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Generating Functions and Integral Representations for the Spherical Functions on Some Classical Gelfand Pairs

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Introduction

Let $\mathbf{F}$ be $\mathbf{R}$,$\mathbf{C}$ or $\mathbf{H}$ and $a \mapsto \overline{a}$ the usual conjugation in $\mathbf{F}$. We define the following quadratic form in $\mathbf{F}^{n+1}$.

$$(x, y)_- = -\overline{x}_0 y_0 + \overline{x}_1 y_1 + \cdots + \overline{x}_n y_n.$$

Let $U(1, n; \mathbf{F})$ be the group of the linear transformations g in $\mathbf{F}^{n+1}$ which satisfy $(gx, gy)_- = (x, y)_-$ for all $x, y \in \mathbf{F}^{n+1}$. We define the group G as follows.

1. If $\mathbf{F}$=$\mathbf{R}$, G is the connected component of the unit element in $U(1, n; \mathbf{R})$, i.e. $G = SO_0(1, n)$.
2. If $\mathbf{F}$=$\mathbf{C}$, G is the group of all the elements $g \in U(1, n; \mathbf{C})$ of determinant one, i.e. $G = SU(1, n)$.
3. If $\mathbf{F}$=$\mathbf{H}$, $G = U(1, n; \mathbf{H})$, i.e. $G = Sp(1, n)$.

Let $B(\mathbf{F}^n)$ be the unit ball in $\mathbf{F}^n$ and $S(\mathbf{F}^n)$ be the unit sphere in $\mathbf{F}^n$. The group G acts transitively on $B(\mathbf{F}^n)$ and $S(\mathbf{F}^n)$ as follows:
for $\xi = {}^t(\xi_1, ..., \xi_n) \in \mathbf{F}^n$ and $g = (g_{pq})_{0 \le p,q \le n} \in G$, we define

$$\xi' = g\xi,$$

where $\xi' = {}^t(\xi'_1, ..., \xi'_n)$, with

$$\xi'_p = (g_{p0} + \sum_{q=1}^{n} g_{pq}\xi_q)(g_{00} + \sum_{q=1}^{n} g_{0q}\xi_q)^{-1}, \quad 1 \le p \le n.$$

Let K be the isotropy group of $O \in B(\mathbf{F}^n)$ in G. Then K is a maximal compact subgroup of G and $G/K \cong B(\mathbf{F}^n)$. Let $G = KAN$ be the corresponding Iwasawa decomposition and M be the centralizer of A in K. Then M is the isotropy group of $e_1 = {}^t(1, 0, ..., 0) \in S(\mathbf{F}^n)$ in K and $K/M \cong S(\mathbf{F}^n)$ is the Martin boundary on $G/K \cong B(\mathbf{F}^n)$. Except for the case of real numbers, K/M is not a symmetric space, but it is known that (K, M) is a Gelfand pair, i.e. the convolution algebra of functions on K bi-invariant by M is commutative. As is well known, the spherical functions on K/M play an important role in the harmonic analysis on G/K.

$$K \cong \begin{cases} SO(n) & \mathbf{F} = \mathbf{R} \\ U(n) & \mathbf{F} = \mathbf{C} \\ Sp(1) \times Sp(n) & \mathbf{F} = \mathbf{H} \end{cases}$$

Let φ be a zonal spherical function of the real case $SO(n)/SO(n-1) \cong S(\mathbf{R}^n)$. Then φ depends only on η_1 $\left(\eta = {}^t(\eta_1, ..., \eta_n) \in S(\mathbf{R}^n)\right)$ and there exists a unique nonnegative integer p such that

$$\varphi(\eta) = C_p^{\frac{n-2}{2}}(\eta_1)/C_p^{\frac{n-2}{2}}(1), \ \eta = {}^t(\eta_1, ..., \eta_n) \in S(\mathbf{R}^n),$$

where $C_p^{\frac{n-2}{2}}$ is the Gegenbauer polynomial. It is well known that a generating function for the Gegenbauer polynomials $C_p^{\frac{n-2}{2}}$, p=0,1,2,..., is given as follows.

$$(1 - 2tz + t^2)^{-\frac{n-2}{2}} = \sum_{p=0}^{\infty} C_p^{\frac{n-2}{2}}(z)t^p, \quad -1 \le z \le 1, \ -1 < t < 1.$$

This fomula also gives a generating function for the zonal spherical functions of $SO(n)/SO(n-1)$.

