefly review what seems to be known about our proposed set-up. Assume G to be simple. If G is not of type A_n then there is a unique coadjoint orbit of minimal dimension; it is a nilpotent one. In the A_n case there is also a one parameter familly of semi-simple coadjoint orbits of minimal dimension. We exclude the A_n-case once for all. Consider the minimal nilpotent orbit. A minimal representation is a unipotent representation associated to this minimal orbit.

In the archimedean case, D.Vogan [V] proved the existence and the unicity of the minimal (spherical) representation, except in the B_n, $n \geq 4$ situation where no such representation seems to exist. He does not describe any explicit construction of this minimal representation. More recently several authors ([K-S], [B-K], [G-W],[T] have given, for suitable classes of real semi-simple Lie groups, explicit models. One key fact is that in many cases, one knows exactly the K-types occuring in this representation. We will not describe these constructions here because the method we follow in [R-S] does not make use of any model. Rather we shall describe the construction of Kazhdan and Savin ([K-S]) which works over any local fied and which looks very much like the construction of Joseph's ideal (see below).

We suppose that G is an adjoint Chevalley group, over an arbitrary local field of characteristic 0 (to be safe...). Suppose furthermore that the root system of G is irreducible of type D_n, E_6, E_7 or E_8 (non A_n simply laced case). Fix a maximal split torus H and let Δ be the root system. Choose a basis Σ of this system. Let β be the highest root; there is a unique simple root α which is not orthogonal to β. Let P be the corresponding maximal parabolic subgroup. Then $P = TMU$ where T is a one dimensional torus, M the semi-simple part of the Levi and U the unipotent radical.

Let $\mathfrak{u}$ be the Lie algebra of U. Then $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$. Here $\mathfrak{u}_2$ is the root space corresponding to β which is the unique root "containing" α twice. The subspace $\mathfrak{u}_1$ is the direct sum of the root subspaces corresponding to the roots γ "containing" α exactly once. If we fix some non zero element X_β in $\mathfrak{u}_2 = \mathfrak{g}^\beta$ then we define a bilinear, antisymmetric form A on $\mathfrak{u}_1$ by

$$[X,Y] = A(X,Y)X_\beta$$

It is easy to check that A is non degenerate so that $\mathfrak{u}$ is an Heisenberg Lie algebra and U an Heisenberg group. Also M leaves the bilinear form A invariant so that we can imbed M into the symplectic group $\mathrm{Sp}(\mathrm{U}_1)$. Finally note that for a root γ as above, $\beta - \gamma$ is a root and it is easy to get natural polarizations from this remark.

Following $K - S$ we fix a non trivial continuous additive character of the local field. Using the Stone Von-Neumann theorem this gives an infinite dimensional representation of the Heisenberg group U. As in W extends it to the metaplectic group. It is easyly proved that the metaplectic representation splits over M, viewed as a subgroup of the symplectic group, so that we have a representation of MU; then we induce this representation to P. Call τ this representation. It is not difficult to show that τ is a (unitary) irreducible representation of P.

Let s_α be a representant in G of the reflection with respect to the simple root α. Also let B be the Borel subgroup corresponding to our choice of simple roots. Then if $B_\alpha = B \cap s_\alpha B s_\alpha$ one can check that the restriction of τ to B_α remains irreducible and also that the representation τ' obtained by twisting τ by the outer automorphism $\mathrm{Int}\, \mathrm{s}_\alpha$ is equivalent to τ. Let C be an intertwinning map. The main result of $K - S$ is the following theorem.

Theorem 1 ([K-S]). — *One can normalize C so that there exists a unique irreducible unitary representation π of G, which extends τ and satisfies $\pi(s_\alpha) = C$*

Of course uniqueness is clear because P and s_α generates G. To prove existence one has to take care of the so called braid relations involving s_α. This is where the assumption simply laced is important. Later Savin [S 1] managed to extend the theorem to G_2, using a global method. However for G_2, one gets a representation of the 3-fold covering of G. Assuming that, over any local field, there is no minimal representation in the B_n case, there remains the F_4 case. It could perhaps be done along the same line as G_2.

