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The Plancherel Formula for Spherical Functions with a One-dimensional K-type on a Simply Connected Simple Lie Group of Hermitian Type

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Our purpose is to generalize Harish-Chandra's inversion formula for the spherical transform on a Riemannian symmetric space $X = G/K$ of the non-compact type to homogeneous line bundles on X. We use the approach by means of spherical functions. It only gives something new for the case when K has non-discrete center. The universal covering of X is a direct product of irreducible symmetric spaces. So we may assume that G is connected, simply connected, simple Lie group and X is a Hermitian symmetric space.

An elementary proof of the Plancherel theorem on X is given by Helgason, Gangolli and Rosenberg ([R], [H2, Chapter IV, Section 7]) based on the idea of change of contour of integration in the proof of the classical Paley-Wiener theorem for the real line. It also leads to a generalization of classical Paley-Wiener theorem. We use this method for homogeneous line bundles on X. When we treat functions on X, residues do not appear during the contour change. However, in our case, we must treat the residues during the contour change. There is a formulation of handling successive residues due to Arthur [Ar], which does not require any explicit knowledge of poles for arbitrary Eisenstein integrals. In our case we can compute residues explicitly and write them in terms of the elementary spherical functions. Our method is elementary and we use no general theorem about the relative discrete series nor the Plancherel measure for G.

This problem was considered by Flensted-Jensen [FJ1, 2] and Muta [M]. They obtain the Plancherel and Paley-Wiener theorems for $\mathfrak{g} = \mathfrak{su}(1,n)$. The Paley-Wiener theorem was obtained by Wallach [Wal] for $G = SU(1,n)$ and by Kawazoe [Ka] for $SU(2,2)$. We mention the papers of Pukanszky [P] and Aoki [Ao] for the universal covering group of $SL(2,\mathbb{R})$.

1. Elementary spherical functions

Let G be a simple, simply connected, connected Lie group with an Iwasawa decomposition

$$G = KAN.$$

We assume that G/K is a Hermitian symmetric space. Let $Z(G)$ denote the center of G. We shall denote the Lie algebras of Lie groups by lower case German script letters, and we will add a subscript c to denote complexification. We have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}.$$

By assumption, $\mathfrak{k}$ is the direct sum of $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}]$ and the center $\mathfrak{k}_a$ of $\mathfrak{k}$, where $\dim \mathfrak{k}_a = 1$. For $x \in G$, let $\kappa(x) \in K$ and $H(x) \in \mathfrak{a}$ be elements uniquely determined by $x \in \kappa(x) \exp H(x) N$.

Let $G_{\mathbb{C}}$ be the connected simply connected Lie group with Lie algebra $\mathfrak{g}_c$ and let $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$ be analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$ and $\mathfrak{k}$ respectively. One-dimensional representations of K are parametrized by $\mathbb{C}$ and denoted by τ_l for $l \in \mathbb{C}$. We normalize τ_l so that it is unitary if and only if $l \in \mathbb{R}$, and it determines a representation of $K_{\mathbb{R}}$ if and only if $l \in \mathbb{Z}$.

Let Σ be the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and let W be its Weyl group. We fix a set of positive roots $\Sigma^+ \subset \Sigma$. Let $\mathfrak{a}_c^*$ be the set of complex linear functions on $\mathfrak{a}$. For $l \in \mathbb{C}$ and $\lambda \in \mathfrak{a}_c^*$ define

$$\phi_{\lambda,l}(g) = \int_{K/Z(G)} e^{-(\lambda+\rho)(H(g^{-1}k))} \tau_l(k^{-1}\kappa(g^{-1}k))dk \ (g \in G),$$

where dk is the invariant measure on $K/Z(G)$ with total measure 1 and ρ is the half the sum of the positive restricted roots including their multiplicities. We call $\phi_{\lambda,l}$ the *elementary spherical function* of type τ_{-l} with parameter λ. Two such functions $\phi_{\lambda,l}$ and $\phi_{\mu,l}$ are identical if and only if $\mu = s\lambda$ for some $s \in W$. The elementary spherical functions on $G_{\mathbb{R}}$ are given by the above formula only for $l \in \mathbb{Z}$.

Let $C^\infty(G/K; \tau_l)$ denote the space of C^∞-functions on G such that $f(xk) = \tau_l(k)^{-1} f(x)$ for all $x \in G$, $k \in K$. Let $\boldsymbol{D}_l$ denote the set of left-invariant differential operators on G that map $C^\infty(G/K; \tau_l)$ into itself. Let $U(\mathfrak{a}_c)^W$ be the set of Weyl group invariant elements in $U(\mathfrak{a}_c)$. We have the Harish-Chandra isomorphism:

$$\bar{\gamma}_l \ : \ \boldsymbol{D}_l \xrightarrow{\sim} U(\mathfrak{a}_c)^W.$$

For $\lambda \in \mathfrak{a}_c^*$ and $l \in \mathbb{C}$ we define an algebra homomorphism $\chi_{\lambda,l}$ of $\boldsymbol{D}_l$ to $\mathbb{C}$ by

$$\chi_{\lambda,l}(D) = \bar{\gamma}_l(D)(\lambda), \ D \in \boldsymbol{D}_l.$$

The elementary spherical function $\phi_{\lambda,l}$ satisfies

$$\phi_{\lambda,l}(e) = 1$$
$$D\phi_{\lambda,l} = \chi_{\lambda,l}(D)\phi_{\lambda,l} \text{ for all } D \in \boldsymbol{D}_l\,,$$

and these conditions characterize $\phi_{\lambda,l}$.

