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On the orbit decomposition of some affine symmetric spaces.

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§ 1. Introduction.

Let G be a reductive Lie group, and σ, τ be commuting involutions on G. We set $G^\sigma = \{g \in G : \sigma(g) = g\}$ and $G^\tau = \{g \in G : \tau(g) = g\}$. Then we know the following double coset decomposition $G^\sigma \backslash G / G^\tau$ as Cartan decomposition.

(1) $G^\theta \backslash G / G^\theta$. (i.e. $\sigma = \tau = \theta$ (Cartan involution))

We also know two generalizations of (1).

(2) $G^\theta \backslash G / G^\tau$. (i.e. τ is an arbitrary involution commuting with θ.)

(3) $G^\sigma \backslash G / G^\sigma$. (i.e. σ is an arbitrary involution.)

((2) is the generalized Cartan decomposition, and (3) is studied in [3].)

But, the generalization combined (2) and (3) ($\sigma \neq \tau$ are arbitrary involutions) is not known except for the case G is compact ([1]). So we will consider the double coset decomposition

$$GL(p,\mathbb{C}) \times GL(q,\mathbb{C}) \backslash GL(n,\mathbb{C}) / GL(r,\mathbb{C}) \times GL(s,\mathbb{C})$$

$(p+q = r+s = n,\ p,q,r,s \in \mathbb{N})$ as one of the examples of the non-compact cases.

§ 2. Approach to the problem.

We will not use the general theory of Lie groups, but the linear algebraic method.

We set

$$\begin{array}{cc} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \begin{matrix} r \\ s \end{matrix} \\ \begin{matrix} p & q \end{matrix} & \end{array} \in GL(n,\mathbb{C}),$$

$$\begin{array}{c} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \\ \begin{matrix} p & q \end{matrix} \end{array} \in GL(p,\mathbb{C}) \times GL(q,\mathbb{C}),$$

$$\begin{array}{c} \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \\ \begin{matrix} r & s \end{matrix} \end{array} \in GL(r,\mathbb{C}) \times GL(s,\mathbb{C}).$$

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We will take proper $P \in GL(p,\mathbb{C})$, $Q \in GL(q,\mathbb{C})$, $R \in GL(r,\mathbb{C})$, $S \in GL(s,\mathbb{C})$ and make

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$$
$$= \begin{pmatrix} PAR & PBS \\ QCR & QDS \end{pmatrix}$$

a canonical form.

(1) Canonical forms of A and D.

We can easily make PAR and QDS canonical forms $\begin{pmatrix} I_a & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & I_d \end{pmatrix}$ respectively by using linear algebra. Here I_k is an identity matrix of degree k, and $a = rankA$, $d = rankD$.

(2) A canonical form of C.

After the procedure above, we will make C a canonical form keeping the canonical forms of A, D invariant. Therefore we make

$$\begin{array}{cc} \begin{pmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{pmatrix} \\ \begin{matrix} q-d & \quad d \end{matrix} \end{array} C \begin{array}{c} \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix} \\ \begin{matrix} a & \quad r-a \end{matrix} \end{array}$$

a canonical form by taking proper $Q_{11}, Q_{21}, Q_{22}, R_{11}, R_{21}, R_{22}$. The entries which appear in this canonical form are also 0 or 1 as in those of A.

(3) A canonical form of D.

If we take a nice canonical form of C in (2), all we should do is to make

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ 0 & P_{22} & P_{23} & P_{24} \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & 0 & P_{44} \end{pmatrix} D \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ 0 & S_{22} & S_{23} & S_{24} \\ 0 & 0 & S_{33} & S_{34} \\ 0 & 0 & 0 & S_{44} \end{pmatrix}$$

a canonical form by taking proper $P_{11},\cdots,P_{44},S_{11},\cdots, S_{44}$. Of course,the sizes of these matrices are due to A, B, C ,and except for P_{22} and S_{33},they are independent of each other. The only one relation between them is $P_{22} = S_{33}^{-1}$. Therefore continuous parameter will appear in the canonical form of D as diagonal entries of Jordan canonical form by contrast with the cases of A,B and C.

