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On the Harish-Chandra C-function for SU(n,1)

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§ 0. Introduction.

Let G be a semisimple Lie group with finite center, K a maximal compact subgroup of G. Let θ be the Cartan involution of G fixing K. Let $G=KAN$ be an Iwasawa decomposion of G and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ the corresponding decomposition of the Lie algebra $\mathfrak{g}$ of G. An element g of G can then be uniquely written as

$g=\kappa(g)\exp H(g)n(g)$ $(\kappa(g)\in K,\ H(g)\in\mathfrak{a},\ n(g)\in N)$. Put $\bar{N}=\theta N$ and let M be the centralizer of A in K. Let τ be a finite dimensional unitary representation of K and denote its representation space by V. The following operator given by the integral

$$(*)\qquad C_\tau(\lambda)=\int_{\bar{N}}\tau(\kappa(\bar{n}))e^{-(\lambda+\rho)(H(\bar{n}))}\,d\bar{n}\qquad(\lambda\in\mathfrak{a}_C^*)$$

is called Harish-Chandra's C-function associated to τ (see [Ha1]). It is well known that the operator $C_\tau(\sigma;\lambda)$ obtained by restricting $C_\tau(\lambda)$ to the irreducible M-component $V_\sigma(\subset V)$, is closely related to the intertwining operator between induced representations (see [Ha1,2]), and also in some special cases it can be represented by a diagonal matrix having diagonal elements in the form of quotients of products of gamma factors with respect to a certain orthogonal basis (cf. [Co], [Wa1]). Though it has been believed for a long time that these phenomena would be true for more general cases, even the computation of the determinant of $C_\tau(\sigma;\lambda)$ has not been easy (cf. [Co], [E-Ta], [Ta], [Wa2]). We give here some affirmative examples to this conjecture in $G=SU(n,1)$, $K\simeq S(U(n)\times U(1))$ case.

§ 1. Notation and preliminaries.

Let $n(n\geq 2)$ be an integer and

$$G=SU(n,1)=\{A\in GL(n+1,\mathbf{C});\ {}^t\bar{A}I_{n,1}A=I_{n,1}\quad\text{and}\quad \det A=1\},$$

where

$$I_{n,1}=\begin{pmatrix} I_n & \\ & -1\end{pmatrix}\in GL(n+1,\mathbf{C})$$

and I_n is the identity matrix of order n. Let

$$\mathfrak{g}=\mathfrak{su}(n,1)=\{X\in\mathfrak{gl}(n+1,\mathbf{C});\ {}^t\bar{X}I_{n,1}+I_{n,1}X=0\quad\text{and}\quad \mathrm{tr}X=0\},$$

$$\mathfrak{k}=\{\begin{pmatrix} A & \\ & -\mathrm{tr}A\end{pmatrix}\ ;\ A\in\mathfrak{u}(n)\}.$$

Let

$$\mathfrak{a}=\{tH;\ t\in\mathbf{R}\},\qquad H=\begin{pmatrix} & & 0 & 1\\ & & \cdot\cdot\cdot & \\ 1 & 0 & & \end{pmatrix},$$

$$\mathfrak{n}=\mathfrak{g}_\alpha+\mathfrak{g}_{2\alpha}\ \text{and}\ \bar{\mathfrak{n}}=\theta\mathfrak{n},$$

where α is a simple root of $(\mathfrak{g},\mathfrak{a})$ which satisfies $\alpha(H)=1$, $\mathfrak{g}_\beta$ is the root subspace of $\mathfrak{g}$ for each root β and θ is the Cartan involution of $\mathfrak{g}$ defined by $\theta X=I_{n,1}XI_{n,1}=-{}^t\bar{X}$ $(X\in\mathfrak{g})$. Then $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$

is an Iwasawa decomposition of $\mathfrak{g}$. Let $K, A, N, \bar{N}$ denote the analytic subgroups of G with Lie algebras $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively. Then $G=KAN$ is the Iwasawa decomposition of G corresponding to the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ and we have

