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量子群版 dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$ とその Capelli Identity

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0: Introduction

This brief note is an announcement of our recent work on a q-analogue of the dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$. As everyone knows, the standard process of making the q-deformations of the universal enveloping algebras of simple Lie algebras is not functorial in the sense that it is not compatible with inclusion maps. This difficulty is the first thing to manage for whoever attempts to define a q-analogue of the dual pairs. Our strategy here is the following: start from the oscillator representation ω; make its nth tensor power representation $\omega^{\otimes n}$; then see what appears as the commutant of the representation. The result is that starting from the oscillator representation ω of $U_q(\mathfrak{sl}_2)$, we will get a representation of a q-deformation of $U(\mathfrak{o}_n)$, which is *different* from the standard one that Drinfeld and Jimbo defined. The q-deformed algebra $U_q(\mathfrak{o}_n)$ appeared here turns out to be what Gavrilik and Klimyk defined in [GK]. This algebra is not a Hopf algebra but plays an important role also in the study of quantized homogeneous spaces in [N3]. Incidentally for the Howe duality $(\mathfrak{sp}_{2m}, \mathfrak{o}_n)$ in the case of Yangian, similar twisted objects introduced in [O] natually come into the story [Na2].

As a quantum counterpart of the spherical harmonics, we have lots to work with this new dual pair. We will, however, restrict ourselves here to stress on one point. The main result of the present report is the Capelli identity, which is an explicit identity between two central elements of these two algebras in duality. Though the Capelli identity we got is classically the easiest one, its direct proof is by far more difficult.

1: The oscillator representation and its tensor powers

Let us first recall the definition of $U_{q^2}(\mathfrak{sl}_2)$. This is an associative algebra over the ground field $\mathbb{K} = \mathbb{Q}(q)$ generated by the elements $e, f, k^{\pm 1}$ with the relations

$$ke = q^4 ek, \quad kf = q^{-4} fk;$$
$$ef - fe = \frac{k - k^{-1}}{q^2 - q^{-2}} = \frac{1}{[2]} \frac{k - k^{-1}}{q - q^{-1}}.$$

Here [2] stands for the basic number $q + q^{-1}$. Their comultiplication rule is given by

$$\begin{cases} \Delta(e) = e \otimes 1 + k^{-1} \otimes e \\ \Delta(f) = f \otimes k + 1 \otimes f \\ \Delta(k) = k \otimes k. \end{cases}$$

The oscillator representation of $U_{q^2}(\mathfrak{sl}_2)$ is realized on the polynomial ring $\mathbb{K}[x]$ of one variable. For this definition we introduce some notations. Let γ be the algebra automorphism of $\mathbb{K}[x]$ given by $\gamma : x \mapsto qx$. The q-difference operator $\partial = \partial_q$ is then defined as

$$\partial = \partial_q = x^{-1}\frac{\gamma - \gamma^{-1}}{q - q^{-1}}.$$

Using these, we define the following three operators:

$$\omega(e) = \frac{x^2}{[2]}; \qquad \omega(f) = \frac{\partial^2}{[2]}; \qquad \omega(k) = q\gamma^2.$$

An easy calculation shows that the commutator $[\partial^2, x^2]$ of ∂^2 and x^2 yields

$$[\partial^2, x^2] = [2]\,\frac{q\gamma^2 - q^{-1}\gamma^{-2}}{q - q^{-1}},$$

so that the ω above gives us a right $U_{q^2}(\mathfrak{sl}_2)$-module structure on $\mathbb{K}[x]$. We call this the oscillator representation, or more precisely a q-oscillator representation. Note that this is not irreducible but breaks into two irreducible components which consist of the polynomials respectively of even and odd degrees.

Since the algebra $U_{q^2}(\mathfrak{sl}_2)$ is a Hopf algebra with the comultiplication Δ given above, we can accordingly define the nth tensor power $\omega^{\otimes n}$ of the representation ω. It is convenient, and even natural from the geometrical point of view, to identify the representation space of $\omega^{\otimes n}$ with the q-commutative ring of n variables.

