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The relation between the η-invariant and the spin representation in terms of the Selberg zeta function

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Abstract. The ratio of the (functional) determinant of the "Cayley transform" of the certain self-adjoint square root of the Hodge Laplacian on the compact oriented odd dimensional hyperbolic manifold and the Selberg zeta function attached to the spin representation can be represented by the exponential of η—invariant, explicitly.

THE ROOT OF THE HODGE LAPLACIAN AND THE η—INVARIANT

Let X be a compact oriented hyperbolic manifold of odd dimension $\ell = 2n-1$, Γ its fundamental group. The universal covering manifold of X is the hyperbolic space $H = H^{2n-1}$ of dimension $2n-1$ which can be written as $G/K = SO_0(2n-1,1)/SO(2n-1)$. Let χ be an unitary representation of Γ on $\mathbb{C}^m$. The q-th (differential) form ω on H with values in $\mathbb{C}^m$ is called the χ-twisted q—form if it satisfies $\gamma^*\omega = \chi(\gamma)\omega$. Let $\Omega^q = \Omega^q_\chi$ be the space of all χ—twisted q—forms on H. Consider the operator A^e on (χ—twisted) even forms on H, $\Omega^{even} = \oplus_{p=0}^{n-1}\Omega^{2p}$ defined on Ω^{2p} by the formula

$$A^e = i^n(-1)^{p+1}(*d - d*),$$

where $i = \sqrt{-1}$, d is the exterior differential and $*$ is the Hodge duality operator defined by the metric. Also we consider the operator A^o on ($\chi-$twisted) odd forms on H, $\Omega^{\mathrm{odd}} = \oplus_{p=1}^{n}\Omega^{2p-1}$ defined on Ω^{2p-1} by the formula

$$A^o = i^n(-1)^{np}((-1)^p * d + d*).$$

Let A denote A^e (resp. A^o). Anyway, it is easily seen that the operator A is formally self-adjoint, elliptic and square A^2 is the Hodge Laplacian $\Delta = d\delta + \delta d$, where δ stands for the formal adjoint of d. Hence A is diagonalizable with real eigenvalues and the eigenvalues of A are square roots of those of Δ. It follows that they can be either positive or negative.

Let T be a linear transformation commuting with our self-adjoint operator A. In this situation, Atiyah-Patodi-Singer introduced the so-called "eta-function"

$$\eta_A(T, s) = \sum_{\lambda \neq 0} (\mathrm{sign}\,\lambda)\,\mathrm{Tr}\,T_\lambda|\lambda|^{-s},$$

where the summation is taken over distinct eigenvalues and T_λ is the transformation induced by T on the λ-eigenspace. Further, they proved the following result : For simplicity, put $\eta(s) = \eta_A(\mathrm{I}, s)$ where I is an identity operator. Then $\eta(s)$ has a meromorphic continuation to the entire complex plane and does not have a pole st zero. From this it follows that $\eta(0)$ is well-defined and is called the η-invariant. It is easy to verify that

$$\begin{aligned}
\eta(s) &= \mathrm{Tr}(A(\Delta)^{-(s+1)/2}) \\
&= \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s+1)/2}\,\mathrm{Tr}(Ae^{-t\Delta})\frac{dt}{t}
\end{aligned}$$

for Re $s \gg 0$. The above result is proved with the help of this formula.

Computation on forms and spin representation

For simplicity, we assume that $\ell = \dim X = 4n-1$ and put $\rho = \frac{\ell-1}{2} = 2n-1$. In this case, note that

$$*d : \Omega^{2n-1} \longrightarrow \Omega^{2n-1}, \qquad d* : \Omega^{2n} \longrightarrow \Omega^{2n}.$$

Define two self-adjoint square roots of Δ as follows :

$$\begin{aligned} B^o &= A^o|_{\text{the space of coclosed forms in } \Omega^{2n-1}} \\ &= *d|_{\text{the space of coclosed forms in } \Omega^{2n-1}}, \\ B^e &= A^e|_{\text{the space of closed forms in } \Omega^{2n}} \\ &= d*|_{\text{the space of closed forms in } \Omega^{2n}}. \end{aligned}$$

It is easy to see that Spec B^o = Spec B^e except for the zero modes.