$$K=\left\{\begin{pmatrix} A & \\ & (\det A)^{-1}\end{pmatrix} ; A\in U(n)\right\},$$

$$A=\left\{\begin{pmatrix} \cosh t & & \sinh t \\ & I_{n-1} & \\ \sinh t & & \cosh t\end{pmatrix} ; t\in\mathbb{R}\right\},$$

$$\bar{N}=\left\{P\begin{pmatrix} 1 & & & & \\ z_1 & 1 & & & \\ \vdots & & \ddots & & \\ z_{n-1} & & & 1 & \\ 1-F & -\bar{z}_1 & \cdots & -\bar{z}_{n-1} & 1\end{pmatrix}P^{-1} ;\quad F=1+\frac{1}{2}\sum_{i=1}^{n-1}|z_i|^2+iu,\ u\in\mathbb{R},\ z_1,\cdots,z_{n-1}\in\mathbb{C}\right\},$$

where

$$P=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & 1 \\ & \sqrt{2}I_{n-1} & \\ -1 & & 1\end{pmatrix}.$$

We can see that $\bar{N}$ can be identified with $\mathbb{C}^{n-1}\times\mathbb{R}$. For any

$$\bar{n}(z,u)=P\begin{pmatrix} 1 & & & & \\ z_1 & 1 & & & \\ \vdots & & \ddots & & \\ z_{n-1} & & & 1 & \\ 1-F & -\bar{z}_1 & \cdots & -\bar{z}_{n-1} & 1\end{pmatrix}P^{-1}\in\bar{N}$$

let $\bar{n}(z,u)=\kappa(\bar{n}(z,u))a(\bar{n}(z,u))n(\bar{n}(z,u))$ be the Iwasawa decomposition of $\bar{n}(z,u)$. Then we can see that

$$a(\bar{n}(z,u)) = P\,\mathrm{diag}(|F|,1,\cdots,1,|F|^{-1})P^{-1}, \tag{1.1}$$

$$\kappa(\bar{n}(z,u))=\begin{pmatrix} (2-F)/|F| & -\sqrt{2}\bar{z}_1/F & \cdots & -\sqrt{2}\bar{z}_{n-1}/F & 0 \\ \sqrt{2}z_1/|F| & 1-|z_1|^2/F & \cdots & -z_1\bar{z}_{n-1}/F & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \sqrt{2}z_{n-1}/|F| & -\bar{z}_1 z_{n-1}/F & \cdots & 1-|z_{n-1}|^2/F & 0 \\ 0 & 0 & \cdots & 0 & F/|F|\end{pmatrix} \tag{1.2}$$

(cf. [Se]). For a real vector space W we denote by W_C its complexification. Let $\mathfrak{k}_1=[\mathfrak{k},\mathfrak{k}]$ and $\mathfrak{z}$ be the center of $\mathfrak{k}$. Then we see that

$$\mathfrak{k}_{1C}=\left\{\begin{pmatrix} X & \\ & 0 \end{pmatrix} ; X\in\mathfrak{sl}(n,C)\right\},$$

$$\mathfrak{z}_C=C\begin{pmatrix} I_n & \\ & -n \end{pmatrix}.$$

Let M be the centralizer of A in K. Then

$$M=\left\{\begin{pmatrix} d & & \\ & X & \\ & & d \end{pmatrix} ; d^2\det X=1,\ X\in U(n-1)\right\}.$$

Let $\mathfrak{m}$ be the Lie algebra of M, $\mathfrak{m}_1=[\mathfrak{m},\mathfrak{m}]$ and put

$$\mathfrak{h}=\left\{\begin{pmatrix} h_1 & & & \\ & \ddots & & \\ & & h_n & \\ & & & 0 \end{pmatrix}\in\mathfrak{gl}(n+1,C) ; \sum_{j=1}^{n} h_j=0\right\},$$

$$\mathfrak{h}_{\mathfrak{m}}=\left\{\begin{pmatrix} 0 & & & & \\ & h_2 & & & \\ & & \ddots & & \\ & & & h_n & \\ & & & & 0 \end{pmatrix}\in\mathfrak{gl}(n+1,C),\ \sum_{j=2}^{n} h_j=0\right\}.$$

