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# On Non-unitary Representations of the Heisenberg Group

KEISAKU KUMAHARA

## 1 Introduction

Let $G$ be a connected, simply connected nilpotent Lie group and $\mathfrak{g}$ be its Lie algebra. Irreducible unitary representations can be described by the orbit method due to A.Kirillov([K]). Let $f$ be an element of the dual space $\mathfrak{g}'$ of $\mathfrak{g}$. Let $G_f$ be the isotropy subgroup of $f$ in $G$ and $\mathfrak{g}_f$ be its Lie algebra. The polarization at $f$ is, by definition([AK]), a complex subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\boldsymbol{C}}$ such that

(1) $\mathfrak{g}_f \subseteq \mathfrak{h}$ and $\mathfrak{h}$ is stable under $\mathrm{Ad}(G_f)$,

(2) $2\dim_{\boldsymbol{C}} \mathfrak{g}_{\boldsymbol{C}}/\mathfrak{h} = \dim_{\boldsymbol{R}} \mathfrak{g}/\mathfrak{g}_f$,

(3) $f|[\mathfrak{h},\mathfrak{h}] = 0$,

(4) $\mathfrak{h} + \overline{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}_{\boldsymbol{C}}$.

A polarization $\mathfrak{h}$ is called real if $\mathfrak{h}=\overline{\mathfrak{h}}$. Let $\mathfrak{h}$ be a real polarization at $f$. Then $f$ defines a one-dimensional representation $\tau_f$ of a subgroup $D = \exp(\mathfrak{g} \cap \mathfrak{h})$ of $G$. We can get an irreducible unitary representation $U^{f,\mathfrak{h}}$ of $G$, which is the induced representation from the representation $\tau_f$ of $D$. The representation $U^{f,\mathfrak{h}}$ is independent of $\mathfrak{h}$ and depends only on the coadjoint orbit containing $f$. And any irreducible unitary representation of $G$ is equivalent to one of $U^{f,\mathfrak{h}}$.

In the present paper we treat the Heisenberg group of $n$-th order. Irreducible unitary representations of the Heisenberg group are parametrized by unitary characters of the center. V.S.Petrosyan([P]) studied the irreducibility of non-unitary representations of the Heisenberg group corresponding to non-unitary characters of the center. To prove the operator irreducibility he used the method of the invariant bilinear forms which is used in [GGV].

In our case, for any linear form which does not vanish on the center all real polarization is abelian. So any element $\Lambda \in \mathfrak{g}'_{\boldsymbol{C}}$ defines a non-unitary character of $\mathfrak{h}$. We study the invariant bilinear forms for non-unitary representations corresponding to non-unitary characters of polarizations.We define a representation $\tau_\Lambda$ of $D$ by $\tau_\Lambda(\exp X) = \exp(\sqrt{-1}\Lambda(X)), X \in \mathfrak{g} \cap \mathfrak{h}$. And we define a non-unitary representation $U^{\Lambda,\mathfrak{g}}$ of $G$ induced from $\tau_\Lambda$. We realize it on the functions space $\mathcal{D}(G/D)$ of $C^\infty$-functions on $G/D$ with compact support. In our case if $\Lambda \neq 0$ on the center of $\mathfrak{g}$, then $G/D \cong \boldsymbol{R}^n$. So we denote $\mathcal{D}(\boldsymbol{R}^n)$ by $\mathcal{D}_\Lambda^{\mathfrak{h}}$ as the representation space of $U^{\Lambda,\mathfrak{h}}$. We get a necessary and sufficient condition to exist an invariant bilinear form on $\mathcal{D}_{\Lambda_1}^{\mathfrak{h}} \times \mathcal{D}_{\Lambda_2}^{\mathfrak{h}}$(Theorem 1). And we also get a necessary and sufficient condition to exist an intertwining operator from $\mathcal{D}_{\Lambda_1}^{\mathfrak{h}}$ to $\mathcal{D}_{\Lambda_2}^{\mathfrak{h}}$(Theoem 2). We call the representation $U^{\Lambda,\mathfrak{h}}$ is operator irreducible if any intertwining operator from

$\mathcal{D}_A^{\mathfrak{h}}$ to itself is a scalar multiple of the identity operator. We prove that $U^{A,\mathfrak{h}}$ is operator irreducible if $A$ is not zero on the center of $\mathfrak{g}$(Theorem 3). Moreover, we determine when the representation $U^{A,\mathfrak{h}}$ is unitary(Theorem 4).