Among several interesting results, Savin [S 2] proved that π is indeed a minimal representation by computing the Gelfand-Kirillov dimension or proving that the wave front set is contained in the closure of the minimal orbit. Finally π is a spherical representation.

Unfortunately there does not seems to exist a nice model like in the classical symplectic case. In particular the vectors invariant under the maximal compact subgroup are not easy to control. Clearly more work is needed but the already known results over archimedean or p-adic fields are enough to test many potential cases of Howe correspondance

For dual pairs we restrict ourselves to the case where both groups are semi-simple. There is an obvious notion of dual pair of semi-simple subalgebras of the Lie algebra of G. Over $\mathbf{C}$ such pairs have just been classified by H.Rubenthaler [R]. At the present stage we may content ourselves with the split case so we do get a lot of examples. Some of them, notably in the exceptional cases, seem so weird that it is hard to believe that a Howe correspondance will always exists. On the other hand, one finds in [R] large famillies of dual pairs built in a uniform and conceptual way. Also [R] exhibits towers of dual pairs with many see-saws.

§ 2. Examples of local Howe correspondance. — The goal of [R-S] is to present evidence that positive answers exist for the precise Howe conjecture stated above, at least for some dual pairs in the archimedean case. We shall work with "direct sums" situations.

We start with a complex simple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ and a basis Σ of the root system Δ. Wexclude the A_n-case.

Consider the extended Dynkin diagram and remove one of the simple roots. In general we are left with two connected components each corresponding to a simple subalgebra. By an old result of Dynkin this is a dual pair. These are very elementary examples. In fact they even do not appear explicitly in [R] where some irreducibility condition is imposed.

As before we call α the root connected to the highest root β and δ the simple root connected to α (for the orthogonal case we take α_3). We shall remove either α or δ. Call $\mathfrak{a}(\alpha) \times \mathfrak{b}(\alpha)$ and $\mathfrak{a}(\delta) \times \mathfrak{b}(\delta)$ those two dual pairs; we suppose that the highest root "belongs" to the $\mathfrak{a}$ subgalgebra. Then in the α case the subalgebra is of type A_1, in the $\delta-$case of type A_2 or A_3.

To check if we can expect a local correspondence we shall start from a remark of [K-K-S]. As a particular case of their work the authors point out that, in the symplectic case, by projecting the minimal coadjoint orbit onto the dual of $\mathfrak{a} \times \mathfrak{b}$ one obtains a correspondance between coadjoint orbits of $\mathfrak{a}$ and $\mathfrak{b}$, at least generically. Thus one can predict the existence of a correspondance of Howe type. The symplectic case was investigated further by J.D.Adams [A]. Our first result is that a such generic correspondance appears in the two above cases.

For each root σ, let H_σ be the coroot and choose a root vector $X_\sigma \in \mathfrak{g}^\sigma$ such that $[X_{-\sigma}, X_\sigma] = H_\sigma$

Let $H \in \mathfrak{h}$ be defined by $\alpha(H) = 1$ and $\sigma(H) = 0$ for $\sigma \in \Sigma - \{\alpha\}$ As we said before, relative to the basis Σ , β is the only root with coordinate 2 It follows that the eigenvalues of $\mathrm{ad}(H)$ are $\{-2, -1, 0, +1, +2\}$ and we obtain a graduation

$$\mathfrak{g} = \bigoplus_{-2}^{+2} \mathfrak{g}_i$$

where $\mathfrak{g}_i$ is the eigenspace for the eigenvalue i. Furthermore

$$\mathfrak{g}_2 = \mathbf{C}X_\beta \qquad \mathfrak{g}_{-2} = \mathbf{C}X_{-\beta}$$

Let Δ_i be the set of roots γ such that $|\gamma|_\alpha = i$; thus $\Delta_2 = \{\beta\}$. The subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is maximal parabolic with Levi component $\mathfrak{g}_0$. The unipotent radical $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is of Heisenberg type.