Then $\mathfrak{h}$ and $\mathfrak{h}_{\mathfrak{m}}$ are Cartan subalgebras of $\mathfrak{k}_{1C}(\simeq\mathfrak{sl}(n,C))$ and $\mathfrak{m}_{1C}$ respectively. Let $\{\alpha_1,\cdots,\alpha_{n-1}\}$ be the fundamental root system of $(\mathfrak{k}_{1C},\ \mathfrak{h})$ defined by

$$\alpha_i(H)=h_i-h_{i+1},\quad H=\begin{pmatrix} h_1 & & & \\ & \ddots & & \\ & & h_n & \\ & & & 0 \end{pmatrix},\quad i=1,2,\cdots,n-1.$$

Then $\{\alpha_2,\cdots,\alpha_{n-1}\}$ is the fundamental root system of $(\mathfrak{m}_{1C},\ \mathfrak{h}_{\mathfrak{m}})$. We now consider the Harish-Chandra C-function $C_\tau(\lambda)$ given by the integral (*) for $SU(n)$-irreducible representation τ, where ρ

denotes the rho function and $d\bar{n}$ denotes an Haar measure on $\bar{N}$. Since $\lambda\in\mathfrak{a}_C^*$ can be written in the form $\lambda=\mu_\lambda\alpha$ $(\mu_\lambda\in C)$ we identify λ with the complex number μ_λ. It is known that the integral (*) converges absolutely for $\lambda=\mu_\lambda\alpha\in\mathfrak{a}_C^*$ such that $\Re e\mu_\lambda>0$ (See e.g. [Wa1, §8.10.16]). Thus ρ is identified with n and (1.1) implies

$$e^{-(\lambda+\rho)H(\bar{n})}=|F(z,u)|^{-\lambda-n}.$$

Let $\lambda_i(i=1,\cdots,n)$ be the linear functionals on $\mathfrak{h}$ defined by

$$\lambda_i(H)=h_i \quad \text{for} \quad H=\begin{pmatrix} h_1 & & & \\ & \ddots & & \\ & & h_n & \\ & & & 0 \end{pmatrix}\in\mathfrak{h}.$$

§3. **The case for fundamental representations** ([E-K-M-W]).

j30 Let $V_0=C^n$ be the complex vector space of the n products of C and consider the vector space $V_0^r=V_0\otimes\cdots\otimes V_0$ (tensor products of r times, $r\geq 2$). As the special linear group SL(n,C) acts on V_0 in natural way, the special unitary group SU(n) does too. We then consider the tensor product representation μ induced from this identity representation. Let $e_i=\begin{pmatrix} 0\\ \vdots \\ 1 \\ \vdots \\ 0\end{pmatrix}$(i-th $(i=1,\cdots,n)$. Let W be the $\mu|_{SU(n)}$ irreducible subspace of V_0^r implying the highest weight vector $e_1\wedge\cdots\wedge e_r$ with the highest weight $\lambda_1+\cdots+\lambda_r$. We denote by τ_r the representation $\mu|_{SU(n)}$ of SU(n) on W. We see that $K=Z_KK_1$, here Z_K denoting the center of K and K_1 the semisimple part of K and also that $K_1\simeq SU(n)$ and $Z_K\simeq T^1$, T^1 denoting the one dimensional torus group $\{e^{i\theta};\theta\in R\}$. Since $\tau_r(z)(z\in Z=Z_K\cap K_1)$ acts as a scalar operator on W, we can write as $\tau_r(z)w=c(z)w$ ($c(z)\in U(1)$, $w\in W$). Take an irreducible unitary representation $\xi_m (m\in \mathbf{Z})$ of Z_K such that $c(z)=\xi_m(z)(z\in Z_K\cap K_1)$, where $\mathbf{Z}$ denotes the set of integers. The cases happen iff $m\equiv r(\mathrm{mod}\ n)$. Let τ be the irreducible unitary representation of K on W such that $\tau|_{Z_K}=\xi_m$ and $\tau|_{K_1}=\tau_r$. Then we have $W = W_1 + W_2$, here W_1 and W_2 denoting the M-invariant and irreducible subspaces implying the M-highest weight vectors $\mathfrak{v}_1=e_1\wedge\cdots\wedge e_r$ and $\mathfrak{v}_2=e_2\wedge\cdots\wedge e_{r+1}$, respectively. In this case, Harish-Chandra's C-function is given as follows:

