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Spectral analysis of a q-difference operator which arises from the quantum SU(1,1) group

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The finite dimensional representations of the quantum groups are studied in detail by many mathematicians and physicists. These representations correspond to the "unitary representations" of the "compact real forms" of the given quantum group. But, we know very little about the infinite dimensional representations which correspond to the unitary representations of the non-compact real forms. One of the simplest non-trivial example will be in the case of quantum $SU(1,1)$ group which we denote by $SU_q(1,1)$.

Let q be a parameter satisfying $0 < q < 1$ and consider the discrete subset

$$(-\infty,0]_{q^2} := \{-q^{2n}; n \in \mathbb{Z}\}$$

of $(-\infty,0]$ with the functions φ on $(-\infty,0]_{q^2}$. For these φ's, the q-differentiation (finite difference) and the q-integration (discrete sum) are defined by

$$[D_{q^2}\varphi](\zeta) := \frac{\varphi(\zeta)-\varphi(q^2\zeta)}{(1-q^2)\zeta} \quad , \zeta \in (-\infty,0]_{q^2} \text{ and}$$

$$\int_{-\infty}^{0} \varphi(\zeta)d_{q^2}\zeta := \sum_{n\in\mathbb{Z}} \varphi(-q^{2n})q^{2n}(1-q^2) .$$

Then we have the Hilbert space $L^2(-\infty,0]_{q^2}$ and the natural Schwartz space $\mathcal{S}(-\infty,0]_{q^2}$ defined by the seminorms $P_{m,n}(\varphi) := \|\zeta^m[D^n_{q^2}\varphi](\zeta)\|_\infty$ for $m,\ n \in \mathbb{N}$. (Note that $\|\ \ \|_\infty$ is the supremum norm.) The operator P obtained as an action of the Casimir element in $U_q(sl(2))$ is given by

$$P := T_{q^2}^{-1}D_{q^2}\zeta(1-q^2\zeta)D_{q^2}$$

where $T_{q^2}^{-1}$ is the inverse of the operator T_{q^2} defined by $[T_{q^2}\varphi](\zeta) := \varphi(q^2\zeta)$.

THEOREM 1. *(1) The operator $(P, \mathcal{S}(-\infty,0]_{q^2})$ is essentially self-adjoint.*

(2) The self-adjoint extension of $(P, \mathcal{S}(-\infty,0]_{q^2})$ is given by $(P, \mathcal{D})$, where

$$\mathcal{D} := \{\varphi \in L^2(-\infty,0]_{q^2} : P\varphi \in L^2(-\infty,0]_{q^2} \ , \lim_{\zeta\to 0} \sqrt{-\zeta}[D_{q^2}\varphi](\zeta) = 0\}.$$

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THEOREM 2. *The spectrum $\sigma(P)$ of the operator P is given by the disjoint union of the continuous spectrum $\sigma_c(P)$ and the point spectrum $\sigma_p(P)$:*

$$\sigma_c(P) = \{\lambda(e^{i\theta}) : 0 \leq \theta \leq \pi\} = \left\{\frac{q^2}{(1+q)^2} \leq \lambda \leq \frac{q^2}{(1-q)^2}\right\},$$
$$\sigma_p(P) = \{\lambda(-q^{2n+1}) : n \in \mathbb{N}\}, \text{ where}$$
$$\lambda(\alpha) := \frac{q^2}{(1-q^2)^2}(1 + q^2 - (\alpha + \alpha^{-1})\, q)$$

The representations corresponding to $\sigma_c(P)$ are the quantum deformations of the principal series. We see that the set $\sigma_c(P)$ is compact and instead, there appears a new series of representations corresponding to $\sigma_p(P)$ which we call "strange series". We then have a formal observation that $\sigma_p(P)$ disappears and the set $\sigma_c(P)$ "converges to" the set $[\frac{1}{4}, \infty)$ if we take the "classical limit $q \to 1$."

It should also be remarked that the operator P is semi-bounded and the compact continuous spectrum lies in the "low energy region". Then the unboundedness of the operator is due to the point spectrum which lies in the "high energy region". This type of phenomena almost never observed in the case of classical differential operators.

For suitable parameters α, β and γ, the basic hypergeometric function is defined by

$$_2\varphi_1\left(\begin{matrix}\alpha & & \beta \\ & \gamma & \end{matrix} : q; \zeta\right) := \sum_{n=0}^{\infty} \frac{(\alpha : q)_n (\beta : q)_n}{(\gamma : q)_n (q : q)_n} \zeta^n$$

where $(\alpha : q)_n := (1-a)(1-aq)\dots(1-aq^{n-1})$. For $\alpha \in \mathbb{C}\backslash\{0\}$, the function of ζ defined by

$$\varphi(\alpha; \zeta) := {}_2\varphi_1\left(\begin{matrix}q\alpha & & q\alpha^{-1} \\ & q^2 & \end{matrix} : q^2; q^2\zeta\right)$$

satisfies $(P - \lambda(\alpha))\varphi(\alpha; \zeta) = 0$.

THEOREM 3. *(1) For $n \in \mathbb{N}$, the eigenfunction associated with the eigenvalue $\lambda(-q^{2n+1})$ is given by $\varphi(-q^{2n+1}; \zeta)$.*

(2) For $\lambda(e^{i\theta}) \in \sigma_c(P)$, $0 \leq \theta \leq \pi$, the generalized eigenfunction associated with the eigenvalue $\lambda(e^{i\theta})$ is given by $\varphi(e^{i\theta}; \zeta)$.

With these explicit eigenfunctions, the Kodaira-Titchmarsh type expansion theorem and the Plancherel theorem are also obtained.

This research will be the beginning part of the whole program of the theory of unitary representations of $SU_q(1,1)$. The details on the spectral theory will soon be published:

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