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Character Variety of Representations of a Finitely Generated Group in SL_2

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This is a partial exposition of [S3-4].

Let Γ be a finitely generated group. We introduce the character variety $Ch(\Gamma, SL_2)$ as a scheme over $\mathbb{Z}$ in order to parametrize conjugacy classes of representations of Γ in SL_2 in a functorial way. Then, we specialize the scaler to the real number field $\mathbb{R}$. Let us explain this briefly.

Let $Hom(\Gamma, SL_n)$ be the functor: $R \in$ {commutative rings with 1} $\mapsto Hom(\Gamma, SL_n(R)) \in$ {sets}. The functor is represented by an affine scheme over $\mathbb{Z}$ (see (1.3) lemma). By abuse of notation, we denote by the same $Hom(\Gamma, SL_n)$ the scheme representing the functor. The group scheme PGL_n acts on $Hom(\Gamma, SL_n)$ by the adjoint action on SL_n. Whether the *universal categorical quotient* $Hom(\Gamma, SL_n)//PGL_n$ (Mumford [Mu]) defined over $\mathbb{Z}$ exists or not seems to be a hard unsolved question. Instead of looking for the quotient space directly, we introduce the *character variety* $Ch(\Gamma, SL_2)$ together with the *invariant morphism* $\pi_\Gamma: Hom(\Gamma, SL_2) \longrightarrow Ch(\Gamma, SL_2)$ and its *discriminant* $D_\Gamma \subset Ch(\Gamma, SL_2)$ defined over $\mathbb{Z}$, for which we show that i) *the inverse image* $\pi_\Gamma^{-1}(D_\Gamma)$ *is a subfunctor of* $Hom(\Gamma, SL_2)$ *consisting of abel or reducible representations*, and ii) *the restriction of* π_Γ *on the complement of* $\pi_\Gamma^{-1}(D_\Gamma)$ *is a principal* PGL_2*-bundle with respect to the étal topology*. So, the functor $Hom^*(\Gamma, SL_2) := Hom(\Gamma, SL_2) \setminus \pi_\Gamma^{-1}(D_\Gamma)$, consisiting of absolutely irreducible representations, has the universal categorical quotient $Ch^*(\Gamma, SL_2) := Ch(\Gamma, SL_2) \setminus D_\Gamma$.

For a finite field $\mathbb{F}_q$, any rational point $t \in Ch^*(\Gamma, SL_2)(\mathbb{F}_q)$ has rational points in $\pi_\Gamma^{-1}(t)$ because of Zen's theorem (Lang). For the real field $\mathbb{R}$, *the variety* $Ch^*(\Gamma, SL_2)(\mathbb{R})$ *decomposes into a disjoint union of two open semialgebraic sets* H_Γ *and* T_Γ *such that* $Hom^*(\Gamma, SL_2(\mathbb{R}))$ *and* $Hom^*(\Gamma, SU(2)))$ *are a pricipal* $PGL_2(\mathbb{R})$*-bundle and a principal* $U(2)/U(1)$*-bundle over* H_Γ and T_Γ, respectively. We may call H_Γ the Teichmüller space for Γ in a sense.

In fact, H_Γ carries a simplectic and Kähler structure for some good Γ.

From the construction, the group $\mathrm{Out}(\Gamma)$ acts naturally on the character variety $\mathrm{Ch}(\Gamma,SL_2)$. Thus, one is naturally lead to look for automorphic forms and functions in a suitable sense on the character variety with respect to $\mathrm{Out}(\Gamma)$ action. This motivated the present work (cf [S2]).

The §1 prepares notation of a representation variety $\mathrm{Hom}(\Gamma,SL_2)/\mathbb{Z}$. The §2 studies invariants of PGL_2-action on $M_2\times M_2$ as the building block. The universal character ring $R(\Gamma,SL_2)$ and the discriminants $\Delta(\alpha,\beta)\in R(\Gamma,SL_2)$ for $\alpha,\beta\in\Gamma$ are introduced in §3. We define $\mathrm{Ch}(\Gamma,SL_2):=\mathrm{Spec}(R(\Gamma,SL_2))$ and $D_\Gamma:=V((\Delta(\alpha,\beta)\ \alpha,\beta\in\Gamma))$. The principal PGL_2-bundle sturcture on the open subset of $\mathrm{Hom}(\Gamma,SL_2)$ over $\mathrm{Ch}(\Gamma,SL_2)$ with repect to the étal toplogy is formulated in §4 Thoerem A. The $PGL_2(\mathbb{R})$ bundle structure over the semialgebraic set of the real character variety $\mathrm{Ch}(\Gamma,SL_2)(\mathbb{R})$ is formulated in §5 Theorem C.

§ 1 Universal representation of a group Γ in SL_n.

This § is devoted for a preparation of notion and terminology for the representation varieties. One is refered to [Pr] [Ba] [L-M] etc.