In Theorem 5 we prove that two representations $U^{A_1,\mathfrak{h}}$ and $U^{A_2,\mathfrak{h}}$ are equivalent if and only if they are on tha same orbit in $(\mathfrak{g}')_C$ of a subgroup $B$ of $G_C$ containing $G$. In §7 we study an invariant bilinear form for representations corresponding to different types of polarizations. Generally, the intertwinig operators are not operartors of $\mathcal{D}$. In §8 we get an intertwinig operator for unitary representations corresponding to different type of polarizations on the Schwartz space $\mathcal{S}$. This is an integral operator whose kernel is exponential of a polynomial of degree 2(Theorem 9).

## 2 Representations of Heisenberg group

Let $G$ be the Heisenberg group of $n$-th order:

$$G = \left\{ g = g(a,b,c) = \begin{pmatrix} 1 & a_1 & \dots & a_n & c \\ & 1 & 0 & & b_1 \\ & & \ddots & & \vdots \\ & & & & b_n \\ 0 & & & & 1 \end{pmatrix} \right\}.$$

Let

$$\mathfrak{g} = \left\{ X = \begin{pmatrix} 0 & x_1 & \dots & x_n & z \\ & 0 & 0 & & y_1 \\ & & \ddots & & \vdots \\ & & & & y_n \\ 0 & & & & 0 \end{pmatrix} \right\}$$

be the Lie algebra of $G$. We denote by $\mathfrak{g}' = \{A = (\lambda, \mu, \nu) | \lambda, \mu \in \boldsymbol{R}^n, \nu \in \boldsymbol{R}\}$ the real dual space of $\mathfrak{g}$ defined by

$$< A, X >= \lambda \cdot x + \mu \cdot y + \nu z.$$

The group $G$ acts on $\mathfrak{g}'$ by coadjoint:

$$< g \cdot A, X >=< A, \mathrm{Ad}(g^{-1})X > .$$

Then

$$g(a,b,c) \cdot (\lambda, \mu, \nu) = (\lambda + \nu b, \mu - \nu a, \nu).$$

For any $A \in \mathfrak{g}'$ we denote by $G_A$ the isotropy subgroup of $A$ in $G$. If $\nu \neq 0$, then $G_A = \{(0,0,c) \in G\}$, which is the center of $G$. The Lie algebra of $G_A$ is

$\mathfrak{g}_\Lambda = \{(0,0,z) \in \mathfrak{g}\}$. The polarization for Heisenberg group at $\Lambda = (\lambda, \mu, \nu)$ for $\nu \neq 0$ is

$$\mathfrak{h}_\infty = \{(x, 0, z) | x, z \in \boldsymbol{C}\}$$

or

$$\mathfrak{h}_k = \{(ky, y, z) | y, z \in \boldsymbol{C}\}$$

for some $k \in \boldsymbol{C}$. It is easy to see that $\mathfrak{h}_\infty$ is real and $\mathfrak{h}_k$ is real if and only if $k \in \boldsymbol{R}$. We assume that $k$ is real. These algebras $\mathfrak{h}_\infty$ and $\mathfrak{h}_k$ are abelian.

Let $D_k = \exp(\mathfrak{g} \cap \mathfrak{h}_k)$. Then $D_k = \{p = (kb, b, c) \in G\}$. Each $\Lambda = (\lambda, \mu, \nu) \in (\mathfrak{g}')_{\boldsymbol{C}}$ defines one-dimensional representation of $D_k$. We denote it by $\tau_\Lambda$:

$$\tau_\Lambda(h) = \tau_\Lambda((kb, b, c)) = \exp\sqrt{-1}\{k\lambda \cdot b + \mu \cdot b + \nu(c - (kb \cdot b)/2)\}$$

for $h = (kb, b, c)$. Let $A = \{(a, 0, 0) \in G\}$. Let $U^{\Lambda,k}$ be a continuous representation of $G$ induced by the representation $\Lambda$ of $D_k$. Let $\mathcal{D}(\boldsymbol{R}^n)$ be the space of infinitely differentiable functions with compact support on $\boldsymbol{R}^n$ with usual topology. We realize $U^{\Lambda,k}$ on $C_c^\infty(A) \cong \mathcal{D}(\boldsymbol{R}^n)$, which we denote by $\mathcal{D}_\Lambda^k$:

$$(2.1) \qquad (U_g^{\Lambda,k}F)(x) = e^{\sqrt{-1}\{(k\lambda+\mu)\cdot b + \nu(c - b\cdot x - (kb\cdot b)/2)\}}F(x - a + kb)$$

for $g = (a, b, c) \in G$ and $F \in \mathcal{D}_\Lambda^k$.