Let $\mathfrak{m} = [\mathfrak{g}_0, \mathfrak{g}_0]$ be the semi-simple part of the Levi component $\mathfrak{g}_0$. Then $\mathfrak{g}_0 = \mathbf{C}H \oplus \mathfrak{m}$. Clearly f is 0 on $\mathfrak{m}$ and $f(H) = 2$.

Let G be the adjoint group of $\mathfrak{g}$ and G_0 the centralizer of H in G. The one dimensional subspace $\mathfrak{g}_2$ is invariant under G_0. Let χ be the character of G_0 given by

$$gX_\beta = \chi(g)X_\beta$$

One checks that $[X_{-\beta}, X_\beta] = H$ so

$$gX_{-\beta} = \chi^{-1}(g)X_{-\beta}$$

Let M be the kernel of χ; it is a semi-simple group with Lie algebra $\mathfrak{m}$. Furthermore M commutes with $\mathrm{Exp}(\mathrm{ad}(\mathfrak{g}_{\pm 2}))$

The group G_0 acts on $\mathfrak{g}_1$ and, by a theorem of Vinberg, has a Zariski open orbit : $\mathfrak{g}_1$ is a prehomogeneous vector space. The same is true for $\mathfrak{g}_{-1}$ which,via the Killing form, we view as the dual space of $\mathfrak{g}_1$. In this context the prehomogeneous space is regular if and only if there exists $Y^+ \in \mathfrak{g}_1$ and $Y^- \in \mathfrak{g}_{-1}$ such that (Y^-, H, Y^+) is an SL(2)-triplet.

Lemma. — *The prehomogeneous spaces $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are regular if and only if Δ is not of type C_ℓ. Furthermore $(X_{-\alpha}+X_{-(\beta-\alpha)}\,,H,\,X_\alpha+X_{\beta-\alpha})$ is an $SL(2)$-triplet.*

Until the end of the section assume that Δ is not of type C_ℓ So $Y^- = X_{-\alpha}+X_{-\beta+\alpha}$ is a generic element of $\mathfrak{g}_{-1}$.
We identify $\mathfrak{g}$ with its dual using the Killing form. The minimal coadjoint orbit is

$$\Omega(\mathfrak{g}) = GX_\beta$$

We remove the root α. Then $\mathfrak{b}(\alpha)$ is simply the semi-simple part $\mathfrak{m}$ of the Levi subalgebra $\mathfrak{g}_0$ and

$$\mathfrak{a}(\alpha) = \mathbf{C}X_{-\beta} \oplus \mathbf{C}H \oplus \mathbf{C}X_\beta$$

is of type A_1. The subgroup $A(\alpha)$ is defined as the subgroup of G generated by $\mathrm{Exp}(\mathbf{C}\mathrm{ad}(X_{\pm\beta}))$ and the subgroup $B(\alpha) = M$ as already been defined. Note that M and $A(\alpha)$ commute. We want to find the "generic" orbits of the adjoint action of $A(\alpha)\times B(\alpha)$ on Ω . By generic we mean that we look only at a non empty Zariski open subset of Ω. Let $Z = \mathrm{Exp}(\mathrm{ad}(Y^-))X_\beta$. Then using the above lemma and the Bruhat decomposition for the maximal parabolic subgroup corresponding to α, it is easy to check that

$$(2-1) \qquad \Omega \approx \mathbf{C}^* A(\alpha) M \mathrm{Exp}(\mathrm{ad}(Y^-))X_\beta$$