The first purpose of this paper is to show that we can also give generating functions for the zonal spherical functions in the complex and the quaternion cases (The conclusions were announced in [6]). In case of $\mathbf{F}$=$\mathbf{C}$,$\mathbf{H}$, however, the zonal spherical functions on K/M are determined by two parameters while those of the real case are determined by one parameter p. So we have to extend the definition of generating function to the case of a system of functions which has two parameters. If $\mathbf{F}=\mathbf{C}$, we consider the function $F(z,w)$ defined in the following as a generating function for functions $G_{pq}(z)$ ($z \in D$, D is a subset of $\mathbf{C}$).

$$F(z,w) = \sum_{p,q=0}^{\infty} G_{pq}(z) w^p \overline{w}^q,$$

where the right hand side absolutely converges for some $w \in \mathbf{C}$, $w \neq 0$. Then we see that

$$G_{pq}(z) = \frac{1}{p!q!}\left[\frac{\partial^{p+q}}{\partial w^p \partial \overline{w}^q} F(z,w)\right]_{w=0}.$$

This is one reason why we consider the function $F(z,w)$ as a generating function. In the generating function expansion, if we put $w = re^{i\theta}$, then we have

$$F(z, re^{i\theta}) = \sum_{p,q=0}^{\infty} G_{pq}(z) r^{p+q} e^{i(p-q)\theta}.$$

We now remark that the function $e^{ik\theta}$ (k is an integer) is a spherical function on $U(1) \cong S(\mathbf{R}^2)$. Therefore the expansion of $F(z,re^{i\theta})$ by the powers of r and the spherical functions of $U(1) \cong S(\mathbf{R}^2)$ can be considered to give a generating function expansion for the functions G_{pq}. This interpretation for generating function will be adapted to the quaternion case, i.e. let $E(z,w)$ be a function defined on a subset of $\mathbf{H} \times \mathbf{H}$ and we set $w = ru$ $\big(r > 0,\ u \in Sp(1)\big)$. When we expand the function $E(z,ru)$ by the powers of r and the spherical functions of $Sp(1) \cong S(\mathbf{R}^4) \cong (Sp(1) \times Sp(1))/\Delta(Sp(1) \times Sp(1))$, $E(z,ru)$ is considered as a generating function for the functions of z which appear as the coefficients in that expansion.

The second purpose of this paper is to give integral representations for the zonal spherical functions on K/M in the complex and the quaternion cases. In particular, those in the complex case will give formulas which are analogous to Rodrigues' formulas.

Suppose that $n \geq 2$ throughout this paper.

1 Complex case

1.1 Generating function for the spherical functions on K/M

Let $H^{(n)}_{p,q}$ denote the space of restrictions to $S(\mathbf{C}^n)$ of harmonic polynomials $f(\xi,\overline{\xi})$ on $\mathbf{C}^n$ which are homogeneous of degree p in ξ and degree q in $\overline{\xi}$. Then it is known that (cf. [2], [4]) $H^{(n)}_{p,q}$ is $U(n)$-irreducible and moreover $L^2\big(S(\mathbf{C}^n)\big) = \bigoplus_{p,q=0}^{\infty} H^{(n)}_{p,q}$. Let $\varphi^{(n)}_{p,q}$ be the zonal spherical function which belongs to $H^{(n)}_{p,q}$. Then a generating function for the functions $\varphi^{(n)}_{p,q}$ is given in the following theorem.