In the same way,

$$(2.2) \qquad (U_g^{\Lambda,\infty}F)(x) = e^{\sqrt{-1}\{\lambda\cdot a + \nu(c + a\cdot x - a\cdot b)\}}F(x - b)$$

for $g = (a, b, c) \in G$ and $F \in \mathcal{D}_\Lambda^k$.

## 3 Invariant bilinear forms on $\mathcal{D}_{\Lambda_1}^k \times \mathcal{D}_{\Lambda_2}^k$

Let $B$ be a non-zero continuous bilinear form on $\mathcal{D}_{\Lambda_1}^k \times \mathcal{D}_{\Lambda_2}^k$ ($k \in \boldsymbol{R} \cup \{\infty\}$) which is invariant under $G$, that is

$$B(U_g^{\Lambda_1,k}F_1, U_g^{\Lambda_2,k}F_2) = B(F_1, F_2)$$

for all $g \in G$, $F_1 \in \mathcal{D}_{\Lambda_1}^k$ and $F_2 \in \mathcal{D}_{\Lambda_2}^k$.

THEOREM 1. *Let $\Lambda_1 = (\lambda_1, \mu_1, \nu_1), \Lambda_2 = (\lambda_2, \mu_2, \nu_2) \in \boldsymbol{C}^{2n+1}$ and $k \in \boldsymbol{R}$. There exist a continuous non-trivial invariant bilinear form on $\mathcal{D}_{\Lambda_1}^k \times \mathcal{D}_{\Lambda_2}^k$ if and only if*

$$(1) \qquad \nu_1 = -\nu_2 \neq 0, \quad \frac{k(\lambda_1 + \lambda_2) + \mu_1 + \mu_2}{\nu_1} \in \boldsymbol{R}^n$$

or

(2) $$\nu_1 = \nu_2 = 0, \quad k(\lambda_1 + \lambda_2) + \mu_1 + \mu_2 = 0.$$

*When* (1) *holds,*

$$B(F_1, F_2) = C \int_{\mathbf{R}^n} F_1(x) F_2(x - \frac{k(\lambda_1 + \lambda_2) + \mu_1 + \mu_2}{\nu_1}) dx$$

*for non zero $C \in \mathbf{C}$. In the case of* (2)

$$B(F_1, F_2) = < B_0, \int_{\mathbf{R}^n} F_1(y) F_2(x + y) dy >$$

*for any distribution $B_0$ on $\mathbf{R}^n$.*

*When $k = \infty$, it suffices only to change $k\lambda_1 + \mu_1$ to $\lambda_1$ and $k\lambda_2 + \mu_2$ to $\lambda_2$.*

Remark that there always exits a continuous invariant bilinear form on $\mathcal{D}^k_\Lambda \times \mathcal{D}^k_{-\Lambda}$.

## 4 Intertwining operators between $\mathcal{D}^k_{\Lambda_1}$ and $\mathcal{D}^k_{\Lambda_2}$

Let $A = A(\Lambda_1, \Lambda_2)$ be a non-trivial intertwinig operator between $\mathcal{D}^k_{\Lambda_1}$ and $\mathcal{D}^k_{\Lambda_2}$, *i.e.* it is a continuous linear mapping of $\mathcal{D}^k_{\Lambda_1}$ to $\mathcal{D}^k_{\Lambda_2}$ such that

$$AU_g^{\Lambda_1, k} = U_g^{\Lambda_2, k} A$$

for all $g \in G$.