Let σ_q be the action of $M = SO(\ell-1) = SO(4n-2)$ on $\wedge^q \mathbb{C}^{\ell-1}$. Then σ_q is irreducible except when $q = \rho$ in which case it decomposes as the sum of the two spin representations σ^+, σ^-. In this situation, Millson has been defined the "odd type" Selberg zeta function $Z_\chi^-(s)$ for $\chi = 1$ and found the η−invariant appeared in the functional equation of this zeta function.

The formula for the determinant

The closed geodesics γ are in a one to one correspondence to the nontrivial conjugacy classes in Γ. Therefore we can write Tr(γ) for a geodesic γ. Further we write m_γ for the holonomy map of the parallel displacement along γ. This m_γ also may be considered as an element of M. We denote by $\ell(\gamma)$ the length of γ and γ_p the primitive closed geodesic underlying γ. Now we define two types of Selberg zeta function attached to the spin representation :

$$Z_\chi^\pm(s) = \exp(-\sum_\gamma \varepsilon_{\gamma,\chi}^\pm \ell(\gamma)^{-1} e^{-s\ell(\gamma)})$$

where

$$\varepsilon_{\gamma,\chi}^\pm = \mathrm{Tr}\,\chi(\gamma)[\mathrm{Tr}\,\sigma^+(m_\gamma) \pm \mathrm{Tr}\,\sigma^-(m_\gamma)]\ell(\gamma_p)\det(\mathrm{I} - m_\gamma e^{-\ell(\gamma)})^{-1},$$

and the summation is taken over all closed geodesics in X. Z_χ^+ is an ordinary type and Z_χ^- is Millson's type Selberg zeta function. After a suitable regularization and a normalization of the (functional) determinant we have the following formulas :

$$\begin{aligned}&\det((B^e)^2 + s^2)\\ &= \det((B^o)^2 + s^2) = \exp\{P_\chi(s) + Q_\chi(s)\} Z_\chi^+(s+\rho)\end{aligned}$$

for some even polynomial Q_χ of degree $\ell - 1$ and the odd polynomial P_χ of degree ℓ which is determined by the degree of χ and the Plancherel measure of H and hence it is computable. This is the usual type formula discussed, for example, by Sarnak. In contrast to this we have also

$$\det(\frac{B^o - is}{B^o + is}) = e^{i\pi(b_{2n-1} - \eta(0))} Z_\chi^-(s+\rho)$$

$$\det(\frac{B^e - is}{B^e + is}) = e^{i\pi(b_{2n} - \eta(0))} Z_\chi^-(s+\rho)$$

where b_q denotes the (χ-twisted) q-th Betti number.

It is clear that the functional equation and the information of the distribution of zeroes (poles) of Selberg zeta functions $Z_\chi^\pm$ follow immediately from these formulas. Moreover, if we consider the product (resp. the quotient) of both sides of these formulas, we can "separate" the spin representations at the spectrum level, and hence we can pick up new Selberg zeta functions exclusively attached to σ^+ (resp. σ^-).

References

[APS] M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry I*, Proc. Camb. Phil. Soc. **77** (1975), 43–69.

[D] A. Deitmar, *The Selberg trace formula and the Ruelle Zeta function for compact hyperbolics*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 101–106.

[DP] E. D'Hoker and D.H. Phong, *On determinants of Laplacians on Riemann surfaces*, Commun. Math. Phys. **104** (1986), 537–545.

[F1] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. math. **84** (1986), 523–540.

[M] J.J. Millson, *Closed geodesics and the $\eta-$invariant*, Ann. of Math. **108** (1978), 1–39.

[Sal] P. Sarnak, *Determinants of Laplacians*, Commun. Math. Phys. **110** (1987), 113–120.

[V] A. Voros, *Spectral functions, special functions and the Selberg zeta function*, Commun. Math. Phys. **110** (1987), 439–465.