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The origins of infinite dimensional unitary representations of Lie groups

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1 Introduction

The systematic investgations of infinite dimensional unitary representations of Lie groups began with the works of V. Bargmann [1947] on SL(2,**R**) and I.M.Gelfand-M.A.Naimark [1947] on SL(2,**C**). The theory of finite dimensional representations of Lie algebras and Lie groups had been established by E. Cartan [1913] and H.Weyl [1925/26], [1935]. However the finite dimensional theory did not develop spontaneously into the infinite dimensional theory. The purpose of this paper is to study the circumstances.

Cartan's theory of representations dealt with the problem to determine all possible irreducible linear (or projective) groups. It was the problem within the framework of Lie groups. In this framework, there was no room to consider infinite dimensional representations, because the general linear group GL(V) are not (finite dimensional) Lie group if V is infinite dimensional.

After 1925, the infinite dimensional representations arose from several problems both in physics and mathematics. Quantum mechanics popularized the notion of Hilbert spaces and operators on them and was the directest incentive for the introduction of the infinite dimensional representations. In mathematics the discovery of the Haar measure enabled the introduction of the regular representation for an arbitrary locally compact group and the theory of rings of operators used unitary representations of discrete groups in order to construct the factors of type II and type III. But these were abstract examples based on the general theory of measures and did not concern the detailed structure of the groups. More detailed studies on infinite dimensional representations of the concrete Lie groups only began with the works of E.Wigner [1939] and P.A.M.Dirac [1945] on the Lorentz group. These two works were the origins of the decisive papers of Bargmann and Gelfand-Naimark cited above.

§2 Finite dimensional representations of Lie groups

E.Cartan [1913] established the theory of finite dimensional representations of Lie algebras over $\mathbb{C}$. As its title showed, Cartan's aim was to get all irreducible linear groups. Cartan [1909] proved that every irreducible linear Lie algebra over $\mathbb{C}$ is either semisimple or the direct sum of a semisimple algebra and a one dimensional center which consists of scalar transformations. Hence we can assume that the Lie algebra $\underline{g}$ is semisimple without loss of generality. Let $\underline{h}$ be a fixed Cartan subalgebra of $\underline{g}$ and put ℓ be the dimension of $\underline{h}$ which is called the rank of $\underline{g}$. A linear form ω on $\underline{h}$ is called a weight of $\underline{g}$ if there exists a vector $x \neq 0$ such that $Hx = \omega(H)x$ for all H in $\underline{h}$. Nonzero weights of the adjoint Lie algebra of $\underline{g}$ are called roots of $\underline{g}$. (The roots and weights were introduced by W.Killing [1988-90].) There exist ℓ roots $\alpha_1,\cdots,\alpha_\ell$ and every root α of $\underline{g}$ can be expressed as a linear combination of $\alpha_1,\cdots,\alpha_\ell$ with the integral coefficients.

Every weight λ is a linear combination of $\alpha_1,\cdots,\alpha_\ell$ with rational coefficients: $\lambda = \Sigma_{i=1}^{\ell} m_i\alpha_i$. We introduce the lexicographic order with respect to $(m_1,\cdots,m_\ell)$ in the set of all weights. The maximal weight of $\underline{g}$ in this order is called highest. Two isomorphic semisimple Lie algebras $\underline{g}$ and $\underline{g}'$ are linearly equivalent if and only if their highest weights λ and λ' coincide with each other (i.e. $\lambda' = \lambda\circ\phi$ where ϕ is an isomorphism of $\underline{g}'$ onto $\underline{g}$). Cartan found that there exist exactly ℓ irreducible linear Lie algebras $\underline{g}_1,\cdots,\underline{g}_\ell$ which are isomorphic to $\underline{g}$ and have the highest weights $\lambda_1,\cdots,\lambda_\ell$ respectively. He showed that every irreducible linear Lie algebra isomorphic to $\underline{g}$ is linearly equivalent to one which has the highest weight $p_1\lambda_1+\cdots+p_\ell\lambda_\ell$ for some non negative integers $p_1,\cdots,p_\ell$ (Cartan's classification theorem). Cartan's proof of classification theorem consisted of two steps. First he proved the existence of linear Lie algebras $\underline{g}_i$'s ($1\leq i\leq\ell$) separately by the concrete realization for each simple Lie algebra. Secondly he proved the existence of a linear Lie algebra $\underline{g}$ with the highest weight $p_1\lambda_1+\cdots+p_\ell\lambda_\ell$ by constructing $\underline{g}$ from $\underline{g}_1,\cdots,g_\ell$. The process is called making the Cartan product. The Cartan product is realized on a subspace of a tensor product of the representation spaces of $\underline{g}_i$'s.The history of the Cartan product was studied by Th.Hawkins[1988]. Cartan's theory gave the complete invariant for irreducible linear Lie algebras, the highest weight and established the very core of the finite dimensional representation theory.