(1.1) Let Γ be a group and $n\in\mathbb{Z}_{\geq 0}$. As in the introduction, let $\mathrm{Hom}(\Gamma,SL_n)$ be the functor: $R\in\{\text{commutative rings with } 1\}\mapsto \mathrm{Hom}(\Gamma,SL_n(R))\in\{\text{sets}\}$. Let us formulate the representability of the functor in the next lemma.

Lemma (the representability of $\mathrm{Hom}(\Gamma,SL_n)$*).*

1. *For the given* Γ *and* $n\geq1$, *there exists a pair* $(A(\Gamma,SL_n),\sigma)$ *of a commutative ring* $A(\Gamma,SL_n)$ *with* 1 *and a representation* $\sigma:\Gamma\longrightarrow SL_n(A(\Gamma,SL_n))$

such that for any commutative ring R *with* 1, *the correspondence*:

(1.1.1) $\qquad \varphi \in \mathrm{Hom}^{\mathrm{ring}}(A(\Gamma,SL_n),R) \mapsto \varphi\circ\sigma \in \mathrm{Hom}^{\mathrm{gr}}(\Gamma,SL_n(R))$

is a bijection.

2. *The pair* $(A(\Gamma,SL_n),\sigma)$ *is unique up to an isomorphism of the ring* $A(\Gamma,SL_n)$ *commuting with the representation* σ.

3. *If* Γ *is a finitely generated group, then* $A(\Gamma,SL_n)$ *is finitely generated algebra over* $\mathbb{Z}$ *and hence it is noetherian.*

The lemma is proven by standard arguments. Here we give an explicit description of $(A(\Gamma,SL_n),\sigma)$. For each $\gamma\in\Gamma$, consider a $n\times n$ matrix:

(1.1.2)
$$\sigma(\gamma) := \left(a_{ij}(\gamma)\right)_{ij=1,\dots,n}\ .$$

Then $A(\Gamma,SL_n)$, called the *universal representation algebra*, is generated by all entries $a_{ij}(\gamma)$ $i,j=1,\cdots,n$, $\gamma\in\Gamma$, and divided by the ideal generated by all entries of the matrices $\sigma(e)-I_n$ (e=the unit of Γ, I_n=the $n\times n$ unit matrix) and $\sigma(\gamma\delta)-\sigma(\gamma)\sigma(\delta)$, and by $\det(\sigma(\gamma))-1$ for $\gamma,\delta\in\Gamma$. That is:

(1.1.3)
$$A(\Gamma,SL_n) := \mathbb{Z}[a_{ij}(\gamma) \text{ for } \gamma\in\Gamma \text{ and } 1\le ij\le n]/I\ ,$$

where
$$I := \left(a_{ij}(e)-\delta_{ij},\ a_{ij}(\gamma\delta)-\textstyle\sum_k a_{ik}(\gamma)a_{kj}(\delta),\ 1-\det(a_{pq}(\gamma))_{pq=1}^{n} \text{ for } 1\le i,j\le n \text{ and } \gamma,\delta\in\Gamma\right).$$

The $\sigma:\gamma\in\Gamma\longrightarrow\sigma(\gamma)\in SL_n(A(\Gamma,SL_n))$ is called a *universal representation* of Γ.

(1.2) Let PGL_n be the group scheme ([SGA III],[D-G]). Its coordinate ring $A(PGL_n)$ is given by the subring $A_0(GL_n)$ of $A(GL_n):=\mathbb{Z}[x_{ij} 1\le i,j\le n]_{\det(X)}$ (here $X:=(x_{ij})_{ij=1}^{n}$) consisting of homogeneous elements of degree 0. The adjoint action of PGL_n on $\mathrm{Hom}(\Gamma,SL_n)$ is given by its dual action:

(1.2.1)
$$\mathrm{Ad} : A(\Gamma,SL_n) \longrightarrow A(\Gamma,SL_n)\otimes_{\mathbb{Z}} A(PGL_n),$$

sending an entry of $\sigma(\gamma)$ to the same entry of $X^{-1}\sigma(\gamma)X := \frac{1}{\det(X)}X^{*}\sigma(\gamma)X$

(1.3) From now on, we switch to $n=2$. Let us list up some relations among the traces (= characters) $\mathrm{tr}(\sigma(\gamma))$ for $\gamma\in\Gamma$. The first one is:

(1.3.1) $$\mathrm{tr}(\sigma(e)) = 2.$$

The next one follows from the Caylay-Hamilton relation: $\sigma(\gamma)^2+\det(\sigma(\gamma))\cdot I_2 = \mathrm{tr}(\sigma(\gamma))\cdot\sigma(\gamma)$. Multiply $\sigma(\gamma^{-1}\delta)$ from right and take traces. So we obtain:

(1.3.2) $$\mathrm{tr}(\sigma(\gamma\delta)) + \mathrm{tr}(\sigma(\gamma^{-1}\delta)) = \mathrm{tr}(\sigma(\gamma))\cdot\mathrm{tr}(\sigma(\delta))$$

for γ, $\delta \in\Gamma$ ([F-K, formulas (2), p.338]). We aim to describe the subring generated by the characters by generators and relations and to show that it is the universal (for any scaler extension) invariant subring by the action (1.2.1) up to localizations by the discriminants $\Delta(\alpha,\beta)$ introduced in §3.

