表現論シンポジウム講演集, 1992
pp.104-108

On a representation of the algebra of invariant differential operators on a homogeneous vector bundle

Shin-ichi AZUMA (東　心一)
School of Science and Engineering
Waseda University

Let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup of G. E_τ denotes the homogeneous vector bundle over G/K associated to $(\tau, V_\tau) \in \hat{K}$, and $D(E_\tau)$ the algebra of G-invariant differential operators on E_τ (cf. [6]).

1 Irreducible representations of $D(E_\tau)$

We set

$$\Gamma_\tau(W) = \mathrm{Hom}_K(V_{\tau^*}, W)$$

for any irreducible $(\mathfrak{g}, K)$-module W (cf. [7]). Here τ^* denotes the contragredient of τ. $z \in U(\mathfrak{g}_c)^K$ acts on $T \in \Gamma_\tau(W)$ by

$$[z.T](v) = z.T(v) \quad (v \in V_{\tau^*}).$$

Proposition 1 *$\Gamma_\tau(W)$ is an irreducible $D(E_\tau)$-module unless trivial.*

A sketch of proof. The claim follows directly from [3] or

$$D(E_\tau) \simeq U(\mathfrak{g}_c)^K / U(\mathfrak{g}_c)^K \cap U(\mathfrak{g}_c)\mathcal{I}^\top \quad ([4, \text{Theorem } 1.3])$$

and

Fact 2 (cf. [3], [7]) *$\Gamma_\tau(W)$ is an irreducible $U(\mathfrak{g}_c)^K$-module unless trivial.*

2 The classification theorem of Langlands

See [1] or [7] for details.

Definition 3 *We say that a triple (P, σ, λ) is Langlands data if P is a standard parabolic subgroup with the langlands decomposition $P = MAN$ and σ is an irreducible unitary representation of M such that $(V_\sigma)_{K\cap M}$ is a tempered $(\mathfrak{m}, K \cap M)$-module and $\lambda \in \mathfrak{a}_c^*$ satisfies $\mathrm{Re}(\lambda, \alpha) > 0$ for all positive roots α of $\mathfrak{a}$ in $\mathfrak{g}$.*

Let $I_{P,\sigma,\lambda}$ be a $(\mathfrak{g}, K)$-module associated to the principal series representation.

Definition 4 *For $\beta_{P,\sigma,\lambda} : I_{P,\sigma,\lambda} \longrightarrow (V_\sigma)_{K\cap M}$ defined by*

$$< \beta_{P,\sigma,\lambda}(f), w >= \int_{\bar{N}} < f(\bar{n}), w > d\bar{n} \quad (w \in V_\sigma),$$

there exists a $(\mathfrak{g}, K)$-homomorphism $j_{P,\sigma,\lambda} : I_{P,\sigma,\lambda} \longrightarrow I_{\bar{P},\sigma,\lambda}$ such that

$$j_{P,\sigma,\lambda}(f)(1) = \beta_{P,\sigma,\lambda}(f).$$

Then we define

$$J_{P,\sigma,\lambda} = \mathrm{Im} j_{P,\sigma,\lambda} \simeq I_{P,\sigma,\lambda}/\mathrm{Ker} j_{P,\sigma,\lambda}.$$

We call $J_{P,\sigma,\lambda}$ a Langlands quotient corresponding to (P, σ, λ).

Remark 5 *When (P, σ, λ) is Langlands data, $\beta_{P,\sigma,\lambda}$ and $j_{P,\sigma,\lambda}$ are well-defined since the above integral converges absolutely.*

Fact 6 (cf. [1], [7]) *For any irreducible $(\mathfrak{g}, K)$-module W, there exists Langlands data (P, σ, λ) such that*

$$W \simeq J_{P,\sigma,\lambda}$$

as a $(\mathfrak{g}, K)$-module.

3 A nontrivial condition of $\Gamma_\tau(J_{P,\sigma,\lambda})$

It is enough to consider $\Gamma_\tau(J_{P,\sigma,\lambda})$ as $\Gamma_\tau(W)$ from §2.