Theorem 1.1 *If $w, z \in \mathbf{C}$, $|w| < 1$, $|z| \le 1$, then*

$$(1 - 2Re(wz) + |w|^2)^{1-n} = \sum_{p,q=0}^{\infty} a^{(n)}_{pq} Q^{(n)}_{pq}(z) w^p \overline{w}^q, \tag{1.1}$$

where

$$Q^{(n)}_{pq}(\eta_1) = \varphi^{(n)}_{pq}(\eta), \quad \eta \in S(\mathbf{C}^n),$$

and

$$a^{(n)}_{pq} = \frac{\Gamma(n+p-1)}{\Gamma(n-1)\Gamma(p+1)} \frac{\Gamma(n+q-1)}{\Gamma(n-1)\Gamma(q+1)}.$$

The series on the right hand side converges absolutely and uniformly for $|z| \le 1$ and $|w| \le \rho$ for each $\rho < 1$.

In the formula (1.1), if we put $w = re^{i\theta}$, then we have

$$(1 - 2rRe(e^{i\theta} z) + r^2)^{1-n} = \sum_{p,q=0}^{\infty} a^{(n)}_{pq} Q^{(n)}_{pq}(z) e^{i(p-q)\theta} r^{p+q}. \tag{1.2}$$

This formula can be interpreted as follows.
The zonal spherical functions $\varphi^{(n)}_{pq}$ appear as the coefficients in the expansion of the left hand side of (1.2) by the powers of r and the spherical functions of $U(1) \cong S(\mathbf{R}^2)$. This interpretation for generating function will be adapted to the quaternion case. See the formula (2.1).

1.2 Generalization of the formula (1.1)

Let ν be a positive number. Suppose that $z, w \in \mathbf{C}$ and $|w|^2 + 2|zw| < 1$. Then we have the following expansion which absolutely converges:

$$(1 - 2Re(wz) + |w|^2)^{-\nu} = \sum_{p,q=0}^{\infty} G_{pq}^{\nu}(z) w^p \overline{w}^q,$$

where the functions G_{pq}^{ν} are polynomials of z and $\overline{z}$.

Lemma 1.1 *The function G_{pq}^{ν} has the following expression:*

$$G_{pq}^{\nu}(z) = \sum_{k=0}^{min(p,q)} \frac{\Gamma(\nu + p + q - k)}{(p-k)!(q-k)!k!\Gamma(\nu)} (-1)^k z^{p-k} \overline{z}^{q-k}.$$

Lemma 1.2 *Suppose that $p \geq q$. Then for $-1 \leq x \leq 1$ we have*

$$G_{pq}^{\nu}(x) = \frac{(-1)^q \Gamma(\nu + p)}{\Gamma(\nu)\Gamma(p - q + 1) q!} x^{p-q} {}_2F_1(-q, \nu + p; p - q + 1; x^2).$$

We remark that

1. $G_{pq}^{\nu}(e^{i\theta} z) = e^{i(p-q)\theta} G_{pq}^{\nu}(z)$ for $\theta \in \mathbf{R}$ and $z \in \mathbf{C}, |z| \leq 1$.
2. $G_{pq}^{\nu}(x) = G_{qp}^{\nu}(x)$ for $-1 \leq x \leq 1$.
3. $G_{pq}^{n-1}(z) = a_{pq}^{(n)} Q_{pq}^{(n)}(z)$ for $z \in \mathbf{C}, |z| \leq 1$.

Proposition 1.1 *The functions G_{pq}^{ν} have the following orthogonality relation:*

$$\int_{|z| \leq 1} G_{pq}^{\nu}(z) \overline{G_{p'q'}^{\nu}(z)} (1 - |z|^2)^{\nu - 1} dx dy = \delta_{pp'} \delta_{qq'} \frac{\pi \Gamma(p + \nu) \Gamma(q + \nu)}{(p + q + \nu) p! q! [\Gamma(\nu)]^2},$$

with $z = x + iy$.