§ 2. PGL_2-invariants for pairs of 2×2 matrices

We study the invariants of the diagonal adjoint action of PGL_2 on the space $M_2\times M_2$ of pair (A,B) of 2×2 matrices. The morphism $\tilde{\pi}$: $M_2\times M_2\longrightarrow\mathbb{A}^5$ given by $(A,B)\mapsto(\mathrm{tr}(A),\mathrm{tr}(B),\mathrm{tr}(AB),\det(A),\det(B))$ is shown to be the universal quotient map ((2.3) Lemma A, B). The discriminant Δ for $\tilde{\pi}$ is introduced ((2.4)lemma C) so that the $\tilde{\pi}$ is a principal PGL_2-bundle on the complement of the discriminant loci $\Delta=0$ ((2.5) Lemma D).

(2.1) Let $M_2\times M_2$ be the space of pairs of 2×2 matrices. The $X\in PGL_2$ acts on $(A,B)\in M_2\times M_2$ from the right diagonaly by letting $(A,B)\cdot Ad(X):=(X^{-1}AX,X^{-1}BX)$. So we have the dual action on the coordinate ring:

(2.1.2) $$Ad : \mathbb{Z}[M_2\times M_2] \longrightarrow \mathbb{Z}[M_2\times M_2] \otimes_{\mathbb{Z}} A(PGL_2)$$

sending an entry of (A,B) to the corresponding entry of $(A,B)\cdot Ad(X)$ where $\mathbb{Z}[M_2\times M_2]$ is the polynomial ring generated by entries of (A,B).

(2.2) Let us consider the morphism

(2.2.1) $$\tilde{\pi} : M_2\times M_2 \longrightarrow \mathbb{A}^5:=\mathrm{Spec}(\mathbb{Z}[\underline{T},\underline{D}]),$$

where $\mathbb{Z}[\underline{T},\underline{D}]$ denotes the polynomial ring $\mathbb{Z}[T_1,T_2,T_3,D_1,D_2]$ of the 5 indeterminates and $\tilde{\pi}$ is associated to the ring homomorphism:

(2.2.2) $$\iota:\ \mathbb{Z}[\underline{T},\underline{D}] \longrightarrow \mathbb{Z}[M_2\times M_2]\ ,$$

given by $\iota(T_1):=\mathrm{tr}(A)$, $\iota(T_2):=\mathrm{tr}(B)$, $\iota(T_3):=\mathrm{tr}(AB)$, $\iota(D_1):=\det(A)$ and $\iota(D_2):=\det(B)$. The morphism $\tilde{\pi}$ is $\mathbb{G}_m$-equivariant and PGL_2-invariant, since $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(AB)$, $\det(A)$ and $\det(B)$ are PGL_2-invariant homogenous polynomials (of degree 1,1,2,2 and 2, respectively). First, we show:

(2.3) *Lemma* A. *$\mathbb{Z}[M_2\times M_2]$ is a free module over $\mathbb{Z}[\underline{T},\underline{D}]$.*

As a consequence of the lemma A, $\mathbb{Z}[M_2\times M_2]$ *is faithfully flat over* $\mathbb{Z}[\underline{T},\underline{D}]$. In particular, the map ι (2.2.2) is universaly injective, and we regard $\mathbb{Z}[\underline{T},\underline{D}]$ as a subring of $\mathbb{Z}[M_2\times M_2]$. More strongly, the next lemma says that it is the universal invariant subring with respect ot the PGL-action.

Lemma B. *Let* R *be any $\mathbb{Z}[\underline{T},\underline{D}]$-algebra with* 1. *Then*

(2.3.1) $$R \simeq \left(R\otimes_{\mathbb{Z}[\underline{T},\underline{D}]}\mathbb{Z}[M_2\times M_2]\right)^{PGL_2}.$$

Here, the PGL_2 *action on* $\mathbb{Z}[M_2\times M_2]$ *is extended to the tensor product by letting* PGL_2 *act trivially on* R.

Remark. 1. It is classical that $\mathbb{Q}[M_2\times M_2]^{PGL_2}$ is generated by traces $\mathrm{tr}(W)$ for $W\in$ {the monoid generated by the A and B} ([G-Y],[Pr],[W]). This is not true for $\mathbb{Z}$-coefficient. E.g.: the relation $2\det(A)=\mathrm{tr}(A)^2-\mathrm{tr}(A^2)$ implies the algebraic dependence of $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A)$ in char=2, whereas $\det(A)$ and $\mathrm{tr}(A)$ are universaly algebraicaly independent (cf. lemma B).