THEOREM 2. *There exists an intertwining operator between $\mathcal{D}^k_{\Lambda_1} \times \mathcal{D}^k_{\Lambda_2}$ if and only if*

(1) $$\nu_1 = \nu_2 \neq 0, \quad \frac{(k\lambda_1 + \mu_1) - (k\lambda_2 + \mu_2)}{\nu_1} \in \mathbf{R}^n.$$

or

(2) $$\nu_1 = \nu_2 = 0, \quad k\lambda_1 + \mu_1 = k\lambda_2 + \mu_2.$$

*In the case of* (1)

$$(A(\Lambda_1, \Lambda_2)F)(x) = CF\left(x + \frac{(k\lambda_1 + \mu_1) - (k\lambda_2 + \mu_2)}{\nu_1}\right), \qquad C \neq 0.$$

*And in the case of* (1) *$A(\Lambda_1, \Lambda_2)$ is a continuous operator on $\mathcal{D}$ which commutes with translations.*

*When $k = \infty$, it suffices only to change $k\lambda_1 + \mu_1$ to $\lambda_1$ and $k\lambda_2 + \mu_2$ to $\lambda_2$.*

In the case (1) of Theorem 2 we put $\Lambda_1 = \Lambda_2$, then we have

$$A(\Lambda_1, \Lambda_2) = CI.$$

Thus we have the following theorem.

THEOREM 3. *The representation* $U^{\Lambda,k}(\Lambda = (\lambda,\mu,\nu))$ *on* $\mathcal{D}^k_\Lambda$ $(k \in \boldsymbol{R} \cup \{\infty\})$ *is operator irreducible if* $\nu \neq 0$.

## 5 Invariant hermitian form on $\mathcal{D}^k_\Lambda$

A hermitian form $H(F_1, F_2)$ on $\mathcal{D}^k_\Lambda$ is said invariant if

$$H(U_g^{\Lambda,k} F_1, U_g^{\Lambda,k} F_2) = H(F_1, F_2)$$

for all $F_1, F_2 \in \mathcal{D}^k_\Lambda$. Let $H(F_1, F_2)$ be a non-trivial continuous hermitian form on $\mathcal{D}^k_\Lambda$.

THEOREM 4. *There exits a non-trivial continuous invariant hermitian form $H$ on $\mathcal{D}^k_\Lambda$ if and only if $\Lambda = (\lambda,\mu,\nu) \in \boldsymbol{R}^{2n+1}$. If $\nu \neq 0$, then*

$$H(F_1, F_2) = C \int_{\boldsymbol{R}^n} F_1(x)\overline{F}_2(x)dx, \qquad C \neq 0.$$

## 6 $G$-and $B_k$-orbits in $(\mathfrak{g}')_C$

Let $\Lambda_1, \Lambda_2 \in (\mathfrak{g}')_C$. Two elements $\Lambda_1 = (\lambda_1, \mu_1, \nu_1)$ and $\Lambda_2 = (\lambda_2, \mu_2, \nu_2)$ are on the same $G-$orbit if and only if $\Lambda_2 = g \cdot \Lambda_1$ for some $g = g(a, b, c) \in G$. Then

$$\lambda_2 = \lambda_1 + \nu_1 b, \qquad \mu_2 = \mu_1 - \nu_1 a \qquad \text{and} \qquad \nu_1 = \nu_2.$$

Hence if $\nu_1 = 0$, then

$$\lambda_1 = \lambda_2, \qquad \mu_1 = \mu_2 \qquad \text{and} \qquad \nu_2 = 0.$$

We assume that $k$ is not infinity. If $\nu_1 \neq 0$, then

$$\nu_1 = \nu_2, \quad \text{and} \quad \frac{(k\lambda_2 + \mu_2) - (k\lambda_1 + \mu_1)}{\nu_1} = kb - a \in \boldsymbol{R}^n.$$

Then $U^{\Lambda_1,k}$ is equivalent to $U^{\Lambda_2,k}$ from Theorem 1. However, even if two representations $U^{\Lambda_1,k}$ and $U^{\Lambda_2,k}$ are equivalent, $\Lambda_1$ and $\Lambda_2$ are not necessarily on the same $G$-orbit.

