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A Robinson-Schensted-type correspondence for a dual pair on spinors

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According to [H], a *dual pair* is a pair of (associative or Lie) algebras (A_1, A_2), both acting on a module V, which satisfies the following conditions:

(1) The actions of A_1 and A_2 commute with each other,

(2) V is completely reducible as an A_i-module ($i = 1, 2$), and

(3) if $\{\lambda_{A_i}\}_{\lambda\in\Lambda_i}$ is a complete set of representatives of the isomorphism classes of irreducible A_i-modules appearing in V, where Λ_i is a certain index set ($i = 1, 2$), then there exists a bijection $\dagger\colon \Lambda_1 \ni \lambda \mapsto \lambda^\dagger \in \Lambda_2$ for which the decomposition of V as an A_1, A_2-bimodule can be described as follows:

$$V \simeq \bigoplus_{\lambda\in\Lambda_1} \lambda_{A_1} \boxtimes \lambda^\dagger_{A_2}. \tag{*}$$

A Robinson-Schensted-type correspondence is connected to certain dual pairs in the following manner. Let (A_1, A_2) be a dual pair on V such that the irreducible representations of A_i (appearing in V) are naturally labelled by certain set of partitions. Suppose V has a natural basis $\mathcal{B}(V)$ indexed by a certain set W of combinatorial objects. Moreover suppose that each λ_{A_i} has a basis $\mathcal{B}(\lambda_{A_i})$ indexed by a certain set $\mathrm{Tab}(\lambda_{A_i})$ of tableaux of shape λ. In this situation, (*) implies that $\#W = \#\left(\coprod_{\lambda\in\Lambda_1} \mathrm{Tab}(\lambda_{A_1}) \times \mathrm{Tab}(\lambda^\dagger_{A_2})\right)$. A Robinson-Schensted-type correspondence is a good bijection between these two sets. If A_i is a reductive Lie algebra, we impose the conditon that the bases should consist of weight vectors, that their weights should be described in terms of the combinatorial objects parametrizing them, and that the correspondence should be weight-preserving.

We can illustrate this in the following diagram.

$$\begin{array}{cccccc}
V & \simeq \bigoplus_{\lambda\in\Lambda_1} & \lambda_{A_1} & \boxtimes & \lambda^{\dagger}_{A_2} & \\
\cup & & \cup & & \cup & \text{(basis)} \\
\mathcal{B}(V) & & \mathcal{B}(\lambda_{A_1}) & & \mathcal{B}(\lambda^{\dagger}_{A_2}) & \\
\updownarrow & & \updownarrow & & \updownarrow & \text{(parametrize)} \\
W & \xrightarrow{\sim} \coprod_{\lambda\in\Lambda_1} & \mathrm{Tab}(\lambda_{A_1}) & \times & \mathrm{Tab}(\lambda^{\dagger}_{A_2}) &
\end{array}$$

The most classical case is the situation of the Weyl reciprocity. In this case we put $A_1 = \mathfrak{gl}(n, \mathbb{C})$, $A_2 = \mathbb{C}[\mathfrak{S}_f]$ and $V = (\mathbb{C}^n)^{\otimes f}$. Λ_1 and Λ_2 are the set of partitions of f into fewer than or equal to n parts, and the map $\dagger$ is the identity. A natural basis is $\mathcal{B}(V) = \{ e_{w_1} \otimes \cdots \otimes e_{w_f} \mid (w_1, \ldots, w_f) \in W \}$, where $W = [n]^f$, $[n] = \{1, 2, \ldots, n\}$, and $\{e_i\}_{i\in[n]}$ is the canonical basis of $\mathbb{C}^n$. Moreover $\mathrm{Tab}(\lambda_{A_1})$ is the set $\mathrm{SSTab}(\lambda; [n])$ of semistandard tableaux, and $\mathrm{Tab}(\lambda_{A_2})$ is the set $\mathrm{STab}(\lambda)$ of standard tableaux. The classical Robinson-Schensted correspondence gives the bijection in this case.

In this talk we will give a Robinson-Schensted-type correspondence for the dual pair of type (C_m, C_n) on spinors [H]. We have $V = \bigwedge \mathbb{C}^{2mn}$, $A_1 = \mathfrak{sp}(2m, \mathbb{C})$ and $A_2 = \mathfrak{sp}(2n, \mathbb{C})$. The set Λ_1 is the set of partitions whose Young diagrams are contained in the m by n rectangle, and the map $\dagger$ is as illustrated in the figure.

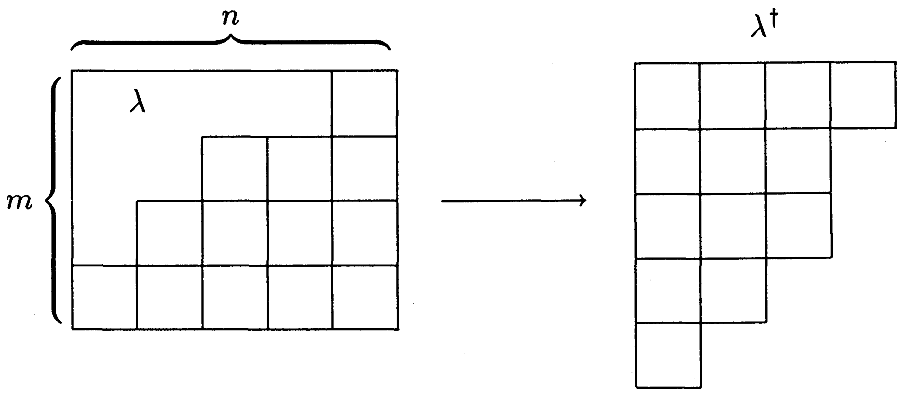

Fig. $(4, 2, 1)^{\dagger}$ for $m = 4$, $n = 5$

The set $\mathrm{Tab}(\lambda_{\mathfrak{sp}(2m,\mathbb{C})})$ is the set of tableaux T of shape λ with entries in the totally ordered set $\{1 < \overline{1} < 2 < \overline{2} < \cdots < m < \overline{m}\}$ satisfying the following conditions:

(Sp1) $T(i,1) \leq T(i,2) \leq \cdots \leq T(i, \lambda_i) \quad (1 \leq i \leq l)$, (Sp2) $T(1,j) < T(2,j) < \cdots < T(\lambda'_j, j) \quad (1 \leq j \leq \lambda_1)$,

(Sp3) $T(i,j) \geq i \quad ((i,j) \in \lambda)$,

where λ'_j is the length of the jth column of λ.

In **[DJM]** Date, Jimbo and Miwa showed, using the quantum deformation $U_q(\mathfrak{gl}(n, \mathbf{C}))$, that the original Robinson-Schensted correspondence can be understood as a limit at $q \to 0$ of the relationship between the two bases of V described above. Moreover Kashiwara showed that a similar structure exists for integrable representations of the quantum deformation of the Kac-Moody Lie algebra constructed from any symmetrizable generalized Cartan matrix. It would be very interesting to interpret our new correspondence in terms of these quantum deformations.

References

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