A function f on G is called *τ_{-l}-spherical* if

$$f(k_1 g k_2) = \tau_l(k_1)^{-1} f(g) \tau_l(k_2)^{-1} \quad \text{for all } g \in G,\ k_1,\ k_2 \in K.$$

Let $\mathcal{D}_l(G)$ denote the space of τ_{-l}-spherical functions which are compactly supported modulo $Z(G)$. For $f \in \mathcal{D}_l(G)$ we define its spherical Fourier transform $f_l^\wedge$ by

$$f_l^\wedge(\lambda) = \int_{G/Z(G)} f(g)\phi_{-\lambda,-l}(g)dg,\ \lambda \in \mathfrak{a}_c^*,$$

where dg is an invariant measure on $G/Z(G)$.

Problem. Find an explicit inversion formula for the spherical Fourier transform.

2. The Harish-Chandra expansion for $\phi_{\lambda,l}$

The root system Σ is of type BC_n or C_n. Let R denote the root system of BC_n:

$$R = \{\tfrac{1}{2}(\pm\beta_j \pm \beta_k), \pm\beta_i, \pm\tfrac{1}{2}\beta_i;\ 1 \le k < j \le r,\ 1 \le i \le r\},$$

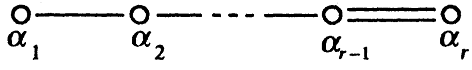
so that $R = \Sigma$ or $R = \Sigma \cup \{\frac{1}{2}\beta_i;\ 1 \le i \le r\}$. We put

$$R^+ = \left\{\tfrac{1}{2}\beta_i,\ \beta_i,\ \tfrac{1}{2}(\beta_j \pm \beta_k)\ ;\ 1 \le i \le r,\ 1 \le k < j \le r\right\}.$$

and let $\Sigma^+ = \Sigma \cap R^+$. We define a subset Σ^0 and Σ_s of Σ by

$$\Sigma_s = \left\{\tfrac{1}{2}(\pm\beta_j \pm \beta_k)\ ;\ 1 \le k < j \le r\right\},$$
$$\Sigma^0 = \Sigma \setminus \{\tfrac{1}{2}\beta_i;\ 1 \le i \le r\},$$

and put $\Sigma_s^+ = \Sigma_s \cap \Sigma^+$, $(\Sigma^0)^+ = \Sigma^0 \cap \Sigma^+$. We put $\alpha_i = \frac{1}{2}(\beta_{r+1-i} - \beta_{r-i})$ $(1 \le i \le r-1)$, $\alpha_r = \frac{1}{2}\beta_1$ if $\frac{1}{2}\beta_i \in \Sigma$ and $\alpha_r = \beta_1$ otherwise. Then $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ is the set of simple roots in Σ^+.



For any $\alpha \in R$ let m_α denote its multiplicity. If $\alpha \notin \Sigma$, we put $m_\alpha = 0$. The multiplicity of long roots β_j $(j = 1, \ldots, r)$ is 1. We put

$$m_{\frac{1}{2}\beta_j} = m\ (j = 1, \ldots, r), \quad m_\alpha = m'\ (\alpha \in \Sigma_s).$$

Let $m_\alpha(l)$ be a deformation of root multiplicities given by

$$m_\alpha(l) = m' \ (\alpha \in \Sigma_s),$$
$$m_{\frac{\beta_i}{2}}(l) = m + 2l, \ m_{\beta_i}(l) = 1 - 2l \ (i = 1, \dots, r)$$

and define

$$\rho(l) = \frac{1}{2} \sum_{\alpha \in R^+} m_\alpha(l)\alpha = \rho - \frac{l}{2} \sum_{j=1}^{r} \beta_j.$$

For $\lambda \in \mathfrak{a}_c^*$ let A_λ be the element of $\mathfrak{a}_c$ determined by $B(H, A_\lambda) = \lambda(H)$ for all $H \in \mathfrak{a}$, where B denotes the Killing form of $\mathfrak{g}_c$. For λ, $\mu \in \mathfrak{a}_c^*$ we put $\langle \lambda, \mu \rangle = B(A_\lambda, A_\mu)$. Since $\{\frac{1}{2}\beta_1, \dots, \frac{1}{2}\beta_r\}$ forms an orthogonal basis of $\mathfrak{a}^*$, any $\lambda \in \mathfrak{a}_c^*$ can be written $\lambda = \sum_{i=1}^r \frac{1}{2}\lambda_i \beta_i$ with

$$\lambda_i = \frac{2\langle \lambda, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \in \mathbf{C} \ (i = 1, \dots, r).$$

We identify $\mathfrak{a}_c^*$ with $\mathbf{C}^r$ by $\lambda \mapsto (\lambda_1, \dots, \lambda_r)$.