Let $G_C$ be the complexification of $G$, *i.e.* the group of elements

$$g = g(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & \alpha_1 & \dots & \alpha_n & \gamma \\ & 1 & 0 & & \beta_1 \\ & & \ddots & & \vdots \\ & & & & \beta_n \\ 0 & & & & 1 \end{pmatrix}$$

$(\alpha,\beta \in \boldsymbol{C}^n, \gamma \in \boldsymbol{C})$. The Lie algebra of $G_{\boldsymbol{C}}$ is $\mathfrak{g}_{\boldsymbol{C}}$. Then $G_{\boldsymbol{C}}$ acts naturally on $(\mathfrak{g}')_{\boldsymbol{C}}$:

$$g(\alpha,\beta,\gamma)\cdot(\lambda,\mu,\nu) = (\lambda+\nu\beta, \mu-\nu\alpha,\nu).$$

Let $B_k$ be a subgroup of $G_{\boldsymbol{C}}$ consisting of elements

$$g = g(a+\sqrt{-1}ku, b+\sqrt{-1}u,\gamma), \qquad a,b,u \in \boldsymbol{R}^n,\ \gamma \in \boldsymbol{C}. \tag{6.1}$$

We assume that $\Lambda_1$ and $\Lambda_2$ satisfiy the condition (1) of Theorem 2. We put

$$\frac{\lambda_1-\lambda_2}{\nu_1} = -b-\sqrt{-1}u,$$
$$\frac{\mu_1-\mu_2}{\nu_1} = a+\sqrt{-1}u,$$

where $a,b,u \in \boldsymbol{R}^n$. We put

$$g = (a+\sqrt{-1}ku, b+\sqrt{-1}u, 0).$$

Then $g\cdot\Lambda_1 = \Lambda_2$.

Conversely, if $g\cdot\Lambda_1 = \Lambda_2$ for $g \in B_k$ of the form (6.1), then it is easy to see that $\Lambda_1$ and $\Lambda_2$ satisfiy the condition (1) of Theorem 2.

Next we assume that $k = \infty$. If $\nu_1 = \nu_2$ and $(\lambda_2-\lambda_1)/\nu_1 \in \boldsymbol{R}^n$, we put $b = (\lambda_1-\lambda_2)/\nu_1$. We define a subgroup

$$B_\infty = \{g(\alpha,b,\gamma) \in G_{\boldsymbol{C}} | \alpha \in \boldsymbol{C}^n, b \in \boldsymbol{R}^n, \gamma \in \boldsymbol{C}\}.$$

Then $U^{\Lambda_1,\infty}$ and $U^{\Lambda_2,\infty}$ are equivalent if and only if $\Lambda_1$ and $\Lambda_2$ belong to the same $B_\infty$-orbit in $(\mathfrak{g}')_{\boldsymbol{C}}$. Thus we have the following theorem.

**Theorem 5.** *Let $k \in \boldsymbol{R}\cup\{\infty\}$. We assume that $\Lambda_1 = (\lambda_1,\mu_1,\nu_1), \Lambda_2 = (\lambda_2,\mu_2,\nu_2) \in (\mathfrak{g}')_{\boldsymbol{C}}$ and $\nu_1 \neq 0, \nu_2 \neq 0$. Then two representations $U^{\Lambda_1,k}$ and $U^{\Lambda_1,k}$ are equivalent if and only if $\Lambda_1$ and $\Lambda_2$ are on the same $B_k$-orbit in $(\mathfrak{g}')_{\boldsymbol{C}}$. Especially, if they are on the same G-orbit, then they are equivalent.*

## 7 Invariant bilinear forms on $\mathcal{D}^0_{\Lambda_1} \times \mathcal{D}^k_{\Lambda_2}$ $\quad (k \neq 0)$

In this section we consider an invariant bilinear form on $\mathcal{D}^0_{\Lambda_1} \times \mathcal{D}^k_{\Lambda_2}$ for $k \neq 0,\infty$. Let $B$ be a non-trivial continuous bilinear form on $\mathcal{D}^0_{\Lambda_1} \times \mathcal{D}^k_{\Lambda_2}$ such that

$$\begin{aligned} &B(F_1,F_2) \\ &\quad = B(e^{\sqrt{-1}\{\mu_1\cdot b+\nu_1(c-b\cdot x)\}}F_1(x-a), \\ &\qquad e^{\sqrt{-1}\{(k\lambda_2+\mu_2)\cdot b+\nu_2(c-b\cdot x-(kb\cdot b)/2\}}F_2(x-a+kb)) \end{aligned}$$

for all $a, b \in \boldsymbol{R}^n, c \in \boldsymbol{R}, F_1 \in \mathcal{D}^0_{\Lambda_1}$ and $F_2 \in \mathcal{D}^k_{\Lambda_2}$.

