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Cohomology of discontinuous subgroups of $\mathbb{Q}$-rank 1 in $Sp_4(\mathbb{R})$

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Introduction.

Let $\mathcal{D}$ be a bounded symmetric domain, $G = (\mathrm{Aut}\,\mathcal{D})^\circ$ be the identity component of the isometries of $\mathcal{D}$, and Γ be an arithmetic subgroup of G such that the quotient $\Gamma\backslash\mathcal{D}$ is non-compact. Then we can associate various cohomology groups to $\Gamma\backslash\mathcal{D}$:

$H^i(\Gamma\backslash\mathcal{D}, \mathbb{Q})$ singular cohomology (without support);

$H^i_c(\Gamma\backslash\mathcal{D}, \mathbb{Q})$ cohomology with compact support;

$H^i_{(2)}(\Gamma\backslash\mathcal{D}, \mathbb{C})$ square-integrable cohomology (in the sense of Borel).

Let $(\Gamma\backslash\mathcal{D})^*$ be the minimal compactification, or the Satake compactification of $\Gamma\backslash\mathcal{D}$. Then by the proof of Zucker's conjecture, it is confirmed

$$H^i_{(2)}(\Gamma\backslash\mathcal{D}, \mathbb{C}) \cong IH^i((\Gamma\backslash\mathcal{D})^*, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

where $IH^i((\Gamma\backslash\mathcal{D})^*, \mathbb{Q})$ is the middle intersection cohomology of $(\Gamma\backslash\mathcal{D})^*$.

It is believed that L^2-cohomology $H^i_{(2)}(\Gamma\backslash\mathcal{D}, \mathbb{C})$ has good accordance with the Selberg trace formula. But if one want to know the congruence zeta function of the reduction of the Shimura variety S associated to $\Gamma\backslash\mathcal{D}$, it is necessary to know the l-adic cohomology $H^i_c(S\otimes\overline{\mathbb{Q}}, \mathbb{Q}_l)$ with compact support.

Thus we are led to consider the following

<u>PROBLEM</u>. (i) What is the difference of $H^i_c(\Gamma\backslash\mathcal{D}, \mathbb{C})$ and $H^i_{(2)}(\Gamma\backslash\mathcal{D},\mathbb{C})$? Or, passing to the Poincaré dual, what is the difference of $H^i(\Gamma\backslash\mathcal{D}, \mathbb{C})$ and $H^i_{(2)}(\Gamma\backslash\mathcal{D},\mathbb{C})$?

We consider the case when $G = Sp_4(\mathbb{R}) = \{g \in M_4(\mathbb{R}) \mid {}^t g J g = J,\ J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0\end{pmatrix}\}$, and $\mathcal{D} = \{ Z \in M_2(\mathbb{C}) \mid Z = {}^tZ,\ \mathrm{Im}\, Z >> 0 \}$. B is a central division algebra over $\mathbb{Q}$, s.t. $B\otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$. And the $\mathbb{Q}$-form of G is given by

$$G_{\mathbb{Q}} = \{ g = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in M_2(B) \mid \begin{pmatrix}\alpha^\iota & \gamma^\iota \\ \beta^\iota & \delta^\iota\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\}$$

Here ι is the main involution of B. We consider arithmetic subgroups Γ in $G_{\mathbb{Q}}$. The group $G_{\mathbb{Q}}$ has $\mathbb{Q}$-rank 1. The quotient $\Gamma\backslash\mathcal{D}$ has only point cusps.

This is the simplest non-trivial case of $\mathbb{R}$-rank 2, because the Hilbert modular case is trivial as far as the problem of comparison between L^2-cohomology and compact support cohomology is concerned.

We want to investigate the cohomology group $H^i(\Gamma\backslash D, \mathbb{C})$ from two view point:

(i) as a mixed Hodge structure;

(ii) Eisenstein cohomology.

We shall show that these two view point has a compatibility. Hence we can describe (i) in terms of the special values of Eisenstein series.

We have some deeper investigation for the second cohomology group $H^2(\Gamma\backslash D, \mathbb{Q})$.

The method of proof uses, in one part, the theory of toroidal compactification; another the theory of Eisenstein cohomology classes.