1.3 Integral representations and Rodrigues' formulas for the functions $Q_{pq}^{(n)}$

Let ν be a positive number. For $z, \xi, \eta \in \mathbf{C}$, we define the function $F_\nu(\xi, \eta, z)$ by

$$F_\nu(\xi, \eta, z) = (1 - \xi z - \eta \bar{z} + \xi\eta)^{-\nu}.$$

If $|z| \leq 1$, then we have

$$|\xi z + \eta \bar{z} - \xi\eta| \leq (|\xi| + 1)(|\eta| + 1) - 1.$$

Thus for a fixed $z \in \mathbf{C}, |z| \leq 1$, the function $F_\nu(\xi, \eta, z)$ is holomorphic on $U = \{(\xi, \eta) \in \mathbf{C}^2; |\xi| < 1/3, |\eta| < 1/3\}$ with respect to ξ, η and has the following Taylor expansion on U:

$$F_\nu(\xi, \eta, z) = \sum_{p,q=0}^{\infty} F_{pq}^{\nu}(z) \xi^p \eta^q,$$

where

$$F_{pq}^{\nu}(z) = \frac{1}{p!q!} \left[\frac{\partial^{p+q}}{\partial \xi^p \partial \eta^q} F_\nu(\xi, \eta, z) \right]_{\xi=\eta=0}.$$

On the other hand, we see that

$$\left[\frac{\partial^{p+q}}{\partial \xi^p \partial \eta^q} F_\nu(\xi, \eta, z) \right]_{\xi=\eta=0} = \left[\frac{\partial^{p+q}}{\partial \overline{w}^q \partial w^p} A(z, w) \right]_{w=0},$$

with $A(z, w) = (1 - wz - \overline{wz} + |w|^2)^{-\nu}$. So we obtain that

$$F_{pq}^{\nu}(z) = G_{pq}^{\nu}(z).$$

In particular, if we put $\nu = n - 1$, then we see that

$$F_{pq}^{n-1}(z) = G_{pq}^{n-1}(z) = a_{pq}^{(n)} Q_{pq}^{(n)}(z).$$

This formula and the Cauchy integral formula suggest the following theorem.

Theorem 1.2 *The function $Q_{pq}^{(n)}$ has the following integral representation and Rodrigues' formula.*

$$\begin{aligned} a_{pq}^{(n)} Q_{pq}^{(n)}(z) &= \frac{(n-1)_p}{2\pi i p!} (|z|^2 - 1)^{2-n} \int_{|\zeta - z| = r} \frac{\zeta^p (\zeta \bar{z} - 1)^{n+q-2}}{(\zeta - z)^{q+1}} d\zeta \\ &= \frac{(n-1)_p}{p!q!} (|z|^2 - 1)^{2-n} \frac{\partial^q}{\partial z^q} \left[z^p (|z|^2 - 1)^{n+q-2} \right], \end{aligned}$$

where $|z| < 1$.

And moreover we can show the following theorem, which gives relations between the functions $Q_{pq}^{(n)}$ and the spherical functions on $SO(2n)/SO(2n-1)$.

Theorem 1.3 *If* $z \in \mathbf{C}, |z| \leq 1$, *then*

$$a_{pq}^{(n)} Q_{pq}^{(n)}(z) = \frac{1}{2\pi} \int_0^{2\pi} C_{p+q}^{n-1}\Big(Re(e^{i\theta} z)\Big) e^{-i(p-q)\theta} d\theta.$$