We denote simply by $\mathcal{A} = \mathbb{K}[x_1, \cdots, x_n]$ the q-commutative ring with the relations $x_i x_j = q x_j x_i$ for $i < j$. Then using the multiplication in $\mathcal{A}$, we identify

$$\mathbb{K}[x_1] \otimes \cdots \otimes \mathbb{K}[x_n] \xrightarrow[\text{積}]{\sim} \mathbb{K}[x_1, \cdots, x_n].$$

Through this identification and the action ω on $\mathbb{K}[x] \simeq \mathbb{K}[x_i]$, we have an action $\omega^{\otimes n}$ of $U_{q^2}(\mathfrak{sl}_2)$ on $\mathcal{A}$. To describe this, we introduce the notations on q-difference operators as follows. Let γ_i be the algebra automorphism of $\mathcal{A}$ given by $\gamma_i : x_j \mapsto q^{\delta_{ij}} x_j$. We put $\gamma = \gamma_1 \cdots \gamma_n$. The q-difference operator $\partial_i = \partial_{i,q}$ is defined as

$$\partial_i = x^{\circ -1}\frac{\gamma_i - \gamma_i^{-1}}{q - q^{-1}}.$$

Here for an element $a \in \mathcal{A}$, we denote its left or right multiplication operator by a or a° respectively.

With this notation, we have

$$\begin{aligned}
\omega^{\otimes n}(e) &= \frac{1}{[2]}(x_1^2 + q^{-1}x_2^2 + \cdots + q^{-n+1}x_n^2) \\
\omega^{\otimes n}(f) &= \frac{1}{[2]}(q^{n-1}\partial_1^2 + q^{n-2}\partial_2^2 + \cdots + \partial_n^2) \\
\omega^{\otimes n}(k) &= q^n \gamma_1^2 \cdots \gamma_n^2 = q^n \gamma^2
\end{aligned}$$

We put for brevity

$$\begin{cases} Q &= x_1^2 + q^{-1}x_2^2 + \cdots + q^{-n+1}x_n^2, \\ \Delta &= q^{n-1}\partial_1^2 + q^{n-2}\partial_2^2 + \cdots + \partial_n^2. \end{cases}$$

2: Commutant of $\omega^{\otimes n}$, the algebra of Gavrilik-Klimyk and Capelli identity

On the space $\mathcal{A}$, we have another action of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ from the left. In terms of L-operators, it can be written as

$$\begin{aligned} L_{ii}^+ &= (L_{ii}^-)^{-1} = \gamma_i \qquad (i = 1, \cdots, n) \\ L_{ji}^+ &= (q - q^{-1})x_i^\circ \partial_j \qquad (j < i) \\ L_{ji}^- &= -(q - q^{-1})x_i^\circ \partial_j \qquad (i < j). \end{aligned}$$

See [**NUW1**] for the definition of $U_q(\mathfrak{gl}_n)$ and the L-operators. Here by abuse of notation, we used the same symbol for an element in $U_q(\mathfrak{gl}_n)$ and its representation. Note that under this action, the multiplication of $\mathcal{A}$ gives a $U_q(\mathfrak{gl}_n)$-homomorphism.

We now set θ_j in $U_q(\mathfrak{gl}_n)$ for $j = 1, \cdots, n-1$ as

$$\theta_j = (q - q^{-1})^{-1} q^{-\epsilon_j}(L_{j\,j+1}^+ - qS(L_{j+1\,j}^-)),$$

where S is the antipode. These elements represent the generators of the q-deformed algebra $U_q(\mathfrak{so}_n)$ of Gavrilik-Klimyk [GK]. In fact, they satisfy the following relations.

$$\begin{cases} [\theta_i, \theta_j] = 0 \quad \text{if} \quad |i - j| > 1 \\ \theta_i^2\theta_j - (q + q^{-1})\theta_i\theta_j\theta_i + \theta_j\theta_i^2 = -\theta_j \quad \text{if} \quad |i - j| = 1. \end{cases}$$

If we let them act on $\mathcal{A}$, then they take the form

$$\begin{aligned} \theta_j &= x_{j+1}^\circ \gamma_j^{-1} \partial_j - x_j^\circ \gamma_{j+1} \partial_{j+1} \\ &= (x_{j+1}\partial_j - x_j\partial_{j+1})\gamma_1 \cdots \gamma_{j-1}\gamma_{j+2}^{-1} \cdots \gamma_n^{-1} \end{aligned}$$

The point is that they commute with the action $\omega^{\otimes n}$ of $U_{q^2}(\mathfrak{sl}_2)$. Furthermore in a suitable algebra of q-difference operators, e.g. $\mathbb{K}[x_i, \partial_i, \gamma_i^{\pm 1}; 1 \leq i \leq n]$, the two algebras form the mutual commutants. With this fact we may well regard the pair $(U_{q^2}(\mathfrak{sl}_2), U_q(\mathfrak{so}_n))$ as a quantum analogue of the dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$.