2. Donkin [D] has shown that $\mathbb{Z}[M_n\times\cdots\times M_n]^{PGL_n}$ is generated by $\mathrm{tr}(\overset{i}{\wedge}W)$ for $i=1,\cdots,n$ and $W\in$ {the monoid generated by $A_1,\cdots,A_m$}, where we denote by $M_n\times\cdots\times M_n$ the space of m-tuple n×n matrices $(A_1,\cdots,A_m)$.

(2.4) We introduce the discriminant $\Delta=\Delta(A,B)$ of the morphism $\tilde{\pi}$. For a matrix $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$, put $A^*:=\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}$ so that $A^*A=AA^*=\det(A)I_2$ and $A+A^*=\mathrm{tr}(A)I_2$.

Definition. The *discriminant* for $\tilde{\pi}$ is the polynomial:

$$(2.4.1) \qquad \Delta(A,B) := \mathrm{tr}(ABA^*B^*) - \mathrm{tr}(AA^*BB^*)$$

$$= T_1^2D_2 + T_2^2D_1 + T_3^2 - T_1T_2T_3 - 4\,D_1D_2\ .$$

A justification for this name is in the next lemma C. Let J be the ideal of $\mathbb{Z}[M_2\times M_2]$ generated by all 5×5 minors of the Jacobian matrix of $\tilde{\pi}$:

$$\frac{\partial(T_1,T_2,T_3,D_1,D_2)}{\partial(a,b,c,d,e,f,g,h)} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ e & g & f & h & a & c & b & d \\ d & -c & -b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & -g & -f & e \end{bmatrix}.$$

Lemma C. *The intersection of the Jacobi ideal* J *with the invariant subring* $\mathbb{Z}[\underline{T},\underline{D}]$ *is a principial ideal generated by the discriminant.*

$$(2.4.2) \qquad J \cap \mathbb{Z}[\underline{T},\underline{D}] = (\Delta).$$

(2.5) Let us denote by D_Δ the divisor of $\mathbb{A}^5=\mathrm{Spec}(\mathbb{Z}[\underline{T},\underline{D}])$ defined by the ideal (Δ). Owing to the (2.4.2), the Jacobian criterion implies the smoothness of the morphism $\tilde{\pi}$ (2.2.1) on the complement of the $\tilde{\pi}^{-1}(D_\Delta)$:

$$(2.5.1) \qquad \tilde{\pi}_\Delta:\ M_2\times M_2\setminus\tilde{\pi}^{-1}(D_\Delta) \longrightarrow \mathbb{A}^5\setminus D_\Delta.$$

More strongly, we prove the next lemma, to which the proof of main theorem A in §4 is reduced.

Lemma D. *The morphism* (2.5.1) *is a principal* PGL_2*-bundle with respect to the étal topology.*

§ 3 The universal character ring $R(\Gamma,SL_2)$.

We introduce the universal character ring $R(\Gamma,SL_2)$ by generators and relations in terms of Γ. The goal is the formula (3.4.3), which is a key in the proof of the main theorem B in §4.

(3.1) *Definition.* The *universal character ring* $R(\Gamma,SL_2)$ *of representations of* Γ *in* SL_2 is generated by the indeterminates $s(\gamma)$ $\gamma\in\Gamma$ and divided by the ideal generated by $s(e)-2$ (e=the unit of Γ) and by $s(\gamma)s(\delta)-s(\gamma\delta)-s(\gamma^{-1}\delta)$ for all $\gamma,\delta\in\Gamma$. That is:

(3.1.1) $R(\Gamma,SL_2) := \mathbb{Z}[s(\gamma),\gamma\in\Gamma]/\Big(s(e)-2,\ s(\gamma)s(\delta)-s(\gamma\delta)-s(\gamma^{-1}\delta)\Big)$.

(3.2) The ring $R(\Gamma,SL_2)$ is finitely generated over $\mathbb{Z}$, if Γ is finitely generated as shown in the next lemma. Such finiteness was asserted for the ring of traces of SL_2 by Fricke [F-K] and proven in [H, 2.f)], [Ho, Theo.3.1.] and [C-S].

Proposition. Let A *be a linearly ordered generater subset of* Γ. *Then* $R(\Gamma,SL_2)$ *is generated by* $G:=\bigcup_{m\in\mathbb{N}}\{s(\alpha_1\cdots\alpha_m) \mid \alpha_i\in A,\ \alpha_1<\cdots<\alpha_m\}$ *over* $\mathbb{Z}$.

