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Character formula for cuspidal unramified series representations of the multiplicative group of division algebras over local fields

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Let D be a division algebra of degree $n \geq 2$ over a non-archemedian local field F, and E be a maximal unramified extension of F in D. Then the multiplicative group $D^\times$ of D is a topological group compact modulo the center $F^\times$ and the multiplicative group $E^\times$ of E is a Cartan subgroup of $D^\times$. Associated with an admissible quasi-character θ of $E^\times$ over F, in the sense of Howe [H], Corwin-Howe [CH], an irreducible representation π_θ of $D^\times$ has been constructed by Howe [H], Corwin-Howe [CH], Moy [M], Gerardin [G2]. In this article, we shall determine the values of the character of π_θ, for a certain admissible quasi-character θ of $E^\times$, on certain regular integral elements t of $E^\times$, by the formula analogous to the famous Weyl's one for irreducible unitary representations of compact Lie groups, as follows

$$ch\,\pi_\theta(t) = (-1)^\varepsilon \sum_{\gamma \in \Gamma} \theta(t^\gamma) / |D(t)|^{1/2}$$

where Γ and $|x|$ denote the Galois group of E over F and the canonical valuation of an element x of F, respectively,

$$D(t) = \prod_{\gamma \neq \gamma'} ((t^\gamma / t^{\gamma'})^{1/2} - (t^{\gamma'} / t^\gamma)^{1/2}),$$

$\varepsilon = 0, 1$ and it is completely determined, in theorem 6 of §3.

§1. Preliminaries.

Let F be a non-archemedian local field. Denote by $\mathcal{O}_F$, P_F, ω and $\bar{F}$ the maximal order of F, the maximal ideal of $\mathcal{O}_F$, an uniformizer of $\mathcal{O}_F$ and the residue field of F. Let p and q be the characteristic and the order of $\bar{F}$ respectively. Let D be a division algebra of degree $n \geq 2$ over F. Denote by $\mathcal{O}_D$, $\mathcal{P}_D$ and Π the maximal order of D, the maximal ideal of $\mathcal{P}_D$, an uniformizer of $\mathcal{O}_D$. Let E be a maximal unramified extension of F in D. We denote by $\mathcal{O}_E$, $\mathcal{P}_E$ and $\bar{E}$ for E the same as those for F and D. Then ω is also an uniformizer of $\mathcal{O}_E$. Denote by v_E be the valuation of E normalized by $v_E(\omega) = 1$.

From now on, we assume that p is odd and n is prime to p.

The following proposition is well known by Weil [We] or Serre [S].

Proposition 1. *Notations and asummptions being as above. D has the following properties:* (i) *We may put $\Pi^n = \omega$.* (ii) *$\{1, \ldots, \Pi^{n-1}\}$ is a basis of D as a left vector space over E, and generates $\mathcal{O}_D$ as a left $\mathcal{O}_E$-module.* (iii) *We have*

$$\Pi^{-1} x \Pi = x^{\sigma} \quad \textit{for any } x \in E,$$

where σ is a generator of the Galois group Γ of E over F. (iv) *The multiplicative group $D^{\times}$ of D is a totally disconected topological group with a fundamental system of neiborhoods $\{1+\mathcal{P}_D^m\}_{m \geq 1}$ of the unit 1. The multiplicative group $E^{\times}$ of E is a Cartan subgroup of $D^{\times}$ and satisfies $E^{\times} \cap (1+\mathcal{P}_D^m) = 1+\mathcal{P}_E^m$ for any integer $m \geq 1$.*

Definition. A quasi-character θ of $E^{\times}$ is *admissible* (*over F*) if θ does not come via the norm from a subextension of E/F.

Remark. In general, another condition is imposed on the definition of the admissibility. But, as E/F is unramified, it can be omitted.

If θ is an admissible quasi-character of $E^{\times}$, by Howe [H1], Moy [M], Waldspurger [Wa], the following facts have been verified: There exist a unique tower of fields

$$F = E_0 \subset E_1 \subset \ \dots \ \subset E_r \subset E_{r+1} = E$$

and quasi-characters

$$\theta_0,\ \theta_1,\ \dots,\ \theta_r,\ \theta_{r+1}$$

of $E_0^\times$, $E_1^\times$, ... ,$E_r^\times$, $E_{r+1}^\times$,respectively, such that

$$(*) \qquad \theta = (\theta_0 \cdot N_{E/E_0})\,(\theta_1 \cdot N_{E/E_1})\,\dots\,(\theta_{r+1} \circ N_{E/E_r})\,\theta_r$$

which is called a Howe factorization of θ, where N_{E/E_i} denotes the norm mapping from E into E_i. There also exist elements