For $D \in U(\mathfrak{g})$, we denote by $\Delta_l(D)$ its (τ_{-l}, τ_{-l})-radial component. Let A_i $(i = 1, \dots, r)$ be a basis of $\mathfrak{a}$ orthonormal with respect to B and let $\Omega_\mathfrak{a} = \sum_i A_i^2$. Let $\mathfrak{a}^+$ be the positive Weyl chamber and put $A^+ = \exp \mathfrak{a}$. We define a function u on A^+ by

$$u(\exp H) = \prod_{i=1}^{r} 2 \cosh \frac{\beta_i(H)}{2} \quad \text{for } H \in \mathfrak{a}^+.$$

By an explicit formula of the Casimir operator of $\mathfrak{g}_c$, we can show that the function $\psi = u^l \phi_{\lambda,l} | A^+$ satisfies the differential equation

$$(2.1) \qquad (\Omega_\mathfrak{a} + \sum_{\alpha \in R^+} m_\alpha(l) \coth(\alpha(H)) A_\alpha)\psi = (\langle \lambda, \lambda \rangle - \langle \rho(l), \rho(l) \rangle)\psi.$$

The function ψ also satisfies

$$(2.2) \qquad (u^l \circ \Delta_l(D) \circ u^{-l})\psi = \chi_{\lambda,l}(D)\psi \quad (D \in \mathbf{D}_l).$$

For "generic" $\lambda \in \mathfrak{a}_c^*$, the function $\phi_{\lambda,l}$ has an expansion:

$$\phi_{\lambda,l} = u^{-l} \sum_{w \in W} \mathbf{c}(w\lambda, l) \Phi_{w\lambda,l} \quad \text{on } A^+.$$

The constant $\mathbf{c}(\lambda, l)$ is Harish-Chancra's $\mathbf{c}$-function which is given by

$$\mathbf{c}(\lambda, l) = \prod_{\alpha \in (\Sigma^0)^+} \mathbf{c}_\alpha(\lambda, l).$$

Here c_α $(\alpha \in (\Sigma^0)^+)$ are given by

$$c_\alpha(\lambda, l) = \frac{2^{m'-1}}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}(m'+1)\right) \frac{\Gamma(\langle \lambda, \alpha_0 \rangle)}{\Gamma\left(\frac{1}{2}m' + \langle \lambda, \alpha_0 \rangle\right)} \quad (\alpha \in \Sigma_s^+),$$

where $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$ and

$$c_{\beta_j}(\lambda, l) = \frac{2^{\frac{1}{2}m+1-\lambda_j} \Gamma\left(\frac{1}{2}m+1\right) \Gamma(\lambda_j)}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2}m+1+\lambda_j+l\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{1}{2}m+1+\lambda_j-l\right)\right)} \quad (1 \leq j \leq r).$$

The function $\Phi_{\lambda,l}$ is a joint eigenfunction of (2.2) on A^+, with asymptotic behavior $\Phi_{\lambda,l}(a) \sim a^{\lambda-\rho(l)}$ $(a \to \infty$ in $A^+)$.

Remark. The function $2^{-rl} u^l \phi_{\lambda,l}|A$ is the hypergeometric function defined by Heckman and Opdam ([HO]) with parameter $2k_\alpha = m_\alpha(l)$ and λ.

If $\mathfrak{g}$ is of tube type (for example $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$), then there is an non-trivial $(\tau_{-l'}, \tau_{-l})$-spherical eigenfunction for l and l' with $l \equiv l' \mod 2$. The restrictions of these functions to A are also given by hypergeometric functions of Heckman and Opdam.

3. The inversion formula

Let $\mathcal{H}_W(\mathfrak{a}_c^*)$ denote the space of W-invariant entire functions on $\mathfrak{a}_c^*$ satisfying the following property:
There exist R and C_N $(N \in \mathbb{N} = \{0, 1, 2, \cdots\})$ such that

$$|F(\lambda)| \leq C_N (1 + |\lambda|)^{-N} e^{R|\mathrm{Im}\lambda|} \; (N \in \mathbb{N},\; \lambda \in \mathfrak{a}_c^*).$$

For any $f \in \mathcal{D}_l(G)$, $f_l^\wedge$ is contained in $\mathcal{H}_W(\mathfrak{a}_c^*)$.

Let $d\lambda$ be the invariant measure on $\sqrt{-1}\mathfrak{a}^*$ given by $d\lambda = d\lambda_1 d\lambda_2 \cdots d\lambda_r$ and let da be the invariant measure on A which is dual to $d\lambda$. We normalize the Haar measure dg on $G/Z(G)$ so that

$$\int_{G/Z(G)} f(g) dg = \int_{A^+} f(a) \delta(a) da$$

for $f \in \mathcal{D}_0(G)$, where

$$\delta(\exp H) = \prod_{\alpha \in \Sigma^+} \left(e^{\alpha(H)} - e^{-\alpha(H)}\right)^{m_\alpha}.$$

Let $\mathcal{D}_W(A)$ denote the space of W-invariant compactly supported C^∞-functions on A. We identify $\mathcal{D}_l(G)$ with $\mathcal{D}_W(A)$ by restriction to A. Then the spherical Fourier transform is given by

$$f_l^\wedge(\lambda) = \int_{A^+} f(a) \phi_{-\lambda,-l}(a) \delta(a) da$$

for $f \in \mathcal{D}_W(A)$.