Corollary 1. *If the group* Γ *is finitely generated, then the* $R(\Gamma,SL_2)$ *is finitely generated over* $\mathbb{Z}$. *Hence it is noetherian.*

(3.3) For α and $\beta\in\Gamma$, define the *discriminant* $\Delta(\alpha,\beta)\in R(\Gamma,SL_2)$ by

(3.3.1) $\Delta(\alpha,\beta):= s(\alpha\beta\alpha^{-1}\beta^{-1}) - 2.$

$$= s(\alpha)^2+s(\beta)^2+s(\alpha\beta)^2-s(\alpha)s(\beta)s(\alpha\beta)-4.$$

This definition of Δ is pararell to that in $\mathbb{Z}[\underline{T},\underline{D}]$ (2.4.1). In fact, we confuse them in a proof of the main theorem B in §4. The polynomial has been studied extensively by R. Fricke [F] and many others.

(3.4) An importance of the discriminant is explained in the next lemma, which plays an essential role in a proof of our main theorem in §4.

Definition. Let M be a $R(\Gamma,SL_2)$-module. A map $h:\Gamma\longrightarrow M$ is called a *form* with values in M, if for any γ and $\delta\in\Gamma$ one has a relation:

(3.4.1) $$h(\gamma\delta) + h(\gamma^{-1}\delta) = s(\gamma)h(\delta).$$

Lemma Let h *be a form with values in* M. *Suppose* $h(e)=h(\alpha)=h(\beta)=h(\alpha\beta)=0$ *for some* α *and* $\beta\in\Gamma$. *Then for any* $\gamma\in\Gamma$ *one has*

(3.4.2) $$\Delta(\alpha,\beta)h(\gamma) = 0 .$$

The composition of h *with the localization* $M\longrightarrow M_{\Delta(\alpha,\beta)}$ *is identically zero.*

Corollay. For any α,β,γ *and* $\delta\in\Gamma$, *one has*

(3.4.3)
$$s(\gamma\delta) = -\frac{1}{\Delta(\alpha,\beta)}(s(\gamma),s(\gamma\alpha),s(\gamma\beta),s(\gamma\alpha\beta))\cdot T\cdot{}^t(s(\delta),s(\alpha\delta),s(\beta\delta),s(\alpha\beta\delta))$$

in the localization $R(\Gamma,SL_2)_{\Delta(\alpha,\beta)}$, *where* $T\in M_4(R(\Gamma,SL_2))$ *such that* $T\cdot(s(\xi_i\xi_j))_{ij=1}^4=-\Delta(\alpha,\beta)I_4$ for $\xi_1=e$, $\xi_2=\alpha$, $\xi_3=\beta$ and $\xi_4=\alpha\beta$.

This is a key lemma to prove that the system $\{s(\gamma)\ \gamma\in\Gamma\}$ satisfies any algebraic relations which is satisfied by the system $\{tr(\sigma(\gamma))\ \gamma\in\Gamma\}$ of characters at the place where $\Delta(\alpha,\beta)$ is invertible for some $\alpha,\beta\in\Gamma$.

(3.5) *Remark.* The study of the algebra of traces (=characters) of representations of a group Γ into SL_2 started by Voigt and Fricke and is developped by many authers Helling, Horowitz, Magnus, Bass, Lubotsky Procesi, Platonov, Vinberg and others. See references of the quoted papers.

Let F_n be a free group generated by n elements. Magnus called the homomorphic image of the ring $R(F_n,SL_2)$ in the ring of functions on the representation space $Hom(F_n,SL_2(\mathbb{C}))$ the ring of Fricke characters [Ma]. We do not know whether the homomorphism has non trivial kernel or not.

§4 The invariant morphism π_Γ

After introducing the character variety $Ch(\Gamma,SL_2)$, the invariant morphism $\pi_\Gamma:Hom(\Gamma,SL_2)\to Ch(\Gamma,SL_2)$ and the discriminant D_Γ in (4.1), the main results are formulated in (4.2) Assetion and (4.3) Theorem A.

(4.1) Let Γ be a group. Put

(4.1.1) $$\mathrm{Ch}(\Gamma, SL_2) := \mathrm{Spec}(R(\Gamma, SL_2)),$$

where $R(\Gamma, SL_2)$ is defined in (3.1). Recall the fact that $\mathrm{Hom}(\Gamma, SL_2)$ is identified with the affine scheme for the universal reperesentaion ring $A(\Gamma, SL_2)$ (cf (1.3) lemma). Then the invariant morphism is defined

(4.1.2) $$\pi_\Gamma : \mathrm{Hom}(\Gamma, SL_2) \to \mathrm{Ch}(\Gamma, SL_2)$$

through the ring homomorphism:

(4.1.3) $$\begin{array}{c} \Phi : R(\Gamma, SL_2) \longrightarrow A(\Gamma, SL_2). \\ s(\gamma) \mapsto tr(\sigma(\gamma)) \end{array}$$

The Φ is well defined (compare (3.1.1) with (1.3.1) and (1.3.2)). The *discriminant subvariety* $\subset \mathrm{Ch}(\Gamma, SL_2)$ of the morphism π_Γ is defined as

(4.1.4) $$D_\Gamma := \bigcap_{\alpha, \beta \in \Gamma} V(\Delta(\alpha, \beta)),$$

where the $\Delta(\alpha, \beta) \in R(\Gamma, SL_2)$ is introduced in (3.3.1).