To define the Casimir element of $U_q(\mathfrak{so}_n)$, we make appropriate elements corresponding not only to the generators but also to the whole Lie algebra $\mathfrak{o}_n$. The elements $\theta_{ij}^\pm$ for $1 \leq i < j \leq n$ are inductively defined as

$$\begin{cases} \theta_{i+1\,i}^\pm = \theta_i, \\ \theta_{ji}^\pm = \theta_{jk}^\pm\theta_{ki}^\pm - q^{\pm 1}\theta_{ki}^\pm\theta_{jk}^\pm \quad (i < k < j). \end{cases}$$

The choice of k does not affect on the definition as long as $i < k < j$, hence well-defined. Though the explicit form of these elements is a little bit complicated, we can write them down:

$$\theta_{ji}^{+} = (x_j\partial_i\gamma_{i+1}^{-1}\cdots\gamma_{j-1}^{-1} - q^{j-i-1}x_i\partial_j\gamma_{i+1}\cdots\gamma_{j-1})\gamma_1\cdots\gamma_{i-1}\gamma_{j+1}^{-1}\cdots\gamma_n^{-1}$$
$$+ (q-q^{-1})\sum_{i<k<j} q^{j-k}x_k^2\partial_i\partial_j(\gamma_1\cdots\gamma_{i-1})^2\gamma_i\gamma_j^{-1}(\gamma_{j+1}^{-1}\cdots\gamma_n^{-1})^2,$$
$$\theta_{ji}^{-} = (x_j\partial_i\gamma_{i+1}\cdots\gamma_{j-1} - q^{i-j+1}x_i\partial_j\gamma_{i+1}^{-1}\cdots\gamma_{j-1}^{-1})\gamma_1\cdots\gamma_{i-1}\gamma_{j+1}^{-1}\cdots\gamma_n^{-1}$$
$$- (q-q^{-1})\sum_{i<k<j} q^{i-k}x_ix_j\partial_k^2(\gamma_1\cdots\gamma_{i-1})^2\gamma_i\gamma_j^{-1}(\gamma_{j+1}^{-1}\cdots\gamma_n^{-1})^2.$$

From this we can see that $\theta_{ij}^{\pm}$ is no more "first order" but is a "second order" difference operator for $|i-j| > 1$. With these elements, the (second order) Casimir element C of $U_q(\mathfrak{so}_n)$ is defined as

$$C = \sum_{i<j} q^{n+1-i-j}\theta_{ji}^{-}\theta_{ji}^{+} = \sum_{i<j} q^{-n-1+i+j}\theta_{ji}^{+}\theta_{ji}^{-}.$$

This element is seen to be in the center of $U_q(\mathfrak{so}_n)$. Our Capelli identity is now in order:

THEOREM. **(Capelli Identity)**

$$Q\,\Delta - \{\gamma\}\{q^{n-2}\gamma\} = C$$

Here we used the notation $\{a\} = (a - a^{-1})/((q - q^{-1})$.

Note that the left-hand side of the identity is essentially the Casimir element of $U_{q^2}(\mathfrak{sl}_2)$ modulo Euler operators. In other words, this identity expresses explicitly the coincidence of the central elements of the two algebras $U_{q^2}(\mathfrak{sl}_2)$ and $U_q(\mathfrak{so}_n)$ which are in duality. This looks analogous in its form to the classical case (see e.g. [**HU**]). But in checking it by a straightforward calculation based on the explicit formulas above, we come across quite a few tricky cancellations.

An analogue of b-function is of course deduced from this similarly as [**NUW1**]:

$$\Delta(Q^s) = \{q^{2s}\}\{q^{n-2+2s}\}Q^{s-1}.$$

Actually in this case such calculation is much easier and is implied direclty from the fact that $\{Q,\ \Delta,\ \gamma\}$ gives a representation of $U_{q^2}(\mathfrak{sl}_2)$.

3: Remarks

(1) Only for $U_{q^2}(\mathfrak{sl}_2)$, the classical notion of q-difference operators will suffice. For the general $U_q(\mathfrak{sp}_{2m})$, we need to employ the q-differential operators that were introduced in [**NUW1**]. In this article we restricted ourselves to the simplest case. However, we can show how the tensor power of the oscillator representation of $U_q(\mathfrak{sp}_{2m})$ look like on the

coordinate ring $\mathcal{A}(Mat_q(m \times n))$ of q-deformed matric space. For simplicity, we give here the formulas for $n = m$ without detailed explanations:

$$\omega^{\otimes n}(L_{ji}) = (q - q^{-1}) \sum_{k=1}^{n} t^{\circ}_{ki} \partial_{kj} \quad (1 \leq i, j \leq n)$$

$$\omega^{\otimes n}(e_n) = \frac{1}{[2]} \sum_{k=1}^{n} q^{k-1} \partial^2_{kn}$$

$$\omega^{\otimes n}(f_n) = \frac{1}{[2]} \sum_{k=1}^{n} q^{k-n} t^{\circ 2}_{kn}$$

$$\omega^{\otimes n}(k_n) = q^{n+2\epsilon_n}.$$

The first one corresponds to the representation of $\mathfrak{gl}_n$ embedded in $\mathfrak{sp}_{2n}$.