$$(3.1) \qquad C_\tau(\lambda)=c\int_{\mathbf{C}^{n-1}\times\mathbf{R}} |F(z,u)|^{-\lambda-n}\ \tau(\kappa(\bar{n}(z,u)))dzd\bar{z}du,$$

where c is a certain constant coming from the normalization of the measures. We choose weight vectors $\mathfrak{v}_3,\cdots,\mathfrak{v}_d$($d=\dim W$) so that $\{\mathfrak{v}_1,\mathfrak{v}_2\cdots,\mathfrak{v}_d\}$ is a basis of W. Then we have the following expressions:

$$(3.2) \qquad \tau(\kappa(\bar{n}))\mathfrak{v}_1= \frac{\bar{F}-|z_r|^2-\cdots-|Z_{n-1}|^2}{|F|}\left(\frac{F}{|F|}\right)^m \mathfrak{v}_1+ \text{other},$$

$$(3.3) \qquad \tau(\kappa(\bar{n}))\mathfrak{v}_2= \frac{F-|z_1|^2-\cdots-|z_r|^2}{F}\left(\frac{F}{|F|}\right)^m \mathfrak{v}_2+ \text{other}.$$

Since $C_\tau(\lambda)$ acts as a scalar operator on each M-irreducible subspace W_i, we denote by $c_i(\lambda)$ the scalar induced by the action of $C_\tau(\lambda)$ on W_i. Then we have

$$(3.4) \qquad c_1(\lambda)=c\int_{\mathbf{C}^{n-1}\times\mathbf{R}}|F|^{-\lambda-n}\left(\frac{F}{|F|}\right)^m|F|^{-1}(\bar{F}-|Z_r|^2-\cdots-|Z_{n-1}|^2)dzd\bar{z}du,$$

$$(3.5) \qquad c_2(\lambda)=c\int_{\mathbf{C}^{n-1}\times\mathbf{R}}|F|^{-\lambda-n}\left(\frac{F}{|F|}\right)^m F^{-1}(F-|Z_1|^2-\cdots-|Z_r|^2)dzd\bar{z}du,$$

respectively.

Theorem 3.1. *The matrix elements* $c_1(\lambda)$, $c_2(\lambda)$ *of* $C_\tau(\lambda)$ *are represented as follows:*

$$(3.6)\qquad c_1(\lambda)= c\frac{(2\pi)^n\cdot 2^{-\lambda-1}\ \Gamma(\lambda)(\lambda-n+m+2r-1)}{\Gamma\left(\frac{\lambda+n-m+1}{2}\right)\Gamma\left(\frac{\lambda+n+m+1}{2}\right)}$$

$$(3.8)\qquad c_2(\lambda)= c\frac{(2\pi)^n\cdot 2^{-\lambda-1}\ \Gamma(\lambda)(\lambda+n-m-2r)}{\Gamma\left(\frac{\lambda+n-m+2}{2}\right)\Gamma\left(\frac{\lambda+n+m}{2}\right)}$$

Remark. In [Se] $c_1(\lambda)$ is computed in another way.

§4. The case for adjoint representation ([E-M-W]).