(4.2) We characterize the inverse image $\pi_\Gamma^{-1}(D_\Gamma)$ of the discriminant.

Assertion. Let $p \in \mathrm{Spec}(A(\Gamma, SL_2))$. *Then* p *belongs to* $\pi_\Gamma^{-1}(D_\Gamma)$, *if and only if the image* $\sigma_p(\Gamma)$ *in* $SL_2(k_p)$ *of the* Γ *is either abelian or reducible, where* k_p *is the fraction field of the integral domain* $A(\Gamma, SL_2)/p$, *and* $\sigma_p : \Gamma \longrightarrow SL_2(k_p)$ *is obtained by a specialization of the universal* σ *at* p.

(4.3) Let us state a main result of the present note.

Theorem A. *The restriction of the invariant morphism* π_Γ *to the complement of the inverse image* $\pi_\Gamma^{-1}(D_\Gamma)$ *of the discriminant is a principal* PGL_2*-bundle with respect to the etal topology defined over* $\mathbb{Z}$.

Proof. By definition of D_Γ, one has an affine open covering $Ch(\Gamma, SL_2)\setminus D_\Gamma = \bigcup_{\alpha,\beta\in\Gamma} Spec(R(\Gamma, SL_2)_{\Delta(\alpha,\beta)})$. Then the proof is reduced to each affine open piece, stated as a consequence of the next Theorem B.

(4.4) For any fixed pair α and β of Γ, consider the PGL_2-equivariant morphism $Hom(\Gamma, SL_2) \longrightarrow M_2 \times M_2$ and a morphism $h_{\alpha\beta}: Ch(\Gamma, SL_2) \longrightarrow \mathbb{A}^5 = Spec(\mathbb{Z}[\underline{T}, \underline{D}])$ defined by the coordinate ring homomorphisms given by

(4.4.1) $$A \mapsto \sigma(\alpha) \quad \text{and} \quad B \mapsto \sigma(\beta),$$

(4.4.2) $$T_1 \mapsto s(\alpha),\ T_2 \mapsto s(\beta),\ T_3 \mapsto s(\alpha\beta),\ D_1 \mapsto 1 \text{ and } D_2 \mapsto 1$$

so that the next diagram becomes commutative

(4.4.3) $$\begin{array}{ccc} Hom(\Gamma,\ SL_2) & \longrightarrow & M_2 \times M_2 \\ \downarrow \pi_\Gamma & & \downarrow \tilde{\pi} \\ Ch(\Gamma,\ SL_2) & \longrightarrow & \mathbb{A}^5 = Spec(\mathbb{Z}[\underline{T}, \underline{D}]) \end{array}$$

We remark that the discriminant $\Delta(\alpha,\beta)$ (3.3.1) is the pull back of $\Delta(A,B)$ (2.4.1). By abuse of notation, we shall denote both of them by Δ.

Theorem B. *The diagram* (4.4.3) *is Cartesian on the complement of the loci* $\Delta=0$. *That is: the localization by* Δ *of the homomorphism*

(4.4.4) $$\Psi_{\alpha,\beta} : R(\Gamma, SL_2) \otimes_{\mathbb{Z}[\underline{T},\underline{D}]} \mathbb{Z}[M_2 \times M_2] \longrightarrow A(\Gamma, SL_2)$$

(obtained from (4.1.3), (4.4.1) *and* (4.4.2)) *is an isomorphism.*

Theorem A follows from theorem B with (2.5) lemma D. Theorem B is proved by constructing the localized inverse homomorphism of $(\Psi_{\alpha,\beta})_\Delta$ of (4.4.4). In view of the representability (1.1) lemma, this is equivalent to give *a representation* σ^* *for a prescribed system of "characters"* $s(\gamma)$ $\gamma\in\Gamma$ *and a pair of matrix* $(A,B)\in M_2\times M_2$ *(over the same point of* $\mathbb{A}^5\setminus\{\Delta=0\}$) *such that* $tr(\sigma^*(\gamma)) = s(\gamma)$ *and* $(\sigma^*(\alpha), \sigma^*(\beta)) = (A,B)$. This is achieved by putting:

$$\sigma^*(\gamma) := -\frac{1}{\Delta(\alpha,\beta)}(I_2, A, B, AB)\cdot T\cdot {}^t(s(\gamma), s(\alpha\gamma), s(\beta\gamma), s(\alpha\beta\gamma)),$$

where T is the matrix of $M_4(R(\Gamma, SL_2))$ in (3.4.3). The fact that this defines a homomorphism is shown by an essential use of the formula (3.4.3).