Remark: We have gotten the algorithm to calculation, but we can not write all representatives of orbits by canonical forms.

§ 3 Example.

In this section, we consider the case of $r = n-1, s = 1$ which we can think as rank 1 case. (We assume $p, q \geq 2$ for simplicity.) By the algorithm stated in §2, we get eight orbits and one family of orbits in this case. Their representatives are

$$\mathcal{O}_1 : \begin{pmatrix} I_{p-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q-1} & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \ (codim \mathcal{O}_1 = q),$$

$$\mathcal{O}_2 : \begin{pmatrix} I_{p-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & I_{q-2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \ (codim \mathcal{O}_2 = 1),$$

$$\mathcal{O}_3 : \begin{pmatrix} I_{p-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_{q-1} & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \ (codim \mathcal{O}_3 = p),$$

$$\mathcal{O}_4 : \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & I_{p-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{q-1} & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \ (codim \mathcal{O}_4 = 1),$$

$$\mathcal{O}_5(x) : \begin{pmatrix} I_{p-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & I_{q-1} & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \ (x \in \mathbb{C}^\times \setminus \{1\}) \ (codim \mathcal{O}_5(x) = 1),$$

$$\mathcal{O}_6 : \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{q-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \ (codim \mathcal{O}_6 = 2p),$$

$$\mathcal{O}_7 : \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & I_{p-1} & 0 & 0 \\ 0 & 0 & I_{q-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ (codim \mathcal{O}_7 = p),$$

$$\mathcal{O}_8 : \begin{pmatrix} I_{p-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & I_q & 0 \end{pmatrix} \ (codim \mathcal{O}_8 = 2q),$$

$$\mathcal{O}_9 : \begin{pmatrix} I_{p-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & I_{q-1} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ (codim \mathcal{O}_9 = q).$$

The codimensions of these orbits are calculated from the isotropy subgroups of their representatives.

§ 4 The map Φ.

The invariants of the orbits which occurs immediately to us are *rankA*, *rankB*, *rankC*, *rankD*. But they are not enough for us to distinguish every orbit as we can see in § 3. Therefore we consider the following map.

LEMMA. *We assume that* $r \geq p \geq q \geq s$.

$$For\ X = \begin{array}{cc} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \begin{array}{c} r \\ s \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array} \in GL(n, \mathbb{C}),$$

$$\begin{cases} (a)\ dim_{\mathbb{C}}(C(KerA) + ImD) = q \\ (b)\ rankA = p \\ (c)\ rankD = s \end{cases}$$

$$\Longrightarrow ImD \cap C(A^{-1}(B(v))) = \{a\ \ point\} for^{\forall} v \in \mathbb{C}^{s}.$$

DEFINITION: We define $\Phi_X \in End(\mathbb{C}^s)$ for $X \in GL(n, \mathbb{C})$ satisfying the conditions (a),(b),(c) in the above lemma.

$$\Phi_X(v) := D^{-1}(ImD \cap C(A^{-1}(B(v))))$$

for ${}^{\forall}v \in \mathbb{C}^s$.

PROPOSITION. *The eigenvalues of* Φ_X *coincide with those of* Φ_{gXh} *for* ${}^{\forall}X \in GL(n, \mathbb{C})$, ${}^{\forall}g \in GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ *and* ${}^{\forall}h \in GL(r, \mathbb{C}) \times GL(r, \mathbb{C})$.

REMARK: This map is due to H.Ochiai.

PROPOSITION. *The orbits of* $GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) \backslash GL(n, \mathbb{C}) / GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$ *are distinguished by rankA, rankB, rankC, rankD and whether* Φ_X *can be defined or not, and (if defined) the eigenvalue of* Φ_X.

REMARK: For general r, s (with $r + s = n$), *rankA*, *rankB*, *rankC*, *rankD* and the eigenvalues of Φ_X are not enough for us to distinguish every $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ -orbit in the $GL(n, \mathbb{C}) / GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$.

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