Let $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^*;\ \langle\lambda,\alpha\rangle > 0 \text{ for all } \alpha \in \Sigma^+\}$. Let η be a point in $-\mathrm{Cl}(\mathfrak{a}_+^*)$ that satisfies $\frac{1}{2}m + 1 - |\mathrm{Re}\ l| > \eta_1$, i.e. $\mathbf{c}(-\lambda,l)^{-1}$ is a regular function of λ for $\mathrm{Re}\ \lambda \in \eta - \mathrm{Cl}(\mathfrak{a}_+^*)$. For any $F \in \mathcal{H}_W(\mathfrak{a}_c^*)$, define

$$F_l^{\vee}(a) = \int_{\eta+\sqrt{-1}\mathfrak{a}^*} F(\lambda)u(a)^{-l}\Phi_{\lambda,l}(a)\mathbf{c}(-\lambda,l)^{-1}d\lambda, \quad a \in A^+. \tag{3.1}$$

The integral converges and is independent of the choice of η and holomorphic in l.

Theorem 1. (inversion formula, first form). *If $\lambda \in \mathfrak{a}_c^*$, $l \in \mathbb{C}$ and $f \in \mathcal{D}_W(A)$, then*

$$f(a) = (f_l^{\wedge})_l^{\vee}(a), \quad a \in A^+.$$

In particular, if $\frac{1}{2}m + 1 - |\mathrm{Re}\ l| > 0$ then we can put $\eta = 0$ and we have

$$f(a) = \frac{1}{|W|}\int_{\sqrt{-1}\mathfrak{a}^*} f_l^{\wedge}(\lambda)\phi_{\lambda,l}(a)(\mathbf{c}(\lambda,l)\mathbf{c}(-\lambda,l))^{-1}d\lambda, \quad a \in A.$$

When $-\frac{m}{2} - 1 < l < \frac{m}{2} + 1$, the method of Gangolli-Helgason-Rosenberg generalizes without essential change to our situation. For general l, Theorem 1 follows by analytic continuation on l.

Theorem 2. (Paley-Wiener theorem). *Let $l \in \mathbb{C}$. The spherical transform $f \mapsto f_l^{\wedge}$ is a bijection of $\mathcal{D}_W(A)$ onto $\mathcal{H}_W(\mathfrak{a}_c^*)$.*

Hereafter we assume that $l \in \mathbb{R}$. For $F \in \mathcal{H}_W(\mathfrak{a}_c^*)$ let

$$F_{\emptyset,l}^{\vee}(x) = \frac{1}{|W|}\int_{\sqrt{-1}\mathfrak{a}^*} F(\lambda)\phi_{\lambda,l}(x)|\mathbf{c}(\lambda,l)|^{-2}d\lambda. \tag{3.2}$$

We define subsets $D_{l,j} \subset \mathbb{R}^j$ $(1 \le j \le r)$ by

$$\begin{aligned} D_{l,j} = \{\lambda \in \mathbb{R}^j\,;\ & \lambda_1 + |l| - \frac{m}{2} - 1 \in 2\mathbb{N},\ \lambda_r < 0 \\ & \lambda_{i+1} - \lambda_i - m' \in 2\mathbb{N}\ (1 \le i \le j-1)\}. \end{aligned}$$

Theorem 3. *Let $l \in \mathbb{R}$. Then $|\phi_{\lambda,l}| \in L^2(G/Z(G))$ if and only if $\mu \in D_{l,r}$ for some $\mu \in W\lambda$. If $\mu \in D_{l,r}$, then the relative discrete series representation of G corresponding to $\phi_{\mu,l}$ is a holomorphic discrete series representation.*

For a subset Θ of Ψ let $\mathbf{c}^{\Theta}(\lambda,l)$ and $\mathbf{c}_{\Theta}(\lambda,l)$ be meromorphic functions given by

$$\mathbf{c}^{\Theta}(\lambda,l) = \prod_{\alpha\in(\Sigma^0)^+\setminus\langle\Theta\rangle} c_\alpha(\lambda,l),$$
$$\mathbf{c}_{\Theta}(\lambda,l) = \prod_{\alpha\in(\Sigma^0)^+\cap\langle\Theta\rangle} c_\alpha(\lambda,l),$$

where $\langle\Theta\rangle = \Sigma \cap \sum_{\alpha\in\Theta} \mathbb{R}\alpha$ and let W_Θ be the subgroup of W generated by the reflection with respect to the roots in $\langle\Theta\rangle$.

We define subsets Θ_j $(j = 0, \ldots, r)$ of Ψ by

$$\Theta_j = \{\alpha_i;\ r - j + 1 \leq i \leq r\}\ (j = 0, \ldots, r).$$

We put $\mathfrak{a}_{\Theta_j} = \sum_{i=j+1}^{r} \mathbb{R}A_{\beta_i}$ and identify $(\mathfrak{a}_{\Theta_j})_c^*$ with $\mathbb{C}^{r-j}$. For fixed j $(1 \leq j \leq r)$ we write $\lambda = (\lambda', \lambda'')$ $(\lambda' \in \mathbb{C}^j,\ \lambda'' \in \mathbb{C}^{r-j})$ and put $d\lambda'' = d\lambda_{j+1} \cdots d\lambda_r$.