§5 Character variety with real coefficients

We specialize the scaler to the real number field $\mathbb{R}$. We determine the image of the invariant morphism $\pi_\Gamma(\mathbb{R})$ (4.1.2) (up to the discriminant) as an open semialgebraic set in the real character variety. This leads to *the* $PGL_2(\mathbb{R})$*-bundle structure on* $\mathrm{Hom}^*(\Gamma, SL_2(\mathbb{R}))$ *over a semialgebraic set* H_Γ *with respect to the classical topology* ((5.5) Theorem C).

(5.1) Consider the invariant map $\tilde{\pi}$ (2.2.1) over the real number field $\mathbb{R}$:

(5.1.1) $$\tilde{\pi} : M_2(\mathbb{R})\times M_2(\mathbb{R}) \longrightarrow \mathbb{A}^5(\mathbb{R}) := \mathrm{Hom}(\mathbb{Z}[\underline{T},\underline{D}),\mathbb{R})$$

and the real loci of the discriminant Δ (2.4):

(5.1.2) $$\tilde{D}_\Delta(\mathbb{R}) := \{\varphi\in\mathbb{A}^5(\mathbb{R}) : \Delta(\varphi) := \varphi(\Delta)=0\}.$$

(5.2) Let us introduce an open semialgebraic subset of $\mathbb{A}^5(\mathbb{R})$:

(5.2.1) $$\tilde{T}_\Delta := \{\delta_1<0,\ \delta_2<0,\ \delta_3<0,\ \Delta<0\} ,$$

where

(5.2.2) $$\delta_1=T_1^2-4D_1,\quad \delta_2=T_2^2-4D_2 \quad\text{and}\quad \delta_3=T_3^2-4D_1D_2.$$

are the discriminants for the characteristic polynomials of A, B and AB respectively. The following facts are shown by direct calculations.

Lemma E. i) $\tilde{T}_\Delta$ *is a convex connected component of* $\mathbb{A}^5(\mathbb{R})\setminus\tilde{D}_\Delta$.

ii) $\mathbb{A}^5(\mathbb{R})\setminus\tilde{T}_\Delta = \tilde{\pi}\Big(M_2(\mathbb{R})\times M_2(\mathbb{R})\Big)$.

Let us decompose the space $\mathbb{A}^5(\mathbb{R})$ into semialgebraic sets:

(5.2.3) $$\mathbb{A}^5(\mathbb{R}) = \tilde{D}_\Delta \amalg \tilde{H}_\Delta \amalg \tilde{T}_\Delta,$$

where $\tilde{H}_\Delta := \mathbb{A}^5(\mathbb{R})\setminus(\tilde{D}_\Delta\amalg\tilde{T}_\Delta)$ is a union of connected components of $\mathbb{A}^5(\mathbb{R})\setminus\tilde{D}_\Delta$. Then the lemma E ii) is paraphrase as: $\mathrm{Image}(\tilde{\pi}) = \tilde{D}_\Delta\amalg\tilde{H}_\Delta$.

(5.3) For any given $\alpha,\beta\in\Gamma$, consider the homomorphism $F_2\longrightarrow\Gamma$ associating the two generators of F_2 to α and β. This induces the diagram (4.4.3):

$$(5.3.1)\qquad \begin{array}{ccc} \mathrm{Hom}(\Gamma,\ SL_2(\mathbb{R})) & \longrightarrow & \mathrm{Hom}(F_2,\ SL_2(\mathbb{R})) \subset M_2(\mathbb{R})\times M_2(\mathbb{R}) \\ \downarrow \pi_\Gamma(\mathbb{R}) & & \downarrow \tilde{\pi}(\mathbb{R}) \\ \mathrm{Ch}(\Gamma,\ SL_2)(\mathbb{R}) & \xrightarrow{h_{\alpha,\beta}} & \mathbb{A}^5(\mathbb{R}) \end{array}$$

where the morphism $h_{\alpha,\beta}$ is defined by (4.4.2) and $\mathrm{Ch}(\Gamma,SL_2)(\mathbb{R}) := \mathrm{Hom}(R(\Gamma, SL_2),\mathbb{R})$ is the real character varity. By a use of the morphism $h_{\alpha,\beta}$, let us introduce a decomposition of the real character variety:

$$(5.3.2)\qquad \mathrm{Ch}(\Gamma,SL_2)(\mathbb{R}) = D_\Gamma(\mathbb{R}) \amalg H_\Gamma \amalg T_\Gamma ,$$

where

$$D_\Gamma(\mathbb{R}) := \{t\in \mathrm{Hom}(R(\Gamma,SL_2),\mathbb{R}) \mid h_{\alpha,\beta}(t)\in D_\Delta(\mathbb{R}) \text{ for } {}^\forall\alpha,\beta\in\Gamma\}$$