Theorem 7 *$\Gamma_\tau(J_{P,\sigma,\lambda}) \neq 0$ if and only if there exists $f \in I_{P,\sigma,\lambda}(\tau^*)$ such that*

$$\int_{\bar{N}} < f(\bar{n}), w > d\bar{n} \neq 0 \quad (\textit{ for some } w \in V_\sigma).$$

A sketch of proof. The assertion follows from the definition of $j_{P,\sigma,\lambda}$ and

$$\begin{aligned} \Gamma_\tau(J_{P,\sigma,\lambda}) \neq 0 \quad &\Leftrightarrow \quad J_{P,\sigma,\lambda}(\tau^*) \neq 0 \\ &\Leftrightarrow \quad I_{P,\sigma,\lambda}(\tau^*) \not\subseteq \mathrm{Ker} j_{P,\sigma,\lambda}. \end{aligned}$$

Corollary 8 *Let P be minimal. Then $\Gamma_\tau(J_{P,\sigma,\lambda}) \neq 0$ if and only if there exists $f \in I_{P,\sigma,\lambda}(\tau^*)$ such that*

$$\int_{\bar{N}} e^{-(\lambda+\rho)H(\bar{n})} < \tau^*(\kappa(\bar{n})^{-1})f(1), w > d\bar{n} \neq 0 \quad (\textit{ for some } w \in V_\sigma).$$

A sketch of proof. The Iwasawa decomposition and the functional equation of $I_{P,\sigma,\lambda}$ imply that

$$\begin{aligned} f(\bar{n}) &= e^{-(\lambda+\rho)H(\bar{n})} f(\kappa(\bar{n})) \\ &= e^{-(\lambda+\rho)H(\bar{n})}\tau^*(\kappa(\bar{n})^{-1})f(1). \end{aligned}$$

Hence the result follows from the theorem above.

4 An application to the Poisson transform

We introduce the Poisson transform following [4]. We assume P is minimal below. Let $\mathcal{B}(G,(\sigma,\lambda))$ be the space of hyperfunctions from G to V_σ such that

$$f(gman) = e^{(\lambda-\rho)H(a)}\sigma(m^{-1})f(g) \quad (g \in G, m \in M, a \in A, n \in N),$$

and $\mathcal{A}(E_\tau)$ the space of analytic sections of E_τ. We identify $\mathcal{B}(F_{\tau,\lambda})$ with $\mathcal{B}(G,(\tau|_M,\lambda))$. $\mathcal{P}_{\tau,\lambda} : \mathcal{B}(F_{\tau,\lambda}) \to \mathcal{A}(E_\tau)$ defined by

$$\mathcal{P}_{\tau,\lambda}(\phi)(g) = \int_K \tau(k)\phi(gk)dk \quad (g \in G, k \in K)$$

is called the Poisson transform for E_τ. Put $H_\sigma = \mathrm{Hom}_M(V_\sigma, V_\tau)$. $\mathcal{P}_{\tau,\sigma,\lambda}$ denotes the restriction of $\mathcal{P}_{\tau,\lambda}$ to $\mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$. If we set $H = \Gamma_\tau(J_{P,\sigma,\lambda})$, then (χ, H) is (equivalent to) a subrepresentation of $(\chi_{\tau,\sigma,-\lambda}, H_\sigma)$ of $D(E_\tau)$. Since $H^* \subseteq H_\sigma^* \simeq H_{\tau,\sigma} = \mathrm{Hom}_M(V_\tau, V_\sigma)$ we can regard $V_\tau \otimes H^* \otimes H$ as a subspace of $\mathcal{B}(F_{\sigma,-\lambda})^\tau \otimes H$ by

$$V_\tau \otimes H^* \otimes H \subseteq V_\tau \otimes H_{\tau,\sigma} \otimes H \simeq \mathcal{B}(F_{\sigma,-\lambda})^\tau \otimes H.$$

Lemma 9 *Let $\sigma \in R_\tau = \{\sigma \in \hat{M} \mid [\tau : \sigma] > 0\}$ and $H = \Gamma_\tau(J_{P,\sigma,\lambda})$. Then $\mathcal{P}_{\tau,\sigma,-\lambda}$ is injective on $V_\tau \otimes H^* \otimes H$ if there exists $f \in I_{P,\sigma,\lambda}(\tau^*)$ such that*

$$\int_{\bar{N}} e^{-(\lambda+\rho)H(\bar{n})} < \tau^*(\kappa(\bar{n})^{-1})f(1), w > d\bar{n} \neq 0 \quad (\textit{ for some } w \in V_\sigma).$$

A sketch of proof. Similar arguments as [4, Lemma 3.4 and Theorem 3.5] complete the proof. It is essencial that H is irreducible unless trivial by its construction.