H.Weyl [1925/26] developed the theory in many ways. His main innovations were the introduction of the global Lie groups and their covering groups, the integral formula for compact semisimple Lie groups, the principle of unitary restriction and the explicit character formula. Using these tools, Weyl found that the proof of Cartan's classification theorem can be reduced to the completeness of the set of all irreducible characters of a compact semisimple Lie group G in the space of all square-integrable central functions on G. This completeness is a corollary to that of all matrix coefficients of irreducible representations in the space $L^2(G)$. The latter was proved by F.Peter-H.Weyl [1927] with the aid of the eigenvalueproblem of integral operators. Thus Weyl could give a unified proof to Cartan's classification theorem in Weyl [1934/35]. This was a remarkable progress from the case by case verification of Cartan.

These were the main results of the finite dimensional representation theory of Lie groups by Cartan and Weyl. It had a beutiful façade of classification theorem. But it lacked a unified method of constructing irreducible representations. Weyl [1925/26],[1939] constructed the irreducible representations of $SL(n,C)$ and $Sp(n,C)$ on the spaces of certain tensors. The Lie algebra $\underline{o}(n,\mathbb{C})$ requires the space of spinors besides those of tensors. The constructions were carried out for each classical Lie groups and Lie algebras separately. A unified method was given later by A.Borel and A.Weil (cf. § 7).

On the other hand E.Cartan [1929] generalized Peter-Weyl theorem to the completeness of the sherical functions on the homogeneous spaces of compact Lie groups. Moreover he gave an analogue of his classification theorem for the irreducible representations realized by spherical functions on compact Riemannian symmetric spaces G/K. He showed that the restricted highest weights of such representations make λ(=rankG/K)-dimensional lattices. He did not determine which irreducible representations are realized by spherical functions on G/K. The determination was obtained by Sugiura [1962].

We remark that Peter-Weyl and Cartan dealt with solely the finite dimensional representations. They neither used nor defined the regular representation of a Lie group. Infinite dimensional vector spaces and function spaces such as L^2, ℓ^2 and C([0,1]) had been studied since the first decade of the 20th century by D.Hilbert, E.Schmidt, Hadamard, Frecht, F. Riesz and E.Fischer and others (cf. J.Dieudonné [1981]). The notion of Banach space was defined in 1922 by S.Banach and N.Wiener. However the infinite dimensional representations did not appear immediately. As long as mathematicians considered the representations of Lie groups as the homomorphisms between two Lie groups, infinite dimensional representations were beyond their scope.

§3 The quantum mechanics

The situation has changed completely since the rise of the quantum mechanics introduced by W.Heisenberg [1925] and E.Schrödinger [1926]. Soon the mathematical equivalence of their theories were recognized. M.Born [1926] gave the probabilistic interpretation of a normalized wave function $\psi(x,y,z,t)$ of a particle that $p_t(A) = \iiint_A |\psi(x,y,z,t)|^2 dxdy dz$ is equal to the probability of the particle to be found in a subset A of R^3 at the time t. The Hilbert spaces of the square integrable functions arose naturally in this context. The mathematical formulation of the quantum mechanics was given by J. von Neumann [1927a], [1927b] and [1932]. In [192 7a] he gave the first definition of an abstract Hilbert space. In [1932] a state of a physical system is represented by a ray in a Hilbert space and a physical quantity is represented by a self-adjoint operator.

In this trend of studies, many applications of group representations to quantum mechanics were found in the period from 1926 to 1932. But most of them concern the finite dimensional representations. We pick up two examples of the infinite dimensional representations.

Weyl [1928] introduced an infinite dimensional representation U of the rotation group SO(3) on the Hilbert space of wave functions. The representation U and its restriction to the subgroup SO(2) explain the azimuthal quantum number and the magnetic quantum number.

Weyl [1927] remarked that the Heisenberg commutation relation

(1) $$PQ - QP = iI$$

for self-adjoint operators P and Q can be derived from the relation

(2) $$U_s V_t U_s^{-1} V_t^{-1} = e^{ist} I, \qquad s,t \in R$$

for the one parameter group $U_s = e^{isP}$ and $V_t = e^{itQ}$ of unitary operators. M.H. Stone [1930] proved that if U_s and V_t satisfies the Weyl relation (2), then they are unitarily equivalent to a direct sum of operators given by

$$(U_s\psi)(q) = \psi(q-s) \quad \text{and} \quad (V_t\psi)(q) = e^{itq}\psi(q).$$

The same results was obtained by von Neumann [1931] independently. Much later, it was found that Stone-von Neumann Theorem is essentially equivalent to give a certain unitary representations of the Heisenberg group. cf. A.A.Kirillov [1962], and R.How [1980].