2 Quaternion case

2.1 Generating function for the spherical functions on K/M

A zonal spherical function φ of K/M depends only on η_1, more precisely on $Re(\eta_1)$ and $|\eta_1|$ $\Big(\eta = {}^t(\eta_1, \ldots, \eta_n) \in S(\mathbf{H}^n)\Big)$, and there uniquely exists a pair of nonnegative integers (p, q) such that

$$\varphi(\eta) = c_{pq} C_p^1\Big(\frac{Re(\eta_1)}{|\eta_1|}\Big) |\eta_1|^p {}_2F_1(-q, p+q+2n-1; p+2; |\eta_1|^2),$$

where

$$c_{pq} = \frac{(-1)^q (p+2)_q}{\Big(2(n-1)\Big)_q} [C_p^1(1)]^{-1}.$$

See Theorem 3.1 in [4], p.144 and the formula (16) in [1], p.170 and we follow the notations in [5]. From now on, we denote φ by $\varphi_{pq}^{(n)}$. When we denote $\{f * \varphi_{pq}^{(n)} | f \in L^2(K/M)\}$ by $H_{p,q}^{(n)}$, $H_{p,q}^{(n)}$ is K-irreducible and moreover $H_k^{(4n)} \cong \bigoplus_{p+2q=k} H_{p,q}^{(n)}$, $L^2\Big(S(\mathbf{H}^n)\Big) = \bigoplus_{p,q=0}^{\infty} H_{p,q}^{(n)}$, where $H_k^{(4n)}$ is the space of restrictions to $S(\mathbf{R}^{4n})$ of harmonic polynomials on $\mathbf{R}^{4n}$ which are homogeneous of degree k. A generating function for the functions $\varphi_{pq}^{(n)}$ is given in the following theorem.

Theorem 2.1 *If* $z \in \mathbf{H}$, $|z| \leq 1$, $u \in Sp(1)$ *and* $0 \leq r < 1$, *then*

$$\begin{aligned} &\int_{Sp(1)} [1 - 2rRe(vzv^{-1}u) + r^2]^{1-2n} dv \\ &= \sum_{p,q=0}^{\infty} \beta_{pq}^{(n)} R_{pq}^{(n)}(z) C_p^1\Big(Re(u)\Big) r^{p+2q}, \qquad (2.\ 1) \end{aligned}$$

where dv is the normalized Haar measure on $Sp(1)$ and

$$R_{pq}^{(n)}(\eta_1) = \varphi_{pq}^{(n)}(\eta), \quad \eta \in S(\mathbf{H}^n),$$

and

$$\beta_{pq}^{(n)} = \frac{p+1}{\Gamma(p+q+2)} \frac{(2n-1)_{p+q}(2n-2)_q}{q!}.$$

The series on the right hand side converges absolutely and uniformly for $|z| \leq 1$, $u \in Sp(1)$ and $r \leq \rho$ for each $\rho < 1$.

The formula (2.1) means that the zonal spherical functions $\varphi_{pq}^{(n)}$ appear as the coefficients in the expansion of the left hand side of (2.1) by the powers of r and the spherical functions of $Sp(1) \cong S(\mathbf{R}^4)$. So we can consider that (2.1) gives a generating function for the functions $\varphi_{pq}^{(n)}$.

2.2 Generalization of the formula (2.1)

We suppose that $\nu > 1$ and define $c_{pq}(\nu)$, $\beta_{pq}(\nu)$ and Φ_{pq}^{ν} by

$$\begin{aligned} c_{pq}(\nu) &= \frac{(-1)^q (p+2)_q}{(\nu-1)_q}[C_p^1(1)]^{-1}, \\ \beta_{pq}(\nu) &= \frac{p+1}{\Gamma(p+q+2)} \frac{(\nu)_{p+q}(\nu-1)_q}{q!}, \\ \Phi_{pq}^{\nu}(z) &= c_{pq}(\nu) C_p^1\Big(\frac{Re(z)}{|z|}\Big)|z|^p {}_2F_1(-q, p+q+\nu; p+2; |z|^2). \end{aligned}$$

We can see the following theorem as a generalization of the formula (2,1).