Before we proceed let us make our goal more specific. If $B_\mathfrak{g}$ is the Killing form of $\mathfrak{g}$ and if $Z \in \mathfrak{g}$ then it defines the linear form $Y \mapsto B_\mathfrak{g}(Z,Y)$. For any semi-simple subalgebra $\mathfrak{s}$ with Killing form $B_\mathfrak{s}$ there is a unique element $p_{\mathfrak{g}/\mathfrak{s}} \in \mathfrak{s}$ such that, for all $Y \in \mathfrak{s}$ we have

$$B_\mathfrak{g}(Z,Y) = B_\mathfrak{s}(p_{\mathfrak{g}/\mathfrak{s}}(Z),Y)$$

Note that the projection $p_{\mathfrak{g}/\mathfrak{s}}$ commutes with the action of $\mathfrak{s}$ and more generally with the action of the normalizer of $\mathfrak{s}$ in G.
Going back to formula (2-1) we put

$$Z = \mathrm{Exp}(\mathrm{ad}(Y^-))X_\beta$$

and, for $t \neq 0$ project tZ on $\mathfrak{a}(\alpha)$ and $\mathfrak{b}(\alpha) = \mathfrak{m}$. We get

$$p_{\mathfrak{g}/\mathfrak{a}(\alpha)}(\Omega) \approx \bigcup_{t \in \mathbf{C}^*} tA(\alpha)p_{\mathfrak{g}/\mathfrak{a}(\alpha)}(Z)$$
$$p_{\mathfrak{g}/\mathfrak{m}}(\Omega) \approx \bigcup_{t \in \mathbf{C}^*} tMp_{\mathfrak{g}/\mathfrak{m}}(Z)$$

We have a correspondance between the (co)-adjoint orbits :

$$tA(\alpha)p_{\mathfrak{g}/\mathfrak{a}(\alpha)}(Z) \qquad \text{and} \qquad tMp_{\mathfrak{g}/\mathfrak{m}}(Z)$$

We can make it explicit.
Let $C_1, \ldots, C_r$ be the connected components of $\Sigma - \{\alpha\}$. With obvious notations

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r \qquad \mathfrak{h} \cap \mathfrak{m} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r.$$

For each i, there is a unique simple root $\delta_i \in C_i$ such that $\langle \alpha, \delta_i \rangle \neq 0$. Define the weight $\varpi_i \in \mathfrak{h}_i^*$ by $\varpi_i(H_{\delta_i}) = 1$ and $\varpi_i(H_\gamma) = 0$ for $\gamma \in C_i - \{\delta_i\}$. On $\mathfrak{a}(\alpha)$ let ϖ the unique fundamental weight. For this case the final result is

Theorem [R-S] (Δ of type B_ℓ with $\ell \geq 3$ or D_ℓ with $\ell \geq 4$ or exceptionnal) *In $\mathfrak{a}(\alpha)$ the coadjoint orbits of $t\varpi$ and $t'\varpi$ coincide if and only if $t = \pm t'$. In $\mathfrak{m} = b(\alpha)$ the coadjoint orbits of $t \sum n(\alpha, \delta_i)\varpi_i$ and $t' \sum n(\alpha, \delta_i)\varpi_i$ coincide if and only if $t = \pm t'$.*
Generically, the projection of the minimal orbitin the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ to the dual $\mathfrak{a}(\alpha)^* \times \mathfrak{m}^*$ of $\mathfrak{a}(\alpha) \times \mathfrak{m}$ induces a correspondance between the coadjoint orbits of

$$t\varpi \qquad \text{and} \qquad \sum tn(\alpha, \delta_i)\varpi_i$$

If we remove the δ root then we get a similar theorem. The proof which is more involved but still elementary goes trough a reduction to a subalgebra of type D_4 where explicit computations can be done. Although we have not checked all the cases we believe that all direct sum dual pairs could be treated along similar lines. In the C_n case it is trivial (but useful) that, for direct sums, the restriction of the oscillator representation is the tensor product of the oscillator representations of A and B which are smaller symplectic groups. This is very atypical. Let us also mention that for A_n, starting with a coadjoint orbit of minimal codimension and any direct sum dual pair (so we just imbed in an obvious way $A_p \times A_q$ into A_{p+q+1}), it is a simple exercice to show the existence of a correspondance;

