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Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G

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M. Berger (1957) classified involutive automorphisms σ of simple Lie algebras $\mathfrak{g}$ and determined the type of the subalgebras $\mathfrak{g}^\sigma$ of fixed points. Now we try to find involutive automorphisms σ for connected exceptional universal linear Lie groups G and determine the group structures of G^σ of fixed points. Our results are as follows.

G	G^σ	σ	
$G_2{}^C$	$(Sp(1, C) \times Sp(1, C))/Z_2$	γ	
$G_2{}^C$	G_2	τ	
G_2	$(Sp(1) \times Sp(1))/Z_2$	γ	
$G_2{}^C$	$G_{2(2)}$	$\tau\gamma$	$\tau\gamma_C$
$G_{2(2)}$	$(Sp(1) \times Sp(1))/Z_2$	γ	
	$(Sp(1, R) \times Sp(1, R))/Z_2 \times 2$		γ
$F_4{}^C$	$(Sp(1, C) \times Sp(3, C))/Z_2$	γ	
	$Spin(9, C)$	σ	
$F_4{}^C$	F_4	τ	
F_4	$(Sp(1) \times Sp(3))/Z_2$	γ	
	$Spin(9)$	σ	

F_4^C	$F_{4(4)}$	$\tau\gamma$	$\tau\gamma_C$	$\tau\sigma\gamma$	
$F_{4(4)}$	$(Sp(1) \times Sp(3))/Z_2$	γ			
	$(Sp(1, R) \times Sp(3, R))/Z_2 \times 2$		γ		
	$(Sp(1) \times Sp(1, 2))/Z_2$			γ	
	$spin(4, 5)$	σ			
F_4^C	$F_{4(-20)}$	$\tau\sigma$	$\tau\sigma'$		
$F_{4(-20)}$	$(Sp(1) \times Sp(1, 2))/Z_2$	γ			
	$Spin(9)$	σ			
	$Spin(8, 1)$		σ		
E_6^C	$(Sp(1, C) \times SL(6, C))/Z_2$	γ			
	$(C^* \times Spin(10, C))/Z_4$	σ			
	F_4^C	λ			
	$Sp(4, C)/Z_2$	$\lambda\gamma$			
E_6^C	E_6	$\tau\lambda$			
E_6	$(Sp(1) \times SU(6))/Z_2$	γ			
	$(U(1) \times Spin(10))/Z_2$	σ			
	F_4	λ			
	$Sp(4)/Z_2$	$\lambda\gamma$			
E_6^C	$E_{6(6)}$	$\tau\gamma$	$\tau\gamma_C$	$\tau\gamma\gamma_C$	$\tau\sigma\gamma$
$E_{6(6)}$	$(Sp(1) \times SU^*(6))/Z_2$	γ			
	$(Sp(1, R) \times SL(6, R))/Z_2 \times 2$		γ		
	$(R^+ \times spin(5, 5)) \times 2$	σ			
	$F_{4(4)}$	λ			
	$Sp(4)/Z_2$	$\lambda\gamma$			
	$Sp(4, R)/Z_2 \times 2$			$\lambda\gamma$	
	$Sp(2, 2)/Z_2 \times 2$				$\lambda\gamma$

$E_6{}^C$	$E_{6(2)}$	$\tau\lambda\gamma$	$\tau\lambda\gamma_C$	$\tau\lambda\gamma_H$	$\tau\lambda\sigma\gamma$	$\tau\lambda\rho$
$E_{6(2)}$	$(Sp(1) \times SU(6))/Z_2$	γ				
	$(Sp(1, R) \times SU(3, 3))/Z_2 \times 2$			γ		
	$(Sp(1) \times SU(2, 4))/Z_2$				γ	
	$(U(1) \times spin(6, 4))/Z_4$	σ				
	$(U(1) \times spin^*(10))/Z_4$					σ
	$F_{4(4)}$	λ				
	$Sp(1, 3)/Z_2$	$\lambda\gamma$				
	$Sp(4, R)/Z_2 \times 2$		$\lambda\gamma$			
$E_6{}^C$	$E_{6(-14)}$	$\tau\lambda\sigma$	$\tau\lambda\sigma'$	$\tau\lambda\rho\gamma$	$\tau\lambda\sigma\gamma_H$	
$E_{6(-14)}$	$(Sp(1) \times SU(2, 4))/Z_2$	γ				
	$(Sp(1, R) \times SU(5, 1))/Z_2$				γ	
	$(U(1) \times Spin(10))/Z_4$	σ				
	$(U(1) \times spin(8, 2))/Z_4$		σ			
	$(U(1) \times spin^*(10))/Z_4$			σ		
	$F_{4(-20)}$	λ				
	$Sp(2, 2)/Z_2 \times 2$	$\lambda\gamma$				
$E_6{}^C$	$E_{6(-26)}$	τ				
$E_{6(-26)}$	$(Sp(1) \times SU^*(6))/Z_2$	γ				
	$R^+ \times Spin(9, 1)$	σ				
	F_4	λ				
	$F_{4(-20)}$	$\lambda\sigma$				
	$Sp(1, 3)/Z_2$	$\lambda\gamma$				
$E_7{}^C$	$(C^* \times E_6{}^C)/Z_3$	ι				
	$SL(8, C)/Z_2$	$\lambda\gamma$				
	$(SL(2, C) \times Spin(12, C)/Z_2$	σ				