Theorem 2.2 *If $\nu > 1$, then we have*

$$\begin{aligned} &\int_{Sp(1)} [1 - 2rRe(vzv^{-1}u) + r^2]^{-\nu} dv \\ &= \sum_{m=0}^{\infty} r^m \sum_{k+2s=m} \beta_{ks}(\nu) C_k^1\big(Re(u)\big)\Phi_{ks}^{\nu}(z), \end{aligned}$$

for $u \in Sp(1)$, $0 \leq r < 1$ and $z \in \mathbf{H}$, $|z| \leq 1$.

Here we give the orthogonality relation of the functions Φ_{pq}^{ν}. We set $\Psi_{pq}^{\nu} = \beta_{pq}(\nu)\Phi_{pq}^{\nu}$.

Proposition 2.1 *For $\nu > 1$, we have*

$$\int_{|z|\leq 1} \Psi^{\nu}_{pq}(z)\Psi^{\nu}_{k\ell}(z)(1-|z|^2)^{\nu-2}dz$$

$$= \delta_{pk}\delta_{q\ell}\frac{\pi^2\Gamma(p+q+\nu)\Gamma(q+\nu-1)}{[\Gamma(\nu)]^2(p+2q+\nu)q!\Gamma(p+q+2)},$$

where $dz = dz_1dz_2dz_3dz_4$, $z = z_1 + z_2 i + z_3 j + z_4 k$.

2.3 Integral representations for the functions $R^{(n)}_{pq}$

We consider integral representations for the spherical functions $R^{(n)}_{pq}$. In what follows, we shall use the following notations for $z \in \mathbf{H}$:

$$z = z_1 + iz_2 + jz_3 + kz_4,$$

where $z_\nu (1 \leq \nu \leq 4) \in \mathbf{R}$.

First of all, in the formula (2,1), if we put $u = e^{i\theta}$ and set $w = re^{i\theta}$, then we have,

$$\int_{Sp(1)} \left[1 - 2Re\left\{\left(z_1 + i(vzv^{-1})_2\right)w\right\} + |w|^2\right]^{1-2n} dv$$

$$= \sum_{p,q=0}^{\infty} \beta^{(n)}_{pq} R^{(n)}_{pq}(z)\Big(\sum_{k=0}^{p} w^{q+k}\overline{w}^{p+q-k}\Big). \qquad (2.\ 2)$$

Let m and ℓ be nonnegative integers and we suppose that $0 \leq \ell \leq m$. Applying the differential operator $\frac{\partial^m}{\partial w^\ell \partial \overline{w}^{m-\ell}}$ to (2,2), we put $w = 0$. Then we obtain that

$$\ell!(m-\ell)! \sum_{\substack{p+2q=m \\ q\leq \ell}} \beta^{(n)}_{pq} R^{(n)}_{pq}(z)$$

$$= \int_{Sp(1)} \left[\frac{\partial^m}{\partial w^\ell \partial \overline{w}^{m-\ell}}\left[1 - 2Re\left\{\left(z_1 + i(vzv^{-1})_2\right)w\right\} + |w|^2\right]^{1-2n}\right]_{w=0} dv.$$

Thus we can conclude that

Theorem 2.3 *If $z \in \mathbf{H}, |z| < 1$, then*

$$\sum_{\substack{p+2q=m \\ q\leq \ell}} \beta^{(n)}_{pq} R^{(n)}_{pq}(z)$$

$$= \frac{(2n-1)_\ell}{2\ell!(m-\ell)!}\int_0^\pi H_{m\ell}(z_1 + i\sqrt{z_2^2+z_3^2+z_4^2}\cos\theta)\sin\theta d\theta,$$

where
$$H_{m\ell}(w) = (|w|^2-1)^{2-2n}\frac{\partial^{m-\ell}}{\partial w^{m-\ell}}[w^\ell(w\overline{w}-1)^{2n+m-\ell-2}],$$
and
$$\frac{\partial}{\partial w} = \frac{1}{2}\Big(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\Big), \quad w = u+iv.$$

Secondly, we shall give relations between the functions $R^{(n)}_{pq}$ and $Q^{(2n)}_{k\ell}$. From the relations, we can see integral representations of the functions $R^{(n)}_{pq}$ by the functions $Q^{(2n)}_{k\ell}$.