is the Zariski closed real discriminant set and

$$H_\Gamma := \{t\in \mathrm{Hom}(R(\Gamma,SL_2),\mathbb{R}) \mid {}^\exists\alpha,\beta\in\Gamma \text{ such that } h_{\alpha,\beta}(t)\in\tilde{H}_\Delta\}$$

$$T_\Gamma := \{t\in \mathrm{Hom}(R(\Gamma,SL_2),\mathbb{R}) \mid {}^\exists\alpha,\beta\in\Gamma \text{ such that } h_{\alpha,\beta}(t)\in\tilde{T}_\Delta\}.$$

are open semialgebraic sets such that $H_\Gamma\cap T_\Gamma=\phi$. (Openness is clear, for $\tilde{T}_\Gamma$ and $\tilde{H}_\Gamma$ are open. Lemma E with the Cartesianness of (5.3.1) (§4 Theorem B) implies $H_\Gamma\cap T_\Gamma=\phi$. Due to the basis theorem of Hilbert, one finds a finite system $\{(\alpha_i,\beta_i)\}_{i\in I}$ such that $D_\Gamma(\mathbb{R})=\bigcap_{i\in I}\{\Delta(\alpha_i,\beta_i)=0\}$. Then for the same index set I, one can show the equalities: $H_\Gamma=\bigcup_{i\in I}h^{-1}_{\alpha_i\beta_i}(\tilde{H}_\Delta)$, $T_\Gamma=\bigcup_{i\in I}h^{-1}_{\alpha_i\beta_i}(\tilde{T}_\Delta)$).

(5.4) The commutativity of (5.3.1) with Lemma E imply that the image of $\pi_\Gamma(\mathbb{R})$ is contained in $D_\Gamma(\mathbb{R})\cup H_\Gamma$. More strongly, since (5.3.1) is Cartesian ((4.4) Theorem B), any point of H_Γ is in the image of $\pi_\Gamma(\mathbb{R})$. So we have:

Lemma F. *Let* $\pi_\Gamma(\mathbb{R})$ *be the real invariant morphisms* (5.1). *Then*

$$(5.4.1)\qquad \mathrm{Image}(\pi_\Gamma(\mathbb{R}))\setminus D_\Gamma(\mathbb{R}) = H_\Gamma .$$

A meaning of the remaining set T_Γ is given by the next lemma. Since $SU(2)\subset SL_2(\mathbb{C})$, the restriction of $\pi_\Gamma(\mathbb{C})$ induces the following map:

$$(5.4.2)\qquad \begin{array}{rcl} u_\Gamma:\ \mathrm{Hom}(\Gamma,\ SU(2)) & \longrightarrow & \mathrm{Ch}(\Gamma,\ SL_2)(\mathbb{R}). \\ \rho & \mapsto & (s(\gamma)\mapsto \mathrm{tr}(\rho(\gamma))) \end{array}$$

Lemma G. *The image set of the morphism* u_Γ *is given by*

(5.4.3) $$\mathrm{Image}(u_\Gamma)\setminus D_\Gamma(\mathbb{R}) = T_\Gamma.$$

(5.5) Combining the above (5.4.1), (5.4.3) with the §4 Theorem A, we summarize the results of the paper as follows.

Theorem C. *Consider the sets of absolutely irreducible representations*

$$\mathrm{Hom}^*(\Gamma,SL_2(\mathbb{R})) := \mathrm{Hom}(\Gamma,SL_2(\mathbb{R}))\setminus\pi_\Gamma^{-1}(D_\Gamma(\mathbb{R})),$$

and

$$\mathrm{Hom}^*(\Gamma,SU(2)) := \mathrm{Hom}(\Gamma,SU(2))\setminus u_\Gamma^{-1}(D_\Gamma(\mathbb{R})) .$$

The restrictions of (4.1.2) *and* (5.4.2) *are a principal* $PGL_2(\mathbb{R})$*-bundle and a prinicipal* $U(2)/U(1)$*-bundle over the open semialgebraic sets* H_Γ *and* T_Γ (5.3.2) *of the real character variety* $Ch(\Gamma,SL_2)(\mathbb{R})$, *respectively.*

$$\pi_\Gamma : \mathrm{Hom}^*(\Gamma,SL_2(\mathbb{R})) \xrightarrow{PGL_2(\mathbb{R})} H_\Gamma$$

$$u_\Gamma : \mathrm{Hom}^*(\Gamma,SU(2)) \xrightarrow{U(2)/U(1)} T_\Gamma$$

The value of a coordinate $s(\gamma)\in R(\Gamma,SL_2)$ *at a point of the base space is equal to the character* $tr(\rho(\gamma))$ *for any representation* ρ *in its fiber.*

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