E_7^C	E_7	$\tau\lambda$						
E_7	$(U(1) \times E_6)/Z_3$	ι						
	$SU(8)/Z_2$	$\lambda\gamma$						
	$(SU(2) \times Spin(12))/Z_2$	σ						
E_7^C	$E_{7(7)}$	$\tau\gamma$	$\tau\sigma\gamma$	$\tau\iota\gamma$	$\tau\lambda\iota\gamma$	$\tau\lambda\iota\gamma_C$	$\tau\lambda\iota\rho$
$E_{7(7)}$	$(R^+ \times E_{6(6)}) \times 2$	ι					
	$(U(1) \times E_{6(2)})/Z_3$				ι		
	$SU(8)/Z_2$	$\lambda\gamma$					
	$SU(4, 4)/Z_2 \times 2$		$\lambda\gamma$				
	$SU^*(8)/Z_2 \times 2$			$\lambda\gamma$			
	$SL(8, R)/Z_2 \times 2$					$\lambda\gamma$	
	$(SL(2, R) \times spin(6, 6))/Z_2 \times 2$			σ			
	$(SU(2) \times spin^*(12))/Z_2$						σ
E_7^C	$E_{7(-5)}$	$\tau\lambda\gamma$	$\tau\lambda\sigma$	$\tau\lambda\sigma'$	$\tau\lambda\rho\gamma$		
$E_{7(-5)}$	$(U(1) \times E_{6(2)})/Z_3$	ι					
	$(U(1) \times E_{6(-14)})/Z_3$		ι				
	$SU(2, 6)/Z_2$	$\lambda\gamma$					
	$SU(4, 4)/Z_2 \times 2$		$\lambda\gamma$				
	$(SU(2) \times Spin(12))/Z_2$		σ				
	$(SU(2) \times spin(8, 4))/Z_2$			σ			
	$(SU(2, R) \times spin^*(12))/Z_2 \times 2$				σ		
E_7^C	$E_{7(-25)}$	τ	$\tau\lambda\iota$	$\tau\lambda\iota\sigma$	$\tau\lambda\iota\rho\gamma$		
$E_{7(-25)}$	$(R^+ \times E_{6(-26)}) \times 2$	ι					
	$(U(1) \times E_6)/Z_3$		ι				
	$(U(1) \times E_{6(-14)})/Z_3$			ι			

	$SU(2, 6)/Z_2$	$\lambda\gamma$			
	$SU^*(8)/Z_2$		$\lambda\gamma$		
	$(SL(2, R) \times spin(2, 10))/Z_2$	σ			
	$(SU(2) \times spin^*(12))/Z_2$				σ
$E_8{}^C$	$(SL(2, C) \times E_7{}^C)/Z_2$	υ			
	$Ss(16, C)$	$\tilde{\lambda}\gamma$			
$E_8{}^C$	E_8	$\tau\tilde{\lambda}$			
E_8	$(SU(2) \times E_7)/Z_2$	υ			
	$Ss(16)$	$\tilde{\lambda}\gamma$			
$E_8{}^C$	$E_{8(8)}$	$\tau\gamma$	$\tau\tilde{\lambda}\upsilon\gamma$	$\tau\sigma\gamma$	$\tau\upsilon\gamma$
$E_{8(8)}$	$(SL(2, R) \times E_{7(7)})/Z_2 \times 2$	υ			
	$(SU(2) \times E_{7(-5)})/Z_2$		υ		
	$Ss(16)$	$\tilde{\lambda}\gamma$			
	$Sso(8, 8)_0 \times 2$			$\tilde{\lambda}\gamma$	
	$Sso^*(16) \times 2$				$\tilde{\lambda}\gamma$
$E_8{}^C$	$E_{8(-24)}$	τ	$\tau\tilde{\lambda}\upsilon$	$\tau\tilde{\lambda}\gamma$	$\tau\tilde{\lambda}\upsilon\sigma$
$E_{8(-24)}$	$(SL(2, R) \times E_{7(-25)})/Z_2 \times 2$	υ			
	$(SU(2) \times E_7)/Z_2$		υ		
	$(SU(2) \times E_{7(-5)})/Z_2$			υ	
	$Sso(4, 12)$	$\tilde{\lambda}\gamma$			σ
	$Sso^*(16) \times 2$		$\tilde{\lambda}\gamma$		