Proposition 2.2 *For $z \in \mathbf{H}$, $|z| \le 1$, we have*
$$\begin{aligned}\beta^{(n)}_{k\ell}R^{(n)}_{k\ell}(z) &= a^{(2n)}_{\ell,k+\ell}\int_{Sp(1)} Q^{(2n)}_{\ell,k+\ell}\big(z_1+i(vzv^{-1})_2\big)dv \\ &\quad -a^{(2n)}_{\ell-1,k+\ell+1}\int_{Sp(1)} Q^{(2n)}_{\ell-1,k+\ell+1}\big(z_1+i(vzv^{-1})_2\big)dv,\end{aligned}$$
with $Q^{(2n)}_{-1,k+1} = 0$ and $a^{(2n)}_{-1,k+1} = 0$.

Then integral representations for the functions $R^{(n)}_{pq}$ are given in the following theorem.

Theorem 2.4 *For $z \in \mathbf{H}$, $|z| \le 1$, we have*
$$\begin{aligned}\beta^{(n)}_{k\ell}R^{(n)}_{k\ell}(z) &= \frac{a^{(2n)}_{\ell,k+\ell}}{2}\int_0^\pi Q^{(2n)}_{\ell,k+\ell}(z_1+i\sqrt{z_2^2+z_3^2+z_4^2}\cos\theta)\sin\theta d\theta \\ &\quad - \frac{a^{(2n)}_{\ell-1,k+\ell+1}}{2}\int_0^\pi Q^{(2n)}_{\ell-1,k+\ell+1}(z_1+i\sqrt{z_2^2+z_3^2+z_4^2}\cos\theta)\sin\theta d\theta \\ &= \int_0^\pi T^{(n)}_{k\ell}(z_1+i\sqrt{z_2^2+z_3^2+z_4^2}\cos\theta)\sin\theta d\theta,\end{aligned}$$
where the functions $T^{(n)}_{k\ell}(\xi)$ are defined for $\xi \in \mathbf{C}$ as follows:
$$\begin{aligned}T^{(n)}_{k\ell}(\xi) &= S^{(n)}_{k\ell}(\xi) - S^{(n)}_{k+2,\ell-1}(\xi), \quad (\ell \ge 1) \\ T^{(n)}_{k0}(\xi) &= S^{(n)}_{k0}(\xi),\end{aligned}$$
and
$$S^{(n)}_{k\ell}(\xi) = \frac{(-1)^\ell(2n-1)_{k+\ell}}{4k!\ell!}(\xi^k+\overline{\xi}^k)_2F_1(-\ell,k+\ell+2n-1;k+1;|\xi|^2).$$

We should notice that our assertions for $\ell = 0$ are nothing but the integral representations for C_k^1 which are well known. See the formula (31) in [1], p.177.

Finaly, we shall give integral representations by the spherical functions on $SO(4n)/SO(4n-1)$.

Theorem 2.5 *If $z \in \mathbf{H}, |z| \leq 1$, then*

$$\beta_{k\ell}^{(n)} R_{k\ell}^{(n)}(z)$$
$$= \frac{1}{2(n-1)\pi} \int_0^{\pi} D_{k+2\ell}^{2n-1}(z_1 \cos\theta, \sqrt{z_2^2 + z_3^2 + z_4^2} \sin\theta) \sin\theta \sin(k+1)\theta d\theta,$$

where

$$D_m^{\lambda}(x, y) = \frac{C_{m+1}^{\lambda-1}(x+y) - C_{m+1}^{\lambda-1}(x-y)}{2y} = (\lambda - 1) \int_{-1}^{1} C_m^{\lambda}(x+ty) dt.$$

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