generically only minimal representations of A and B will occur. Note also that the orthogonal cases (and direct sums) could be worked out along the same lines, using Witt's theorem. Also we tested examples of "tensor product" type and in particular obtained an orbital correspondance for the $G_2 \times A_1$ pair in F_4. On the contrary, it is unclear whether all "tensor product " pairs for the orthogonal groups will give rise to correspondances. Strangely enough, for the problems at hand, the exceptional cases seem to be much more well behaved.
The conclusion is that the orbit method gives a rather easy way to predict Howe's type correspondence. Unfortunately, at least for the present time, there it is not known how to prove the collapsing of the centers from the orbital correspondance.

§ 3 The collapsing of the centers. — Next, as explained above, we want to find a map Ψ between the centers of the enveloping algebras of $\mathfrak{a}$ and $\mathfrak{b}$ such that $\Psi(Z) - Z$ belongs to the kernel of the minimal representation. We do get such a Ψ, in a completely explicit way, for our two examples. The key point is that the kernel of π is under control, thanks to a paper of A.Joseph [J]. Let us recall the main result of [J].
We keep the notations of the beginning of § 2; in particular $\mathfrak{g}$ is not of type A_ℓ but, for the present time at least the case C_ℓ is not excluded. We consider the Heisenberg subalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and also the subalgebra

$$\mathfrak{r} = \mathbf{C}H \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

which is the image of $\mathrm{ad}(X_\beta)$ and may be identified with the tangent space to the minimal orbit Ω at the point X_β. Following Joseph we put $E = X_\beta$ and introduce the localization

$$\mathcal{A} = \mathcal{U}(\mathfrak{r})_E$$

in E of the envelopping algebra $\mathcal{U}(\mathfrak{r})$ of $\mathfrak{r}$.

Theorem (A.Joseph [J]). — *There exists a unique algebra homomorphism Φ of the envelopping algebra $\mathcal{U}(\mathfrak{g})$ into $\mathcal{A}$ which is the identity on $\mathfrak{r}$*

The kernel of this map is the unique completely prime two-sided ideal whose characteristic variety is the closure of the minimal orbit J. It is called Joseph's ideal.
Still following [J], consider the Heisenberg subalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Suppose that we split Δ_1 as a union of two disjoint subsets Γ_1 and Γ_2 such that $\gamma \in \Gamma_1$ implies that $\beta - \gamma \in \Gamma_2$. Let

$$V_1 = \oplus \mathfrak{g}^\gamma \qquad \text{for} \quad \gamma \in \Gamma_1$$

and put

$$\mathcal{S} = \bigoplus_{r \in \mathbf{Q}} S(V_1)E^r$$

We define a representation π of $\mathfrak{r}$ in $\mathcal{S}$ by

$$\begin{aligned}
\pi(X_\gamma) &= \text{multiplication by } X_\gamma \quad \text{if } \gamma \in \Gamma_1 \\
\pi(F_\gamma) &= -E\frac{\partial}{\partial X_\gamma} \quad \text{if } \gamma \in \Gamma_1 \\
\pi(E) &= \text{multiplication by } E \\
\pi(H)X_{\gamma_1}\dots X_{\gamma_j}E^r &= (j+2r)X_{\gamma_1}\dots X_{\gamma_j}E^r
\end{aligned}$$

The representation π extends to a representation of the algebra $\mathcal{A}$. Composing with Φ we get a representation of $\mathfrak{g}$. This construction is parallel to the one in [K-S]. We view it at the Verma analogue of the minimal unitary representation (more correctly we should pick up a representation contained in π)
Up to a constant, that we eventually managed to compute, Joseph gives a completely explicit formula for Φ. In particular we get the