For $F \in \mathcal{H}_W(\mathfrak{a}_c^*)$ we define functions $\check{F}_{\Theta_j,l}(x)$ $(j = 1, \ldots, r)$ by

$$\check{F}_{\Theta_j,l}(x) = \sum_{\lambda' \in D_{l,j}} \frac{d_j(\lambda', l)}{|W_{\Theta_{r-j}}|} \int_{(\lambda',0)+\sqrt{-1}\mathfrak{a}^*_{\Theta_j}} F(\lambda)\phi_{\lambda,l}(x)|c^{\Theta_j}(\lambda, l)|^{-2} d\lambda''$$

for $j = 1, \ldots, r-1$ and

$$\check{F}_{\Theta_r,l}(x) = \sum_{\lambda \in D_{l,r}} d_r(\lambda, l) F(\lambda)\phi_{\lambda,l}(x),$$

where

$$d_j(\lambda', l) = (-2\pi\sqrt{-1})^j \operatorname*{Res}_{\mu_j=\lambda_j} \cdots \operatorname*{Res}_{\mu_2=\lambda_2} \operatorname*{Res}_{\mu_1=\lambda_1} (c_{\Theta_j}(\mu, l)^{-1} c_{\Theta_j}(-\mu, l)^{-1}).$$

Theorem 4. (inversion formula, second form). *Let $l \in \mathbb{R}$. For $f \in \mathcal{D}_l(G)$ we have*

$$f(x) = (\hat{f_l})\check{}_l(x) = \sum_{j=0}^{r} (\hat{f_l})\check{}_{\Theta_j,l}(x),\ x \in G.$$

Define the measure γ on $\bigsqcup_{j=0}^{r} D_{l,j} + \sqrt{-1}\mathfrak{a}^*_{\Theta_j}$ by

$$\int_{\bigsqcup_{j=0}^{r} D_{l,j}+\sqrt{-1}\mathfrak{a}^*_{\Theta_j}} g(\lambda)d\gamma(\lambda) = \sum_{j=0}^{r} \check{g}_{\Theta_j,l}(e).$$

Theorem 5. (Plancherel formula). *Let $l \in \mathbb{R}$. For all $f \in \mathcal{D}_W(A)$,*

$$\int_{A+} |f(a)|^2 \delta(a) da = \int_{\bigsqcup_{j=0}^{r} D_{l,j}+\sqrt{-1}\mathfrak{a}^*_{\Theta_j}} |\hat{f_l}(\lambda)|^2 d\gamma(\lambda).$$

Remark. We can define the Fourier transform on G/K_s and obtain the inversion formula as a corollary of Theorem 4.

The main point in the proof of Theorem 4 is the computation of residues during the contour change.

4. Universal cover of $SL(2,\mathbb{R})$

In order to get the feeling what is required in general, we prove Theorem 4 when G is the universal covering group of $SL(2,\mathbb{R})$. Let H and Z be elements of $\mathfrak{g}$ given by

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad Z=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Then $\mathfrak{a}=\mathbb{R}H$ and $\mathfrak{k}=\mathfrak{k}_a=\mathbb{R}Z$. We identify $\mathfrak{a}_c^*$ with $\mathbb{C}$ by $\lambda\mapsto\lambda(H)$. Let $l\in\mathbb{R}$ and define $\tau_l(\exp\theta Z)=e^{il\theta}$ $(\theta\in\mathbb{R})$. The elementary spherical function is given by

$$\phi_{\lambda,l}(\exp tH)=(\cosh t)^{-l}{}_2F_1\left(\tfrac{1}{2}(1-l+\lambda),\tfrac{1}{2}(1-l-\lambda);1;-(\sinh t)^2\right).$$

We put $a=\exp tH,\ t>0$. For $\lambda\notin\mathbb{Z}$, we have

$$(\cosh t)^l\phi_{\lambda,l}(a)=c(\lambda,l)\Phi_{\lambda,l}(a)+c(-\lambda,l)\Phi_{-\lambda,l}(a) \tag{4.1}$$

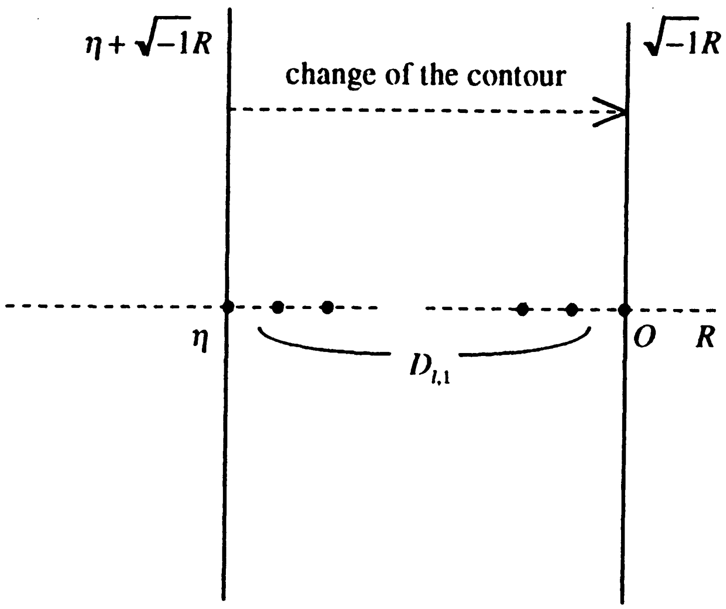
where

$$c(\lambda,l)=\frac{2^{1-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}(\lambda+1+l)\right)\Gamma\left(\frac{1}{2}(\lambda+1-l)\right)}. \tag{4.2}$$

Since $\lambda\mapsto\Phi_{\lambda,l}$ has no poles in the region $\operatorname{Re}\lambda\le 0$, the poles of the integrand of (3.1) in the region $\operatorname{Re}\lambda\le 0$ are possibly the poles of $c(-\lambda,l)^{-1}$. The set of poles of $c(-\lambda,l)^{-1}$ in the region $\operatorname{Re}\lambda\le 0$ is

$$D_{l,1}=\{2i+1-|l|;\ i=0,1,\ldots,\ 2i+1-|l|<0\} \tag{4.3}$$

and all are simple poles. Let us illustrate the process with a diagram.