§4 Arising of infinite dimensional representations

In the 1930's the infinite dimensional representations of topological groups became gradually to be the object of investigations. M.H.Stone [1932] gave the spectral decomposition of a one parameter group of unitary operators. The theorem gave the irreducible decomposition of a unitary representation of the additive group $\mathbb{R}$ of real numbers in an abstract form. It was used later by V.Bargmann [1947] to construct the representation of the Lie algebra from the given unitary representation of the Lorentz group.

The discovery of the Haar measure by A.Haar [1933] was the starting point of many investigations on locally compact groups. Haar [1933] defined the right regular representation of an arbitrary (separable) locally compact group. Using Haar's work, J.von Neumann [1933] solved the Hilbert's fifth problem for compact groups.

F.J.Murray-von Neumann [1936] and von Neumann [1940] constructed the factors of type II and III using a unitary representation of a countable discrete group acting ergodically on a measure space.

In the meanwhile there appeared a series of works which suggested the limitation of the finite dimensional unitary representations. A topological group G is called maximally almost perodic if each two distinct points in G are separated by a finite dimensional unitary representation of G. Compact groups and loclly compact abelian groups are maximally almost periodic by the works of Peter-Weyl [1927], Häar [1933] and L.S.Pontrjagin [1934], van Kampen [1935] respectively. Conversely H.Freudenthal [1936] proved that if a connected locally compact group G is maximally almost periodic then G is the direct product of a compact group and a vector group $\mathbb{R}^n$. Moreover von Neumann-E.Wigner [1940] proved that every non compact connected simple Lie group has no finite dimensional irreducible unitary representations except the identity representation. On the other hand, I.M.Grlfand-D.A. Raikov [1943] proved that every locally compact group G has a sufficiently many irreducible unitary representations, that is, every two distinct points of G are separated by an irreducible unitary representation of G which may be infinite dimensional. These facts clearly indicated the necessity for the infinite dimensional unitary representations.

§5 The Lorentz group

The significance of getting all irreducible unitary representations of the Lorentz group was first pointed out in a paper of E.Wigner[1939]. In order to classify all possible types of relativistic wave equations of a free particle, he desired to determine all irreducible unitary representations U of the inhomogeneous Lorentz group G. The group G is a semidirect product of the proper Lorentz group L and the group $\mathbb{R}^4$ of translations. He showed that such a representation U is determined uniquely up to the unitary equivalence by each orbit $\underline{O}$ of the Lorentz group L on $\mathbb{R}^4$ and an irreducible unitary representation M of the stationary subgroup L_0 of L corresponding to a point p in the orbit $\underline{O}$. The orbits are parametrized by the value of the Minkowski metric

$$P = \langle p,p \rangle = -p_1{}^2-p_2{}^2-p_3{}^2+p_4{}^2$$

of the point $p\varepsilon\underline{O}$. The orbits are divided into four classes according to the value of P:

I $\quad P > 0, \quad L_0 = SO(3)$,

II $\quad P = 0,\ p \neq 0, \quad L_0 = M(2)$ (2-dim. Euclidean motion group)

III $\quad P = 0,\ p = 0, \quad L_0 = L = SO_0(3,1)$,

IV $\quad P < 0, \quad L_0 = SO_0(2,1)$.

Except the case $L_0 = SO(3)$, the irreducible unitary representations of $L_0 = M(2)$, $SO_0(3,1)$ and $SO_0(2,1)$ were not known at that time. Wigner left the problem of determining these representations.

P.A.M.Dirac [1945] found some infinite dimensional unitary irreducible representations of the Lorentz group $L = SO_0(3,1)$ and pointed out some applications to quantum mechanics. The papers of Wigner and Dirac were the origin of the works of Bargmann [1947] and Gelfand-Naimark [1947]. Also inspired by Dirac, Harish-Chandra [1947] gave an infinitesima classification of infinite dimensional irreducible representations of the Lorentz group L.

§6 SL(2,$\mathbb{R}$) and SL(2,$\mathbb{C}$)

The systematic investigations of global unitary representations of the Lorentz groups were achieved by Bargmann [1947] and Gelfand-Naimark [1947]. They studied SL(2,$\mathbb{R}$) and SL(2,$\mathbb{C}$) respectively which are the 2-fold covering groups of the proper Lorentz groups $SO_0(2.1)$ and $SO_0(3.1)$. The methods of the two papers were rather different from each other. But both papers obtained the following results for the group G=SL(2,R) or SL(2,$\mathbb{C}$):

(i) The principal and complementary series of the irreducible unitary representations of G are constructed on some function spaces. The principal series of SL(2,$\mathbb{R}$) is divided into the continuous series and the discrete series. The latter is characterized by the square integrability of their matrix coefficients. The principal series of SL(2,$\mathbb{C}$) and the continuous series of SL(2,$\mathbb{R}$) consist of the representations induced from one dimensional unitary representations of their minimally parablolic subgroups.

