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In the field of (unitary) representation theory, my working area which concerns with representations of non-semisimple and non-group object may be minor and someone would regard it a negligible set, even worse, the empty set. Here I'd like to present a counte-example to such opinions.

As is well-known, Heisenberg group comes out of the canonical commutation relations which is a fundamental constituent in quantum theory. Although originally it describes a system of finite freedom, later developments necessitate the system of infinite freedom (quantum field theory). The representation theoretic structures of the latter is, however, very much complicated and completely different from the former. To make the explanation clear, let me introduce some notations. Given a real vector space V with a non-degenerate alternating bilinear form B, we can formarlly construct a $*$-algebra $\mathcal{A}(V,B)$ which is generated by elements in V by the relations

$$xy - yx = \sqrt{-1}B(x,y), x^* = x, \quad x, y \in V.$$

If $\dim V = \infty$, $\mathcal{A}(V,B)$ is known to have a non-type I representation. Hence its representation theoretical structure is highly complicated and is imposible to describe. However, we can take up a class of (not necessarylly type I) representations, called quasi-free representation, which are important from a physical point of view . Quasi-free representations are parametrized by a positive inner product $S(\ ,\ )$ on V satisfying

$$S(x,y) - \overline{S(x,y)} = \sqrt{-1}B(x,y).$$

So let us denote the quasi-free representation associated to S by π_S. Now we can prove the following fact

Theorem. *Two quasi-free representations* π_{S_1}, π_{S_2} *generate isomorphic von-Neumann algebras if and only if the following two conditions are satisfied.*

(i) *The inner products induce the same topology on* V.

(ii) *If we take an inner product* $(\ ,\ )$ *inducing the topology given in (i) and define positive operators* P_1, P_2 *by*

$$S_j(x, y) = (x, P_j y) \quad (j = 1, 2),$$

then the operator $\sqrt{P_1} - \sqrt{P_2}$ *is in the Hilbert-Schmidt class.*

In a finite-dimensional space, any operator is in the Hilbert-Schmidt class and the above result reduces to the uniqueness of (irreducible) representations of the canonical commutation relations. On the contrary, in the infinite-dimensional case, there are infinitely many non-quasiequivalent representations which makes it difficult to understand the representation theoretical structure of quantized field theory.