We put $f_l^{\wedge}=F$. Then $F_l^{\vee}(a)$ equals the sum of

$$\int_{\sqrt{-1}\mathbb{R}}(\cosh t)^{-l}F(\lambda)\Phi_{\lambda,l}(a)c(-\lambda,l)^{-1}d\lambda \tag{4.4}$$

and

$$\sum_{z\in D_{l,1}} (-2\pi\sqrt{-1})F(z)(\cosh t)^{-l}\Phi_{\lambda,l}(a)\operatorname*{Res}_{\lambda=z}(c(-\lambda,l)^{-1}). \tag{4.5}$$

With a change of variables we see that (4.4) equals $F^{\vee}_{\emptyset,l}(a)$. Let $z\in D_{l,1}$. If $l\notin\mathbb{Z}$, then $z\notin\mathbb{Z}$ and we can substitute z in (4.1). Thus we have

$$\phi_{z,l}(a)=c(z,l)(\cosh t)^{-l}\Phi_{z,l}(a).$$

The above equation holds for all l by analytic continuation. Thus (4.5) equals $F^{\vee}_{\Psi,l}(a)$. By the Cartan decomposition and continuity, we have

$$\begin{aligned} f(x)=&\frac{1}{2}\int_{\sqrt{-1}\mathbb{R}} f_l^{\wedge}(\lambda)\phi_{\lambda,l}(x)|c(\lambda,l)|^{-2}d\lambda\\ &+\sum_{z\in D_{l,1}}(-2\pi\sqrt{-1})f_l^{\wedge}(z)\phi_{z,l}(x)\operatorname*{Res}_{\lambda=z}(c(\lambda,l)^{-1}c(-\lambda,l)^{-1}),\ x\in G. \end{aligned}$$

5. Higher rank

We give the main idea of the proof of Theorem 4 for higher rank by looking at the case $\mathfrak{g}=\mathfrak{sp}(2,\mathbb{R})$. Assume $l\in\mathbb{R}$. For $\mathfrak{g}=\mathfrak{sp}(2,\mathbb{R})$, the space $\mathfrak{a}$ has dimension 2 and $\Psi=\{\alpha_1=\frac{1}{2}(\beta_2-\beta_1),\ \alpha_2=\beta_1\}$. The Weyl group W has order 8. We identify $\mathfrak{a}_c^*$ with $\mathbb{C}^2$ by $\mathbb{C}\ni(\lambda_1,\lambda_2)\mapsto\frac{1}{2}(\lambda_1\beta_1+\lambda_2\beta_2)\in\mathfrak{a}_c^*$. Harish-Chandra's c-function is given by

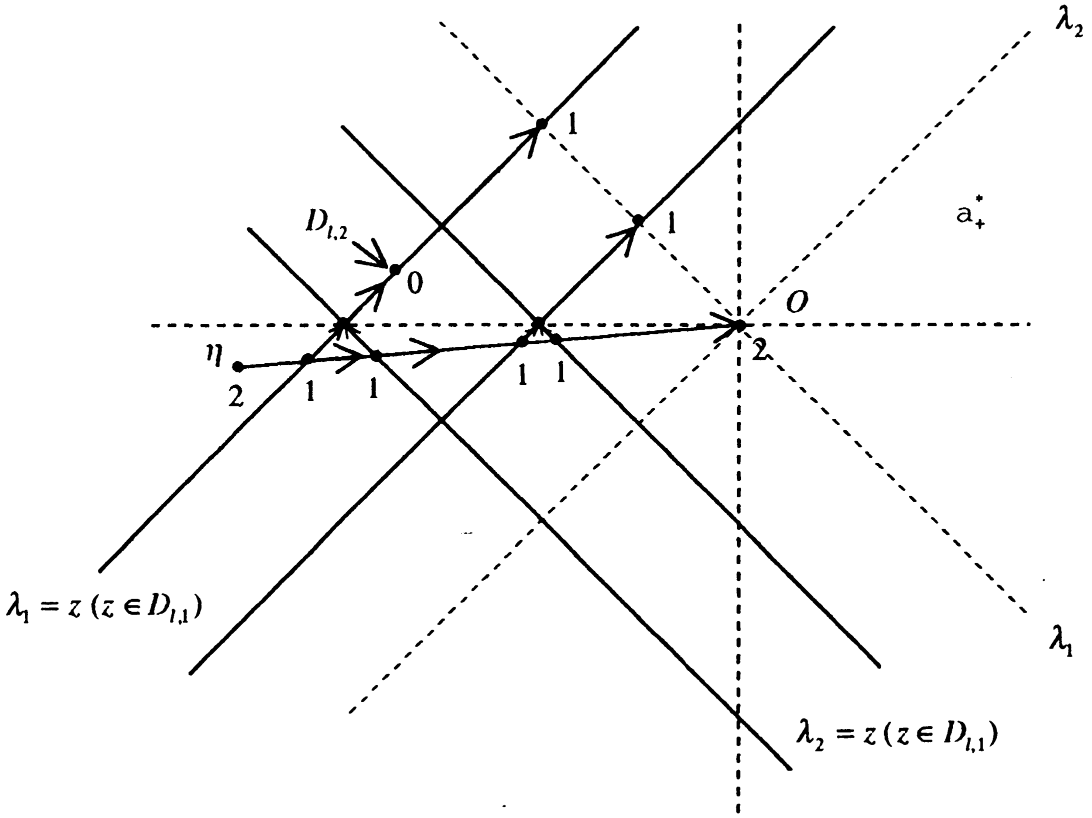
$$c(\lambda,l)=c_1(\lambda_1+\lambda_2)c_1(\lambda_2-\lambda_1)c_2(\lambda_1,l)c_2(\lambda_2,l), \tag{5.1}$$