Lemma. — *Let* $X \in \mathfrak{m}$ *such that* $\mathrm{ad}(X)V_i \subset V_i$. *Then*

$$\pi(\Phi(X)) = \mathrm{ad}(X) + \frac{1}{2}\mathrm{Tr}\big(\mathrm{ad}(X)_{V_1}\big)\mathrm{Id}$$

In the above formula $\mathrm{ad}(X)$ is extended to $S(V_1)$ as a derivation and then to $\mathcal{S}$ in the obvious way.
We can now describe the strategy of [R-S]. Let us look at the case where we remove δ. For example for E_6, one has $\alpha = \alpha_2$ and $\delta = \alpha_4$. Then $\mathfrak{b}(\alpha)$ is of type A_5 and δ is the middle root. The parabolic subalgebra of A_5 corresponding to this middle root has a commutative nilpotent radical, on which $\mathfrak{b}(\delta)$ which is of type $A_2 \times A_2$ acts (it a prehomogeneous vector space, regular and of commutative type). The point is that the invariant theory of such PV is very well known (see [M-R-S]). We have

$$\beta = 2\alpha + 3\delta + \dots$$

where the dots correspond to the other simple roots. If $\gamma \in \Delta_1$ then

$$\gamma = \alpha + n\delta + \cdots, \qquad \beta - \gamma = \alpha + (3-n)\delta + \cdots$$

Thus we can define Γ_1 to be the set of roots in Δ_1 such that the above integer n is 0 or 1. With this choice the above lemma is valid for $X \in \mathfrak{b}(\delta)$ and we can find all the highest weight vectors for $\mathfrak{b}(\delta)$. This subspace is invariant under $\mathfrak{a}(\delta)$.
At this stage umpleasant computations appear. We want to find vectors which are highest weight vectors for $\mathfrak{b}(\delta)$ and $\mathfrak{a}(\delta)$. The subalgebra is of type A_2 and a possible choice of simple roots is $\{-\beta.\alpha\}$. However $\pi(X_\alpha)$ is just multiplication by X_α so in order to get highest weight vectors we had to choose as simple roots $\{\beta-\alpha, -\beta\}$. But then the operators $\pi(X_{-\beta}), \pi(X_{\beta-\alpha})$ are hard to compute. With patience it can be done and, for the highest weights vectors we end up with simple expressions.
But then those joint highest weights vectors are eigenvectors for the centers of the envelopping algebras of $\mathfrak{a}(\delta)$ and $\mathfrak{b}(\delta)$ and there are sufficiently many of them to get the collapsing theorem. All this is valid for the types F_4, E_6, E_7, E_8 and also, with slight modification for B_n, D_n.
To state the final result for the cases F_4, E_6, E_7, E_8 ,let τ be th highest root of $\mathfrak{m} = \mathfrak{b}(\alpha)$ and define ε by $\beta = 2\alpha + \delta + \varepsilon + \tau$; then ε is a root. We check that $t_1 H_\delta + t_2 H_\varepsilon + t_3 H_\tau$ belongs to $\mathfrak{b}(\delta) \cap \mathfrak{h}$ if and only if $t_1 + t_2 + t_3 = 0$
We define a map ψ

$$\psi : \mathfrak{a}(\delta) \cap \mathfrak{h} \longrightarrow \mathfrak{b}(\delta) \cap \mathfrak{h}$$

by

$$\psi\Big(x_1 H_{-\beta} - x_2 H_\alpha\Big) = \frac{6}{k-3}\Big(x_2 H_\delta + x_1 H_\varepsilon + x_3 H_\tau\Big) + \Big(H_\delta - H_\tau + 12\frac{H_\rho}{\|H_\rho\|^2}\Big)$$