Now consider the complex vector space $V=\mathfrak{k}_C$ which is isomorphic to $\mathfrak{gl}(n,C)$. Then we see that $V = \mathfrak{k}_{1C}\oplus \mathfrak{z}_C$ is the Ad(K)-irreducible decomposition. Futhermore $V = \bigoplus_{i=0}^{4} V_i$ is the Ad(M)-irreducible decomposition, where

$$V_0= \mathfrak{z}_C ,$$

$$V_1= \{z\begin{pmatrix}1-n & & \\ & I_{n-1} & \\ & & 0\end{pmatrix} ;\ z\in C\},$$

$$V_2= \{\begin{pmatrix}0 & z_{12}\cdots & z_{1n} & 0\\ & & 0 & \end{pmatrix} ;\ z_{12},\cdots,z_{1n}\in C\},$$

$$V_3= \{\begin{pmatrix}0 & \\ z_{21} & \\ \vdots & 0\\ z_{n1} & \\ 0 & \end{pmatrix} ;\ z_{21},\cdots,z_{n1}\in C\},$$

$$V_4= \{\begin{pmatrix}0 & & \\ & X & \\ & & 0\end{pmatrix} ;\ X\in\mathfrak{sl}(n-1,C)\}.$$

We put

$$X_0=\begin{pmatrix} I_n & \\ & -n \end{pmatrix},\quad X_1=\frac{1}{1-n}\begin{pmatrix} 1-n & & \\ & I_{n-1} & \\ & & 0 \end{pmatrix},\quad X_2=\begin{pmatrix} 0\cdots 0 & 1 & 0 \\ & O & \end{pmatrix},$$

$$X_3=\begin{pmatrix} 0 & \\ 1 & \\ 0 & O \\ \vdots & \\ 0 & \end{pmatrix},\quad X_4=\begin{pmatrix} & 0 & 0 \\ & 1 & 0 \\ O & 0 & 0 \\ & \vdots & \vdots \\ & 0 & 0 \end{pmatrix}.$$

In our case Harish-Chandra's C-function is given as follows:

$$(4.1)\qquad C(\lambda)=c\int_{\mathbf{C}^{n-1}\times\mathbf{R}}|F(z,u)|^{-\lambda-n}\,\mathrm{Ad}\kappa(\bar{n}(z,u))\,dzd\bar{z}du,$$

where c is a certain constant coming from the normalization of measures. Paying attention that $C(\lambda)$ acts as a scalar operator on each M-irreducible subspace V_i, we denote by $c_i(\lambda)$ the scalar induced by the action of $C(\lambda)$ on V_i. Then we have the following.

Theorem 4.1. *We have the following expressions:*

$$(4.2)\qquad c_0(\lambda)=\frac{(2\pi)^n\cdot 2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+n}{2}\right)^2},$$

$$(4.3)\qquad c_1(\lambda)=\frac{(2\pi)^n\cdot 2^{-\lambda-2}\Gamma(\lambda)(\lambda-n)^2}{\Gamma\left(\frac{\lambda+n+2}{2}\right)^2},$$

$$(4.4)\qquad c_2(\lambda)=c_3(\lambda)=\frac{(2\pi)^n\cdot 2^{-\lambda-1}\Gamma(\lambda)(\lambda-n+1)}{\Gamma\left(\frac{\lambda+n+3}{2}\right)\Gamma\left(\frac{\lambda+n-1}{2}\right)},$$

$$(4.5)\qquad c_4(\lambda)=\frac{(2\pi)^n\cdot 2^{-\lambda}\Gamma(\lambda)(\lambda+n-2)}{\Gamma\left(\frac{\lambda+n+2}{2}\right)\Gamma\left(\frac{\lambda+n-2}{2}\right)(\lambda+n)}.$$

Remark. In [Se] $c_2(\lambda)$, $c_3(\lambda)$ are computed in another way.

§5 Other examples.

First of all we recall the Young diagram.