where

$$c_1(z)=\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}z\right)}{\Gamma\left(\frac{1}{2}(z+1)\right)},\ c_2(z,l)=2^{1-z}\frac{\Gamma(z)}{\Gamma\left(\frac{1}{2}(z+1+l)\right)\Gamma\left(\frac{1}{2}(z+1-l)\right)}.$$

In the coordinates λ_1,λ_2, the positive chamber in $\mathfrak{a}^*$ is given by $0\le\lambda_1\le\lambda_2$. The function $\lambda\mapsto\Phi_{\lambda,l}$ has no poles in the region $\operatorname{Re}\lambda\in-\operatorname{Cl}(\mathfrak{a}_+^*)$. By (5.1), the singular hyperplanes of $c(-\lambda,l)^{-1}$ which meet the negative chamber $-\operatorname{Cl}(\mathfrak{a}_+^*)$ are $\lambda_1=z$ and $\lambda_2=z$ for $z\in D_{l,1}$, where $D_{l,1}$ is given by (4.3). By (5.1), we may put $\eta\in-\operatorname{Cl}(\mathfrak{a}_+^*)$ sufficiently near the line $\lambda_1=\lambda_2$.

We illustrate the process with the diagram. For simplicity we assume that $3<l\le5$, that is $D_{l,1}=\{-l+1,-l+3\}$ and $D_{l,2}=\{(-l+1,-l+2)\}$. Each large dot stands for an integral over an imaginary space of dimension 0, 1 or 2, which lies above the dot.



We move the contour of integration from $\eta + \sqrt{-1}\mathfrak{a}$ to $\sqrt{-1}\mathfrak{a}$. As we cross each of the singular hyperplanes, we take a residue consisting of an integral over an imaginary space of dimension 1. We put $F = f_l^{\wedge}$. Then $F_l^{\vee}(a)$ $(a \in A^+)$ equals the sum of

$$(5.2) \qquad \int_{\sqrt{-1}\mathfrak{a}^*} u(a)^{-l} F(\lambda) \Phi_{\lambda,l}(a) c(-\lambda, l)^{-1} d\lambda$$

and

(5.3)

$$\sum_{z \in D_{l,1}} (-2\pi\sqrt{-1}) \left(\int_{(z,z)+\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_1 = z} (u(a)^{-l} F(\lambda) \Phi_{\lambda,l}(a) c(-\lambda, l)^{-1}) d\lambda_2 \right.$$
$$\left. + \int_{(z,z)+\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_2 = z} (u(a)^{-l} F(\lambda) \Phi_{\lambda,l}(a) c(-\lambda, l)^{-1}) d\lambda_1 \right)$$

We shall group integrals in (5.3) by the action of s_{α_1}. We put

$$(5.4) \qquad u(a)^l \Phi_{\lambda,l}^{\{\alpha_1\}}(a) = c_1(\lambda_2 - \lambda_1) \Phi_{(\lambda_1,\lambda_2),l}(a) + c_1(\lambda_1 - \lambda_2) \Phi_{(\lambda_2,\lambda_1),l}(a).$$

Then by a change of variable, (5.3) equals

(5.5)

$$\sum_{z \in D_{l,1}} (-2\pi\sqrt{-1}) \int_{(z,z)+\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_1 = z} (F(\lambda) \Phi_{\lambda,l}^{\{\alpha_1\}}(a) c_1(\lambda_2 - \lambda_1)^{-1} c(-\lambda, l)^{-1}) d\lambda_2.$$

Proposition 6. *If $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \notin \mathbb{Z}$, then*

$$\phi_{\lambda,l}(a) = \sum_{\bar{w}\in\{e,s_{\alpha_1}\}\backslash W} c^{\{\alpha_1\}}(w\lambda, l)\Phi^{\{\alpha_1\}}_{w\lambda,l}(a) \quad (a \in A^+)$$

where $c^{\{\alpha_1\}}(\lambda, l) = c_1(\lambda_1 + \lambda_2)c_2(\lambda_1, l)c_2(\lambda_2, l)$. Moreover $\Phi^{\{\alpha_1\}}_{\lambda,l}(a)$ is a regular function of λ for $\operatorname{Re}\lambda \in -\mathrm{Cl}(\boldsymbol{R}^2_+)$.