where $x_1 + x_2 + x_3 = 0$, where ρ is the half sum of the positive roots of $\mathfrak{b}(\delta)$, the norm is relative to the Killing form of $\mathfrak{b}(\delta)$ and H_ρ is defined by the same formula as the coroots and where k is some integer (9 for E_6). This is an affine, one to one, first degree map from $\mathfrak{a}(\delta) \cap \mathfrak{h}$ into $\mathfrak{b}(\delta) \cap \mathfrak{h}$. The linear part of ψ agrees exactly with the map given by the orbit method of § 2 .
Let $P \in S(\mathfrak{b}(\delta) \cap \mathfrak{h})$; it is a polynomial function on the dual space but, using the Killing form of $\mathfrak{b}(\delta)$ we consider it as a polynomial function on $\mathfrak{b}(\delta) \cap \mathfrak{h}$. Then $\Xi(P)$ defined by $\Xi(P) = P \circ \psi$ is a polynomial function on $\mathfrak{a}(\delta) \cap \mathfrak{h}$ which, using the Killing form of $\mathfrak{a}(\delta)$ we consider as an element of $S(\mathfrak{a}(\delta) \cap \mathfrak{h})$. Also let j stands for the Harish-Chandra isomorphism from the center of the envelopping algebra to the symmetric invariants of the Weyl group.

Theorem [R-S]($\mathfrak{g}$ of type F_4, E_6, E_7 or E_8). — *The map Ξ is an algebra homomorphism and it maps the Weyl group invariants in $S(\mathfrak{b}(\delta) \cap \mathfrak{h})$ into the*

Weyl group invariants in $S(\mathfrak{a}(\delta) \cap \mathfrak{h})$.
Let

$$\Theta : \mathcal{Z}\Big(\mathcal{U}\big(\mathfrak{b}(\delta)\big)\Big) \longrightarrow \mathcal{Z}\Big(\mathcal{U}\big(\mathfrak{a}(\delta)\big)\Big)$$

be defined by

$$\Theta = j_{\mathfrak{a}(\delta)}^{-1} \circ \Xi \circ j_{\mathfrak{b}(\delta)}$$

Then Θ *is an algebra homomorphism and, for any* $Z \in \mathcal{Z}\Big(\mathcal{U}\big(\mathfrak{b}(\delta)\big)\Big)$

$$Z - \Theta(Z) \in J$$

There is a similar result for the B_n, D_n cases and also for the $\alpha-$case. The map is always affine, the linear part being the one predicted by the orbital decomposition. However there is a "tail" whose significance certainly requires further investigation. Its appearance should be meaningful from the point of view of $\theta-$liftings and also from the point of view of the orbit method.
Also is is very tempting to ask whether one can build a Howe type correspondance for Verma modules.
The proofs of the collapsing theorems involve rather heavy-handed computations. There is another approach (see [G-W], [S 3]). Suppose we have a dual pair $A \times B \subset G$ in a complex setup but with a real form $A_{\mathbf{R}} \times B_{\mathbf{R}} \subset G_{\mathbf{R}}$ such that $A_{\mathbf{R}} \times B_{\mathbf{R}}$ is a maximal compact subgroup or at least a large compact subgroup. Because the $K-$types of a minimal representation are known, it is possible to decompose the restriction of π to $A_{\mathbf{R}} \times B_{\mathbf{R}}$. from there it is not difficult to get the collapsing theorems.
Anyway the real challenge is to check the collapsing theorems, over any local field, for the series of dual pairs appearing in [R], at least in the split cases and with a unified proof...
In conclusion, and although there is a need to test more examples of the tensor product type (there is a wealth of candidates in [R]), it seems likely that many dual pairs will give rise to Howe's correspondances and $\theta-$liftings. Whether or not these liftings will be non-trivial examples of functoriality or produce new automorphic forms of particular interest remains to be seen. At the very least this line of work should give new insight into the structure of the exceptional groups.

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