	$r_1 = 5$	1	2	3	4	5
T:	$r_2 = 3$	6	7	8		
	$r_3 = 1$	9				

T is a figure of boxes, which we put in key shape so that the first boxes in each row are put from the first and each row has as many as or more boxes the lower row does. Assume T has ν rows with lengths $r_1, \cdots, r_\nu$ $(r_1 \geq r_2 \geq \cdots \geq r_\nu > 0)$. Put then $r = r_1 + \cdots + r_\nu$. We insert freely the numbers $1, \cdots, r$ in those boxes. Let $\mathfrak{S}_r$ denote the group of permutations of the set $\{1, \cdots, r\}$ and $\mathfrak{R}_T$ (resp. $\mathfrak{L}_T$) be the subgroup consisting of those $\sigma \in \mathfrak{S}_r$ which permute the numbers in each row (resp. column) of T. Put

$$a_T = \sum_{\sigma \in \mathfrak{R}_T} \sigma, \qquad b_T = \sum_{\sigma \in \mathfrak{L}_T} \varepsilon_\sigma \cdot \sigma,$$

where ε_σ is the sign of σ. The operator $e_T = b_T a_T$ is called the Young operator. Let $V_0^r = \oplus_i V_{0i}^r$ be the SU(n)-irreducible decomposition of V_0^r. Then there is a Young diagram T_i for each V_{0i}^r such that $V_{0i}^r = e_{T_i}(V_0^r)$. We assign $e_1, \cdots, e_n$ to those boxes of T so that the i-th row has $p_i (0 \leq p_i \leq r_i)$ copies of e_i from the left and $q_i (0 \leq q_i \leq r_i)$ copies of e_{i+1} in the rest with $p_i + q_i = r_i$, $p_i \geq q_i$. It is known that any finite demensional irreducible representation τ of SU(n) corresponds to such a Young diagram T.

We consider now a new Young diagram T' obtained from T as follows. T' has ν or $\nu-1$ rows with lengths $r_1'=q_1+p_2, \cdots, r_{\nu-1}'=q_{\nu-1}+p_\nu$, $r_\nu'=q_\nu$ and each box of i-th row has e_{i+1} only $(1\le i\le\nu)$. Take the SU(n)-irreducible representation τ determined by T and decompose $V_T^r=e_T(V_0^r)$ into M-irreducible components. Then we see that $e_Te_{T'}(V_0^r)$ is an M-irreducible subspace of V_T^r.

Example 1. ($r_1=\cdots=r_\nu=1$ case)

We consider the following two Young diagrams T_1, T_2.

$$T_1:\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline r\\\hline\end{array} \qquad T_1':\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline \vdots\\\hline r\\\hline\end{array} \qquad T_2=T_2':\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline \vdots\\\hline r+1\\\hline\end{array}$$

(1) For $T=T_1$, since $a_T=1$, $b_T=\sum_{\sigma\in\mathfrak{L}_T}\varepsilon_\sigma\cdot\sigma$, $a_{T'}=1$, $b_{T'}=\sum_{\sigma'\in\mathfrak{L}_{T'}}\varepsilon_{\sigma'}\cdot\sigma'$, we have

$$e_T=\sum_{\sigma\in\mathfrak{L}_T}\varepsilon_\sigma\cdot\sigma, \qquad e_{T'}=\sum_{\sigma'\in\mathfrak{L}_T}\varepsilon_{\sigma'}\cdot\sigma'.$$

Thus

$$e_Te_{T'}(e_1\otimes\cdots\otimes e_\nu)=e_1\wedge\cdots\wedge e_\nu.$$

(2) For $T=T_2$, since $a_T=a_{T'}=1$, $b_T=b_{T'}=\sum_{\sigma\in\mathfrak{L}_T}\varepsilon_\sigma\cdot\sigma$, we have

$$e_T=e_{T'}=\sum_{\sigma\in\mathfrak{L}_T}\varepsilon_\sigma\cdot\sigma.$$

Thus

$$e_Te_{T'}(e_2\otimes\cdots\otimes e_{\nu+1})=e_2\wedge\cdots\wedge e_{\nu+1}.$$

These are the cases of fundamental representation of SU(n) described in §3.