Definitions of exceptional Lie groups and main involutions

Let V be an R-vector space and $V^C = \{ u + iv \mid u, v \in V \}$ its complexification. The complex conjugation of V^C is denoted by $\tau : \tau(u + iv) = u - iv$. For an R-linear mapping $f : V \to V$, its complexification $f^C : V^C \to V^C$ is written by the same notation f. For a mapping $f : V \to V$, V_f denotes $\{ v \in V \mid f(v) = v \}$.

Let S be a group and σ an automorphism of G. G^σ denoted $\{ g \in G \mid \sigma(g) = g \}$. The inner automorphism $\tilde{s}$ induced by $s \in G$ is briefly denoted by s.

G_2 : Let $\mathfrak{C} = H \oplus He$ be the Cayley algebra with the multiplication $(m + ae)(n + be) = (mn - \bar{b}a) + (a\bar{n} + bm)e$ and $\mathfrak{C}^C$ its complexification. Define $\gamma : \mathfrak{C} \to \mathfrak{C}$ by

$$\gamma(m + ae) = m - ae$$

and put $\mathfrak{C}' = (\mathfrak{C}^C)_{\tau\gamma}$. Now define

$$G_2{}^C = \{ \alpha \in \mathrm{Iso}_C(\mathfrak{C}^C) \mid \alpha(xy) = (\alpha x)(\alpha y) \},$$

$$G_2 = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y) \} = (G_2{}^C)^\tau,$$

$$G_{2(2)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{C}') \mid \alpha(xy) = (\alpha x)(\alpha y) \} = (G_2)^{\tau\gamma}.$$

F_4 : Let $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C}) = \{ X \in M(3, \mathfrak{C}) \mid X^* = X \}$ be the Jordan algebra with the multiplication $X \circ Y = \frac{1}{2}(XY + YX)$ and $\mathfrak{J}^C$ its complexification. Let $I_1 = \mathrm{diag}(-1, 1, 1) \in M(3, R)$ and define $\sigma : \mathfrak{J} \to \mathfrak{J}$ by

$$\sigma(X) = I_1 X I_1$$

and put $\mathfrak{J}' = (\mathfrak{J}^C)_{\tau\gamma}$, $\mathfrak{J}_1 = (\mathfrak{J}^C)_{\tau\sigma}$. Now define

$$F_4{}^C = \{ \alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y \},$$

$$F_4 = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y \} = (F_4{}^C)^{\tau},$$
$$F_{4(4)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{J}') \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y \} = (F_4{}^C)^{\tau\gamma},$$
$$F_{4(-20)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{J}_1) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y \} = (F_4{}^C)^{\tau\sigma}.$$

E_6 : In $\mathfrak{J}^C$, we define Hermitian inner products $<X, Y>$, $<X, Y>_\gamma$, $<X, Y>_\sigma$ are defined as

$$<X, Y> = (\tau X, Y), \quad <X, Y>_\gamma = (\tau\gamma X, Y), \quad <X, Y>_\sigma = (\tau\sigma X, Y)$$