(2) Behind the fact that the Casimir element C is central, we have a multiplicative structure and reflection equations. We will briefly explain them. Let us introduce the following four matrices:

$$\Theta^+ := S(L^+) J^t S(L^-), \qquad \Theta^- := {}^t L^+ J L^-;$$
$$\tilde{\Theta}^+ := S(L^-) J^t S(L^+), \qquad \tilde{\Theta}^- := {}^t L^- J L^+.$$

Here $L^{\pm}$ is the L-operators in matix form and $J = \mathrm{diag}(q^{n-1}, q^{n-2}, \cdots, 1)$, which correspond to the quadratic form defining the quantum $\mathfrak{o}_n$. Then the entries of those four matrices give the $\theta^{\pm}_{ji}$'s. More precisely they are triangular matrices whose dialgonal components coincide with J and subdiagonals θ_j's and

$$\Theta^+_{ij} = -(q - q^{-1}) q^{n-j} \theta^+_{ji} \quad \text{if} \quad (i < j),$$
$$\Theta^-_{ij} = (q - q^{-1}) q^{n-j-1} \theta^-_{ij} \quad \text{if} \quad (i > j).$$

Furthermore $\Theta^{\pm}$ and $\tilde{\Theta}^{\mp}$ can be transformed to each other under an involution that transforms $L^{\pm}$ to $\tilde{L}^{\pm} = {}^t S(L^{\mp})$.

With these matrices, the Casimir element C of $U_q(\mathfrak{so}_n)$ is given by

$$\mathrm{Tr}(\Theta^- - J)(J - \Theta^+) = (q - q^{-1})^2 q^{n-2} C.$$

From this expression and the reflection equations below, we can show that the Casimir element C is central. More generally with this formulation, we can have higher degree central elements of $U_q(\mathfrak{so}_n)$ as follows:

THEOREM. *Put $X = \Theta^- \Theta^+$ and $X^{[m]} = X(J^{-2}X)^{m-1}$ for a positive integer m. Then*

$$(\tilde{\Theta}^+_2 J_2^{-2})^{-1} \, {}^{t_1}R^+_{12} X^{[m]}_1 \, {}^{t_2}R^-_{12} (\tilde{\Theta}^+_2 J_2^{-2}) = R^+_{12} X^{[m]}_1 J_2^2 (R^+_{12})^{-1} J_2^{-2}.$$

Here ${}^{t_1}R^{\pm}_{12}$ or ${}^{t_2}R^{\pm}_{12}$ denotes respectively the transposition of $R^{\pm}_{12}$ with respect to the first or the second component in the tensor space. Take here the trace $\mathrm{Tr}^{(1)}$ for the first component, then

$$(\tilde{\Theta}^+ J^{-2})^{-1} \, \mathrm{Tr}(X^{[m]}_1)(\tilde{\Theta}^+ J^{-2}) = \mathrm{Tr}(X^{[m]}_1).$$

In particular, this shows that $\mathrm{Tr}\, X^{[m]}$ is in the center of $U_q(\mathfrak{so}_n)$.

For the proof we use the following reflection equations:

LEMMA. **(Reflection Equations)** *We have the following:*

$$\Theta_2^{-\,t_1} R_{12}^{+} \Theta_1^{-} R_{12}^{\pm} = R_{12}^{\pm} \Theta_1^{-\,t_1} R_{12}^{+} \Theta_2^{-},$$
$$\Theta_2^{+\,t_1} R_{21}^{+} \Theta_1^{+} R_{21}^{\pm} = R_{21}^{\pm} \Theta_1^{+\,t_1} R_{21}^{+} \Theta_2^{+},$$
$$\Theta_1^{+\,t_2} R_{12}^{-} \tilde{\Theta}_2^{+} R_{12}^{+} = R_{12}^{+} \tilde{\Theta}_2^{+\,t_2} R_{12}^{-} \Theta_1^{+},$$

and

$$^{t_1}R_{12}^{+} \Theta_1^{-} R_{12}^{+} \tilde{\Theta}_2^{+} J_2^{-2} = \tilde{\Theta}_2^{+} J_2^{-2} R_{12}^{+} \Theta_1^{-\,t_1} R_{12}^{+}.$$

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