5 A description in terms of C-function

Lemma 9 gives a connection between C-function and the Poisson transform (cf. [5]). We introduce the notation in [2]. For $\tau \in \hat{K}$ and $\lambda \in \mathfrak{a}_c^*$, we set

$$B_\tau(\bar{P} : P : \lambda) = \int_{\bar{N}} \tau(\kappa(\bar{n})^{-1}) e^{-(\lambda+\rho)H(\bar{n})} d\bar{n}.$$

Since $B_\tau(\bar{P} : P : \lambda)$ commutes with the action of M, we put

$$B_\tau^\sigma(\bar{P} : P : \lambda) = B_\tau(\bar{P} : P : \lambda)|_{V_\tau(\sigma)}$$

for $\sigma \in \hat{M}$. $\lambda \mapsto B_\tau^\sigma(\bar{P} : P : \lambda)$ is called Harish-Chandra's C-function. For $v \in V_\tau$ and $T \in \mathrm{Hom}_M(V_\tau, V_\sigma)$, we define $f_{v,T} : G \to V_\sigma$ by

$$f_{v,T}(kan) = e^{-(\lambda+\rho)H(a)} T(\tau(k^{-1})v) \quad (k \in K, a \in A, n \in N).$$

Then $V_\tau \otimes \mathrm{Hom}_M(V_\tau, V_\sigma) \to I_{P,\sigma,\lambda}(\tau)$, $v \otimes T \mapsto f_{v,T}$ gives a K-module isomorphism. We identify $I_{P,\sigma,\lambda}(\tau)$ with $V_\tau \otimes \mathrm{Hom}_M(V_\tau, V_\sigma)$ by this isomorphism. Note that $T(v) = f_{v,T}(1)$ under this identification.

Theorem 10 *Under the assumption of Lemma 9, $\mathcal{P}_{\tau,\sigma,-\lambda}$ is injective on $V_\tau \otimes H^* \otimes H$ if $B^\sigma_{\tau^*}(\bar{P} : P : \lambda)$ is non-zero as a operator.*

A sketch of proof. We have

$$\begin{aligned} & \int_{\bar{N}} e^{-(\lambda+\rho)H(\bar{n})} < \tau^*(\kappa(\bar{n})^{-1}) f_{v,T}(1), w > d\bar{n} \\ = & \int_{\bar{N}} e^{-(\lambda+\rho)H(\bar{n})} < T(\tau^*(\kappa(\bar{n})^{-1})v), w > d\bar{n} \\ = & < T(B_{\tau^*}(\bar{P} : P : \lambda)v), w > . \end{aligned}$$

Hence the claim follows from Lemma 9 since T is M-linear and $w \in V_\sigma$.

References

[1] A.Borel and N.R.Wallach, *Continuous cohomology, discrete subgroupes, and representations of reductive groups*, Annals of Math.Studies, Study **94**, Princeton University Press, Princeton, 1980.

[2] M.Eguchi and S.Tanaka, *The explicit representation of the determinant of Harish-Chandra's C-function in $SL(3,\mathbf{R})$ and $SL(4,\mathbf{R})$ cases*, Hiroshima Math.J.**22**(1992), 57-93.

[3] J.Lepowsky, *Algebraic results on representations of semisimple Lie groups*, Trans.Amer.Math.Soc., **176**(1973), 1-44.

[4] K.Minemura, *Invariant differential operators and spherical sections on a homogeneous vector bundle*, Tokyo J. Math. Vol.**15** No.1(1992), 231-245.

[5] N.Shimeno, *Eigenspaces of invariant differential operators on a homogeneous line bundle on a Riemannian symmetric space*, J.Fac.Sci.Univ.Tokyo Sect.IA,Math.**37**(1990), 201-234.

[6] N.R.Wallach, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York, 1973.

[7] N.R.Wallach, *Real Reductive Groups I*, Academic Press, 1988.Pure and Appl.Math.**132**.