We change the contour of the integration (5.5) from $(z, z) + \sqrt{-1}\mathbb{R}$ to $(z, 0) + \sqrt{-1}\mathbb{R}$. From the above proposition, the singularities of the integrand of (5.5) are possibly the singularities of the function

$$\lambda_2 \mapsto \operatorname*{Res}_{\lambda_1=z}(c_1(\lambda_2 - \lambda_1)^{-1}c(-\lambda, l)^{-1})$$

in the region $z \leq \operatorname{Re}\lambda_2 \leq 0$. By the explicit formula of c-function, we can show that the set of those singularities is $\{v \in \mathbb{R};\ (z, v) \in D_{l,2}\}$, where

$$D_{l,2} = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2;\ \lambda_1 + |l| - 1 \in 2\mathbb{N},\ \lambda_2 - \lambda_1 - 1 \in 2\mathbb{N},\ \lambda_2 < 0\}.$$

Then (5.5) equals the sum of

(5.6)
$$\sum_{z\in D_{l,1}} (-2\pi\sqrt{-1}) \int_{(z,0)+\sqrt{-1}\mathbb{R}} \operatorname*{Res}_{\lambda_1=z}(F(\lambda)\Phi^{\{\alpha_1\}}_{\lambda,l}(a)c_1(\lambda_2 - \lambda_1)^{-1}c(-\lambda, l)^{-1})d\lambda_2.$$

and

$$\sum_{(z,v)\in D_{l,2}} \operatorname*{Res}_{\lambda_2=v}(\operatorname*{Res}_{\lambda_1=z}(F(\lambda)\Phi^{\{\alpha_1\}}_{\lambda,l}(a)c_1(\lambda_2 - \lambda_1)^{-1}c(-\lambda, l)^{-1})). \tag{5.7}$$

It remains to show that (5.2), (5.6) and (5.7) can be written by elementary spherical functions. By changes of variables, we see that (5.2) equals $F^{\vee}_{\emptyset,l}(a)$.

If $z \in D_{l,1}$, then we have

$$c^{\{\alpha_1\}}((-z, -\lambda_2), l) = 0 \text{ and } c^{\{\alpha_1\}}((-z, \lambda_2), l) = 0. \tag{5.8}$$

If $l \notin \mathbb{Z}$, then by Proposition 6 and (5.8) we have

$$\phi_{(z,\lambda_2),l}(a) = c^{\{\alpha_1\}}((z, -\lambda_2), l)\Phi^{\{\alpha_1\}}_{(z,-\lambda_2),l}(a) + c^{\{\alpha_1\}}((z, \lambda_2), l)\Phi^{\{\alpha_1\}}_{(z,\lambda_2),l}(a)$$

for $\lambda_2 \in \sqrt{-1}\mathbb{R} \setminus \{0\}$. Above equality holds for all l by analytic continuation. By a change of variable we then see that (5.6) equals $F^{\vee}_{\{\alpha_2\},l}(a)$.

If $(z,v) \in D_{l,2}$, then we have $c^{\{\alpha_1\}}((z,-v),l) = 0$ in addition to (5.8). If $l \notin \frac{1}{2}\mathbb{Z}$, then by by Proposition 6 we have

$$\phi_{(z,v),l}(a) = c^{\{\alpha_1\}}((z,v),l)\Phi^{\{\alpha_1\}}_{(z,v),l}(a).$$

Above equality holds for all l by analytic continuation. Thus (5.7) equals $\check{F}_{\Psi,l}(a)$. Thus for $x \in A^+$ we have

$$\begin{aligned} f(x) =& \check{F}_{\emptyset,l}(x) + \check{F}_{\{\alpha_2\},l}(x) + \check{F}_{\Psi,l}(x) \\ =& \frac{1}{8}\int_{\sqrt{-1}\mathfrak{a}^*} \hat{f_l}(\lambda)\phi_{\lambda,l}(x)|c(\lambda,l)|^{-2}d\lambda \\ &+ \frac{1}{2}\sum_{z\in D_{l,1}} (-2\pi\sqrt{-1}) \operatorname*{Res}_{\lambda_1=z}(c_2(\lambda_1,l)c_2(-\lambda_1,l)) \\ &\times \int_{\sqrt{-1}\mathbb{R}} \hat{f_l}(z,\lambda_2)\phi_{(z,\lambda_2),l}(x)|c^{\{\alpha_2\}}((z,\lambda_2),l)|^{-2}d\lambda_2 \\ &+ \sum_{(z,v)\in D_{l,2}} (-2\pi\sqrt{-1})^2 \operatorname*{Res}_{\lambda_2=v}(\operatorname*{Res}_{\lambda_1=z}(c(\lambda,l)^{-1}c(-\lambda,l)^{-1}))\hat{f_l}(z,v)\phi_{(z,v),l}(x), \end{aligned}$$

where $c^{\{\alpha_2\}}(\lambda,l) = c_1(\lambda_1+\lambda_2)c_1(\lambda_2-\lambda_1)c_2(\lambda_2,l)$. Since both sides are τ_{-l}-spherical C^∞-functions on G, the theorem follows from the Cartan decomposition $G = K\mathrm{Cl}(A^+)K$.

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