(ii) Any irreducible unitary representation of G is unitarily equivalent to one of the representations described in (i) or the identity representation.

(iii) The matrix coefficients of the principal series representations constitute a complete system in some sense. Gelfand -Naimark proved an analogue of Plancherel theorem on Fourier transform for the group SL(2,$\mathbb{C}$). Similar result for SL(2,$\mathbb{R}$) was proved by Harish-Chandra [1952].

The works of Bargmann and Gelfand-Naimrk [1947] clearly showed the fruitfuness of the studies on the unitary representations of semisimple Lie groups. Immediately after their works, the important investigations appeared one after another in tnis new field. Thus the theory of infinite dimensional unitary representations of Lie groups has been established.

§7 Approaches from pure mathematics

As stated above the theory of the infinite dimensional unitary representations was initiated under the stimulus from the quantum mechanics. However several investigations in pure mathematics were approaching the infinite dimensional representations. In this last section, we study one of them, the induced representations.

The induced representation was a general method which construct a representation U^L of a finite group G from a representation L of a subgroup H of G. The method was invented by G.Frobenius in [1898]. The extension of the notion to compact groups was achieved by A.Weil [1940] (written in 1936).He proved the Frobenius reciprocity law $(U^L:V) = (V|H:L)$ for any irreducible representation V of a compact group G and a closed subgroup H of G. A special case in which L is the identity representation was proved by E.Cartan [1929]. Even if L is finite dimensional, the induce representation U^L is infinite dimensional unless the index of H in G is finite. Hence the notion of the induced representation was a natural entrance to the world of the infinite dimensiona representations. Weil did not define the induced representation for an arbitrary locally compact group. To achieve this generalization, one must overcome two difficulties. One of them arose from the fact that the factor space G/H may have no G-invariant measure. Instead of it, Bargmann and Gelfand-Naimark [1947] used some quasiinvariant measures to define U^L. The general existence proof of the quasi-invariant measure on G/H for an arbitrary locally compact group G and its closed subgroup H was given by J.Dieudonné [1948] and G.W.Mackey [1952].

Another difficulty was more serious. The induced representation U^L of a locally compact group G is not necessarily a direct sum but a direct integral of irreducible representations. This fact was fatal to prove the Frobenius reciprocity law, because the multiplicity of an irreducible representation has lost the usual meaning. On the other hand, Bargmann and Gelfand-Naimark constructed the induced unitary representations of SL(2,$\mathbb{R}$) and SL(2,$\mathbb{C}$) which are irreducible. This discovery has changed the meaning of the induced representations. Now they are the most powerful tool to construct a certain irreducible unitary representations. This situation are entirely different from the cases of compact and finite groups.

After the works of Bargmann and Gelfand-Naimark, the general definition of the induced representation for locally compact groups was given by G.W.Mackey [1952]. As an application of his theory, Mackey determined the irreducible unitary representations of a semidirect product of two locally compact groups H and N where Nis an abelian normal subgroup. this is a generalization of Wigner's orbital analysis. By this method he determined all irreducible unitary representations of the Euclidean motion group M(n). The same result was obtained by S.Ito [1952/53].

After these studies on infinite dimensional representations appeared, A.Borel and A.Weil proved a fundamental theorem on finite dimensional representations. Borel-Weil theorem, published by J.P.Serre [1954], asserts that every finite dimensional irreducible holomorphic representation of a connected complex semi-simple Lie group G is holomorphically induced from 1-dimensional holomorphic representation of a Borel subgroup B of G.

On the other hand, every irreducible unitary representation of G in the principal series is unitarily induced from 1-dimensional unitary representation of B. The parallelism between two theories is very remarkable. But the parallelism was found after the theory of the infinite dimensional representations had been established.

In his commentary in [1979] on his book [1949], A.Weil recollected his works and remarked as follows: "Above all I tried to open the way to a generalization of representation theory of finite groups and compact groups. However not only I did not attain the Land of Promise, the infinite dimensional representations, but also I could not catch a glympse of the landscape. I was discouraged too early when I saw that the matrix coefficients of the finite dimensional representations of a non-compact simple Lie group is not square integrable. This fact denys the hope to look for the representations in $L^2(G)$ and to go forward to the unknown world starting from the known country. Perhaps such a bold march might require the unprejudiced spirit of a phisicist. In fact, they were Dirac, Wigner and Bargmann who opened the way concerning the Lorentz group."

This frank remark of Weil clearly indicated the difficulty to open the way to the infinite dimensional representations. Under the circumstances the finite dimensional representation theory could not develop spontaneously into the infinite dimensional theory but required the stimulus from the quantum mechanics.

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