respectively, where $(X, Y) = \mathrm{tr}(X \circ Y)$. Now define

$$E_6{}^C = \{ \alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \mid \det\alpha X = \det X \},$$
$$E_6 = \{ \alpha \in E_6{}^C \mid <\alpha X, \alpha Y> = <X, Y> \} = (E_6{}^C)^{\tau\lambda},$$
$$E_{6(6)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{J}') \mid \det\alpha X = \det X \} = (E_6{}^C)^{\tau\gamma},$$
$$E_{6(2)} = \{ \alpha \in E_6{}^C \mid <\alpha X, \alpha Y>_\gamma = <X, Y>_\gamma \} = (E_6{}^C)^{\tau\lambda\gamma},$$
$$E_{6(-14)} = \{ \alpha \in E_6{}^C \mid <\alpha X, \alpha Y>_\sigma = <X, Y>_\sigma \} = (E_6{}^C)^{\tau\lambda\sigma},$$
$$E_{6(-26)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{J}) \mid \det\alpha X = \det X \} = (E_6{}^C)^{\tau}$$

where $\lambda : E_6{}^C \to E_6{}^C$ is the outer automorphism of $E_6{}^C$ defined as $\lambda(\alpha) = {}^t\alpha^{-1}$ (${}^t\alpha$ is the transpose of α with respect to (X, Y)).

E_7 : Put $\mathfrak{P} = \mathfrak{J} \oplus \mathfrak{J} \oplus R \oplus R$ and let $\mathfrak{P}^C$ be its complexification and put $\mathfrak{P}' = (\mathfrak{P}^C)_{\tau\gamma}$. Define $\lambda, \iota : \mathfrak{P}^C \to \mathfrak{P}^C$ by

$$\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi), \quad \iota(X, Y, \xi, \eta) = (-iX, iY, -i\xi, i\eta)$$

respectively. In $\mathfrak{P}^C$, we define Hermitian inner products $<P, Q>$, $<P, Q>_\gamma$ by

$$<P, Q> = <X, Z> + <Y, W> + (\tau\xi)\zeta + (\tau\eta)\omega, \quad <P, Q>_\gamma = <\gamma P, Q>$$

respectively, where $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$. For $P, Q \in \mathfrak{P}^C$, we can define C-linear mapping $P \times Q : \mathfrak{P}^C \to \mathfrak{P}^C$. Now define

$$E_7 = \{ \alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \},$$
$$E_7 = \{ \alpha \in E_7{}^C \mid \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \} = (E_7{}^C)^{\tau\lambda},$$
$$E_{7(7)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{P}') \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \} = (E_7{}^C)^{\tau\gamma},$$
$$E_{7(-5)} = \{ \alpha \in E_7{}^C \mid \langle \alpha P, \alpha Q \rangle_\gamma = \langle P, Q \rangle_\gamma \} = (E_7{}^C)^{\tau\lambda\gamma},$$
$$E_{7(-24)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{P}) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \} = (E_7{}^C)^{\tau}.$$

E_8 : Put $\mathfrak{e}_8{}^C = \mathfrak{e}_7{}^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$. $\mathfrak{e}_8{}^C$ has the structure of Lie algebra of type E_8 under the suitable Lie blacket $[R_1, R_2]$ and has the suitable Hermitian inner product $\langle R_1, R_2 \rangle$. Define $\omega : \mathfrak{e}_8{}^C \to \mathfrak{e}_8{}^C$ by

$$\omega(\Phi, P, Q, r, s, t) = (\Phi, Q, -Q, -r, -t, -s)$$

and put $\tilde{\lambda} = \lambda\omega$. Put $\mathfrak{e}_{8(8)} = (\mathfrak{e}_8{}^C)_{\tau\gamma}$, $\mathfrak{e}_{8(-24)} = (\mathfrak{e}_8{}^C)_{\tau}$. Now define

$$E_8{}^C = \{ \alpha \in \mathrm{Iso}_C(\mathfrak{e}_8{}^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2] \},$$
$$E_8 = \{ \alpha \in E_8{}^C \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle \} = (E_8{}^C)^{\tau\tilde{\lambda}},$$
$$E_{8(8)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{e}_{8(8)}) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2] \} = (E_8{}^C)^{\tau\gamma},$$
$$E_{8(-24)} = \{ \alpha \in \mathrm{Iso}_R(\mathfrak{e}_{8(-24)}) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2] \} = (E_8{}^C)^{\tau}.$$

Remark. We have the natural inclusion $G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$. So

$$\gamma \in G_2 \subset F_4 \subset E_6 \subset E_7, \quad \sigma \in F_4 \subset E_6 \subset E_7 \subset E_8, \quad \lambda \in E_7 \subset E_8.$$

($\nu \in E_8$ is the element -1 of center of E_7).

References

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