Example 2. ($r_1=p+q=r$, $0\le p\le r$, $0\le q\le r$ case)

We consider the case that T consists of p copies of e_1 from the left and q copies of e_2 in the rest.

$$T:\ \overbrace{\boxed{e_1}\boxed{e_1}\boxed{\cdots\cdots}\boxed{e_1}}^{p}\overbrace{\boxed{e_2}\boxed{\cdots\cdots}\boxed{e_2}}^{q}$$

Since $a_T=\sum_{\sigma\in\mathfrak{R}_T}\sigma$, $b_T=1$, $a_{T'}=\sum_{\sigma'\in\mathfrak{R}_{T'}}\sigma'$, $b_{T'}=1$, we have

$$e_T=\sum_{\sigma\in\mathfrak{R}_T}\sigma,\qquad e_{T'}=\sum_{\sigma'\in\mathfrak{R}_{T'}}\sigma'.$$

Thus for $\mathfrak{v}_{pq}=\overbrace{e_1\otimes\cdots\otimes e_1}^{p}\otimes\overbrace{e_2\otimes\cdots\otimes e_2}^{q}\in V_0^r$, we have

$$e_Te_{T'}(\mathfrak{v}_{pq})=q!\,e_T(\mathfrak{v}_{pq})\quad(\text{say}=w_{pq}).$$

Let τ be the SU(n)-irreducible representation corresponding to T. Then for $k=(k_{ij})\in SU(n)$ we have

$$\tau(k)w_{pq}=\left(\sum_{i=0}^{p}\binom{p}{i}\binom{q}{i}k_{11}^{p-i}k_{22}^{q-i}(k_{12}k_{21})^{i}\right)w_{pq}+\text{other}$$

and

$$\tau(\kappa(\bar{n}))w_{pq}$$
$$=\left(\sum_{i=0}^{p}\binom{p}{i}\binom{q}{i}\left(\frac{2-F}{|F|}\right)^{p-i}\left(\frac{F-|z_1|^2}{F}\right)^{q-i}\left(\frac{-2|z_1|^2}{F|F|}\right)^{i}\right)w_{pq}+\text{other}.$$

Remark. In SU(2) case, any finite dimensional irreducible representation of SU(2) corresponds to a Young diagram of this type.

Example 3. ($r_1=2$, $r_2=1$ case)

We consider T as the following figure.

$$T: \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Since $a_T=1+(1\ 2)$, $b_T=1-(1\ 3)$, we have

$$e_T=(1-(1\ 3))(1+(1\ 2)).$$

(1) We consider T, T' as the following figures.

$$T: \quad \begin{array}{|c|c|}\hline e_1 & e_1 \\ \hline e_2 \\ \cline{1-1}\end{array} \qquad\qquad T': \quad \begin{array}{|c|}\hline e_2 \\ \hline\end{array}$$

Since $a_{T'}=1$, $b_{T'}=1$, $e_{T'}=1$, we have

$$e_Te_{T'}(e_1\otimes e_2\otimes e_1)=(e_1\wedge e_2)\otimes e_1.$$

Thus

$$\tau(\kappa(\bar{n}))e_Te_{T'}(e_1\otimes e_2\otimes e_1)= \tau(\kappa(\bar{n}))((e_1\wedge e_2)\otimes e_1).$$

Computing the last expression, we see that the coefficient of $(e_1\wedge e_2)\otimes e_1$ is

$$(k_{11}k_{22}-k_{12}k_{21})k_{11}= \left(\frac{\bar{F}-|z_2|}{|F|}\right)\left(\frac{2-F}{|F|}\right),$$

and that the value of the integral (*) coincides with $c_2(\lambda)$ in (4.4).