THEOREM 6. *Let $k \neq 0, \infty$. If $\nu_2 = -\nu_1$, then there exists a continuous invariant bilinear form on $\mathcal{D}^0_{\Lambda_1} \times \mathcal{D}^k_{\Lambda_2}$. It is of the form*

$$B(F_1, F_2) = C \iint_{\boldsymbol{R}^{2n}} e^{\frac{\sqrt{-1}}{2k}\{\nu_1|x-y|^2 - 2(k\lambda_2+\mu_1+\mu_2)(x-y)\}} F_1(x)F_2(y)dxdy,$$

*where $C$ is a non-zero constant.*

## 8 Application to intertwinig operators of the unitary representations $U^{\Lambda,k}$

We assume that $\Lambda$ is real. Then the representation $U^{\Lambda,k}$ can be realized on the Schwartz space $\mathcal{S}(\boldsymbol{R}^n)$ of rapidly decreasing functions by (2.1) and (2.2). We denote $\mathcal{S}(\boldsymbol{R}^n)$ by $\mathcal{S}^k_\Lambda$ as the representation space of $U^{\Lambda,k}$. Then we can prove tha following theorems by the same way of the proof of Theorem 1 and Theore m 6.

THEOREM 7. *Let $\Lambda_1 = (\lambda_1, \mu_1, \nu_1), \Lambda_2 = (\lambda_2, \mu_2, \nu_2) \in \boldsymbol{R}^{2n+1}$ and $k \in \boldsymbol{R}$. There exist a continuous non-trivial invariant bilinear form on $\mathcal{S}^k_{\Lambda_1} \times \mathcal{S}^k_{\Lambda_2}$ if and only if*

(1)
$$\nu_1 = -\nu_2 \neq 0$$

*or*

(2)
$$\nu_1 = \nu_2 = 0, \quad k(\lambda_1 + \lambda_2) + \mu_1 + \mu_2 = 0.$$

*When (1) holds,*

$$B(F_1, F_2) = C \int_{\boldsymbol{R}^n} F_1(x)F_2(x - \frac{k(\lambda_1 + \lambda_2) + \mu_1 + \mu_2}{\nu_1})dx$$

*for non zero $C \in \boldsymbol{C}$. In the case of (2)*

$$B(F_1, F_2) = < B_0, \int_{\boldsymbol{R}^n} F_1(y)F_2(x + y)dy >$$

*for any tempered distribution $B_0$ on $\boldsymbol{R}^n$.*

THEOREM 8. *Let $k \neq 0, \infty$ and $\Lambda_1 = (\lambda_1, \mu_1, \nu_1), \Lambda_2 = (\lambda_2, \mu_2, \nu_2) \in \boldsymbol{R}^{2n+1}$. If $\nu_2 = -\nu_1$, then there exists a continuous invariant bilinear form on $\mathcal{S}^0_{\Lambda_1} \times \mathcal{S}^0_{\Lambda_2}$. It is of the form*

$$B(F_1, F_2) = C \iint_{\boldsymbol{R}^{2n}} e^{\frac{\sqrt{-1}}{2k}\{\nu_1|x-y|^2 - 2(k\lambda_2+\mu_1+\mu_2)(x-y)\}} F_1(x)F_2(y)dxdy,$$

*where $C$ is a non-zero constant.*

THEOREM 9. *Let $k \neq 0, \infty$, $\Lambda_1 = (\lambda_1, \mu_1, \nu_1), \Lambda_2 = (\lambda_2, \mu_2, \nu_2) \in \boldsymbol{R}^{2n+1}$. We assume that $\nu_1 = \nu_2 \neq 0$. Then any non-trivial intertwinig operator of $\mathcal{S}^k_{\Lambda_2}$ to $\mathcal{S}^0_{\Lambda_1}$ is given by*

$$(A(\Lambda_2, \Lambda_1)F)(x) = C \int_{\boldsymbol{R}^n} e^{-\frac{\sqrt{-1}}{2k}\{\nu_1|x-y|^2 - 2(k\lambda_2 - \mu_1 + \mu_2)(x-y)\}} F(y)dy$$

*for non-zero constant $C$.*

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Keisaku KUMAHARA
Department of Mathematics
Faculty of General Education
Tottori University
Tottori 680
JAPAN