$$c_1 \in E_1^\times,\ \dots,\ c_r \in E_r^\times,\ c_{r+1} \in E_{r+1}^\times$$

such that each c_i is multiplicatively generic over $E_{i-1}^\times$ (due to H. Hijikata), it means that the residue field $\bar{E}_i$ is generated by the canonical image of $\omega^{-v_E(c_i)} c_i$ over the residue field $\bar{E}_{i-1}$. For each i, denote by $f(\theta_i)$ the conductoral exponent of θ_i, that is, θ_i is trivial on $\mathcal{P}_E^{f(\theta_i)}$ and not trivial on $\mathcal{P}_E^{f(\theta_i)-1}$. Put $f(\theta_i) = m_i + 1$ for $i = 1, \dots, r+1$. Then we have

$$m_1 > \dots > m_r > m_{r+1} \geq 0,$$

and $m_i = -v_E(c_i)$ for $i = 1, \dots, r+1$. Let $\psi = \psi_F$ be an additive quasi-character of F whose conductoral exponent is 1. We fix this ψ from now on. Put $\psi_{E_i} = \psi \cdot Tr_{E/E_i}$ for $i = 1, \dots, r$, where Tr_{E/E_i} denotes the trace mapping from E into E_i. Then, for $i = 1, \dots, r$, we have

$$\theta_i(1+x) = \psi_{E_i}(c_i x) \quad \text{for any } x \in \mathcal{P}_{E_i}^{[m_i/2]}$$

where [ ] denotes the greatest integer function.

§2. Construction of irreducible representations π_θ

Throughout this section, let θ be an admissible quasi-character of $E^\times$ with a Howe factorization (*) of §1. Let E_i, θ_i, c_i and m_i be as in §1.

Assume that θ is minimal, that is, $\theta_0 \equiv 1$, and that $m_{r+1} \geq 1$.

For $i = 0, 1, \dots, r$, denote by D_i the centralizer of E_i in D. The following properties of D_i follow from prop.1 of §1. Put

$$s_i = [E:E_i] \quad \text{for } i = 0, 1, \dots, r.$$

where $[E:E_i]$ denotes the extension degree of E over E_i. Then every D_i is a division algebra of degree s_i over E_i. Denote also by $\mathcal{O}_{D_i}$ and $\mathcal{P}_{E_i}$ the maximal order of D_i and the maximal ideal of $\mathcal{O}_{D_i}$, respectively. Put

$$P_i = \mathcal{P}_{D_i} \text{ and } d_i = [E_i:E] \quad \text{for } i = 1, \dots, r.$$

Then every Π^{d_i} is an uniformizer of $\mathcal{O}_{D_i}$, so that $P_i = \Pi^{d_i}\mathcal{O}_{D_i} = \mathcal{O}_{D_i}\Pi^{d_i}$. Denote by Γ and Γ_i the Galois groups of E over F and over E_i, respectively. Let σ be a generator of Γ. Then each Γ_i is generated by σ^{d_i}. Every element x of D can be expressed in the form $x = \sum_{i=0}^{n-1} x_i \Pi^i$, $x_i \in E$. Thus, under this expression, we have

$$D_i = \{ \sum_{j=0}^{n-1} x_j \Pi^{j d_i}, \ x_j \in E \}.$$

For $i = 1, \dots, r$, put

$$l_i = m_i s_{i-1} + 1,$$
$$l'_i = [l_i/2],$$
$$l''_i = l_i - l'_i.$$

Associated with θ, we define a subgroup K_θ of $D^\times$ by

$$K_\theta = E^\times (1+P_r^{l'_{r+1}}) (1+P_{r-1}^{l'_r}) \dots (1+P_0^{l'_1}).$$

Put

$$H = F^\times (1+P_r^{l'_{r+1}}) (1+P_{r-1}^{l'_r}) \dots (1+P_0^{l'_1}),$$

and for $i = 1, \dots, r$, put

$$H_i = F^\times (1+P_r^{l'_{r+1}}) \dots (1+P_i^{l'_{i+1}}) (1+P_{i-1}^{l''_i}) (1+P_{i-2}^{l'_{i-1}}) \dots (1+P_0^{l'_1}).$$

Then every H_i is a normal subgroup of H and $K_\theta = E^\times H$.

Now we shall construct irreducible representation π_θ associated with θ by the following procedure: We define a linear representation, i.e. quasi-character η_i of H_i, and extend it to a representation ρ_i of K_θ, for each i. Thus we get a representation of K_θ by

$$\rho_\theta = \rho_1 \otimes \dots \otimes \rho_{r+1},$$

and hence a representation of $D^\times$ by

$$\pi_\theta = Ind_{K_\theta}^{D^\times} \rho_\theta.$$

For the structure of H and H_i, $i = 1, \dots, r$, we obtain the following proposition by prop.1 of §1 and the above observations of D_i.