(2) We consider T, T' as the following figures.

$$T: \quad \begin{array}{|c|c|}\hline e_1 & e_2 \\ \hline e_2 \\ \cline{1-1}\end{array} \qquad\qquad T': \quad \begin{array}{|c|c|}\hline e_2 & e_2 \\ \hline\end{array}$$

Since $a_{T'}=1+(1\ 2)$, $b_{T'}=1$, $e_{T'}=1+(1\ 2)$, we have

$$e_Te_{T'}(e_1\otimes e_2\otimes e_2)= 2((e_1\wedge e_2)\otimes e_2).$$

Thus

$$\tau(\kappa(\bar{n}))e_Te_{T'}(e_1\otimes e_2\otimes e_2)= 2\tau(\kappa(\bar{n}))((e_1\wedge e_2)\otimes e_2).$$

Computing the last expression, we see that the coefficient of $(c_1\wedge e_2)\otimes c_2$ is

$$(k_{11}k_{22}-k_{12}k_{21})k_{22}= \left(\frac{\bar{F}-|z_2|^2}{|F|}\right)\left(\frac{F-|z_1|^2}{F}\right),$$

and that the value of the integral (*) coincides with $c_4(\lambda)$ in (4.5).

(3) We consider T, T' as the following figures.

$$T: \quad \begin{array}{|c|c|} \hline e_1 & e_2 \\ \hline e_3 & \\ \cline{1-1} \end{array} \qquad\qquad T': \quad \begin{array}{|c|} \hline e_2 \\ \hline e_3 \\ \hline \end{array}$$

Since $a_{T'}=1$, $b_{T'}=1-(1\ 3)$, $e_{T'}=1-(1\ 3)$, we have

$$e_{T'}(e_1\otimes e_3\otimes e_2)= e_1\otimes e_3\otimes e_2- e_1\otimes e_2\otimes e_3 ,$$

$$e_Te_{T'}(e_1\otimes e_3\otimes e_2)=(e_1\wedge e_3)\otimes e_2+ 2(e_2\wedge e_3)\otimes e_1-(e_1\wedge e_2)\otimes e_3.$$

Thus

(5.1) $\tau(\kappa(\bar{n}))e_Te_{T'}(e_1\otimes e_3\otimes e_2)$

$= \tau(\kappa(\bar{n}))((e_1\wedge e_3)\otimes e_2)+ 2\tau(\kappa(\bar{n}))((e_2\wedge e_3)\otimes e_1)- \tau(\kappa(\bar{n}))((e_1\wedge e_2)\otimes e_3).$

Computing the last expression, we see that the coefficient of $(e_1\wedge e_3)\otimes e_2$ is

$$(k_{11}k_{22}-k_{12}k_{21})k_{22}+ 2(k_{12}k_{33}-k_{13}k_{32})k_{21}-(k_{11}k_{32}-k_{31}k_{12})k_{23}$$
$$= \frac{1}{F|F|}((\bar{F}-|z_1|^2)(F-|z_1|^2)- 4|z_1|^2+|z_1|^2|z_2|^2)$$
$$= \frac{1}{F|F|}(|F|^2- 6|z_1|^2),$$

and that the integral value of each term in (5.1) coincides with each other. We find also that the value of the integral (*) coincides with $c_1(\lambda)$ in (4.3).

(4) We consider T, T' as the following figures.

$$T=T':\quad \begin{array}{|c|c|}\hline e_2 & e_2 \\ \hline e_3 & \\ \cline{1-1}\end{array}$$

Since $a_{T'}=1+(1\ 2)$, $b_{T'}=1-(1\ 3)$, $e_{T'}=(1-(1\ 3))(1+(1\ 2))$, we have

$$e_T e_{T'}(e_2 \otimes e_3 \otimes e_2) = 4(e_2 \wedge e_3) \otimes e_2.$$

Thus

$$\tau(\kappa(\bar{n}))e_T e_{T'}(e_2 \otimes e_3 \otimes e_2) = 4\tau(\kappa(\bar{n}))((e_2 \wedge e_3) \otimes e_2).$$

Computing the last expression, we see that the coefficient of $(e_2 \wedge e_3) \otimes e_2$ is

$$(k_{22}k_{33}-k_{23}k_{32})k_{22} = \frac{1}{F^2}(F-|z_1|^2-|z_2|^2)(F-|z_1|^2),$$

and that the value of the integral (*) coincides with $c_3(\lambda)$ in (4.4).

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