Proposition 2. *Let assumptions and notations be as above. For* $j = 1, \dots, n-1$, *denote by* $\iota(j)$ *the minimal integer among* $i = 1, \dots, r$ *such that* d_i *devides* j, *and put* $\iota(0) = r$. *Then*

(i) *Put* $H = F^\times(1 + \sum_{j=0}^{n-1} P_E^{\nu(j)} \Pi^j)$. *Then*

$$\nu(j) = \begin{cases} m_{\iota(j)+1}/2 & \text{if } m_{\iota(j)+1} \text{ is even}, \\ (m_{\iota(j)+1} + 1)/2 & \text{if } m_{\iota(j)+1} \text{ is odd and } j < d_{\iota(j)}[(s_{\iota(j)}+1)/2], \\ (m_{\iota(j)+1} - 1)/2 & \text{if } m_{\iota(j)+1} \text{ is odd and } j \geq d_{\iota(j)}[(s_{\iota(j)}+1)/2]. \end{cases}$$

(ii) *Put* $H_i = F^\times(1 + \sum_{j=0}^{n-1} P_E^{\nu'(j)} \Pi^j)$. *Then* $\nu(0) = [(m_{r+1} + 2)/2]$. *If* n *is odd, or if* n *is even and* $i \neq \iota(n/2)$, *we have* $\nu'(j) = \nu(j)$ *for any* $j \geq 1$. *If* n *is even and* $i = \iota(n/2)$, *we have*

$$\nu'(j) = \begin{cases} m_{\iota(j)+1}/2 & \text{if } m_{\iota(j)+1} \text{ is even}, \\ (m_{\iota(j)+1} + 1)/2 & \text{if } m_{\iota(j)+1} \text{ is odd and } j < d_{\iota(j)}[(s_{\iota(j)}+2)/2], \end{cases}$$

$\quad\lfloor\ (m_{\iota(j)+1}-1)/2 \quad$ *if* $m_{\iota(j)+1}$ *is odd and* $j \geq d_{\iota(j)}[(s_{\iota(j)}+2)/2]$.

Corollary. *Let assumptions and notations be as in prop.2. If* $m_{\iota(j)+1}$ *is even, or if* $m_{\iota(j)+1}$ *and* $s_{\iota(j)}$ *are odd, we have* $\nu'(j) = \nu(j)$. *If* $m_{\iota(j)+1}$ *and* $s_{\iota(j)}$ *are even (so* n *is even), we have* $\nu'(j) = \nu(j) + 1$ *or* $\nu(j)$ *according to* $j = n/2$ *or* $\neq n/2$.

This corollary follows directly from prop.2.

We define a function η_i *of* H_i, $i = 1, \dots, r$, by the formula:

$$\eta_i(k) = \theta_i(\nu_{D_i/E_i}(k)) \quad for\ k \in F^\times(1+P_r^{l'_{r+1}}) \dots (1+P_i^{l'_{i+1}}),$$

$$\eta_i(1+x) = \psi_{E_i}(\tau_{D_i/E_i}(c_i x)) \quad for\ 1+x \in (1+P_{i-1}^{l'_i})(1+P_{i-2}^{l'_i}) \dots (1+P_0^{l'_i}).$$

where ν_{D_i/E_i} and τ_{D_i/E_i} denote the reduced norm and trace in D_i, as a division algebra over E_i, respectively. We can show that each η_i is well-defined and a quasi-character of H_i.

We define a representation ρ_i of K_θ, which is an extension of η_i, in two cases where $K_\theta = E^\times \cdot H_i$ and $K_\theta \neq E^\times \cdot H_i$.

Case 1. $K_\theta = E^\times \cdot H_i$. By prop.2 and its corollary, this holds if and only if n is odd, if n is even and $i \neq \iota(n/2)$, or if n is even and $s_{i(n/2)}$ and $m_{\iota(n/2)}$ are even. We define a function ρ_i of K_θ by the formula:

$$\rho_i(tg) = \theta_i(N_{E/E_i}(t))\eta_i(g) \quad \text{for } t \in E^\times,\ g \in H_i.$$

We can show that η_i is well-defined and a linear representation of K_θ.

Case 2. $K_\theta \neq E^\times \cdot H_i$. Again by prop.2 and its corollary, this holds if and only if n is even, $i = \iota(n/2)$, s_i is even and m_i is odd. Put

$$W = H/H_i.$$

Then the mapping $x \mapsto 1+x$ from $P_{i-1}^{l'_i}$ to H induces an isomorphism