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Towards Kac-Moody Lie groups

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Introduction. Let $\mathfrak{g}_{\mathbf{R}}$ be a Kac-Moody algebra over the real number field $\mathbf{R}$ with a symmetrizable generalized Cartan matrix (GCM), and $\mathfrak{h}_{\mathbf{R}}$ the Cartan subalgebra of $\mathfrak{g}_{\mathbf{R}}$. Then, the Kac-Moody algebra $\mathfrak{g}$ ove $\mathbf{C}$ corresponding to the same GCM and its Cartan subalgebra $\mathfrak{g}$ are given by

$$\mathfrak{g} = \mathbf{C}\otimes_{\mathbf{R}}\mathfrak{g}_{\mathbf{R}} \text{ and } \mathfrak{g} = \mathbf{C}\otimes_{\mathbf{R}}\mathfrak{h}_{\mathbf{R}},$$

respectively. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$, and $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^{\alpha}$ the root space decomposition. There exists an (essentially unique) involutive anti-linear anti-automorphism $\mathfrak{g} \ni x \longrightarrow x^{*} \in \mathfrak{g}$ such that

$$h^{*} = h \quad \text{for } \forall h \in \mathfrak{h}, \quad (\mathfrak{g}^{\alpha})^{*} = \mathfrak{g}^{-\alpha} \quad \text{for } \forall\alpha \in \Delta.$$

We denote by $\mathfrak{k}$ the real form $\{x \in \mathfrak{g};\ x + x^{*} = 0\}$ of $\mathfrak{g}$. In the finite-dimensional case, this is nothing but a compact real form of $\mathfrak{g}$.

The purpose of this note is to present a method for construction of "Lie groups" associated with (suitable compelitions of) this real form $\mathfrak{k}$, as groups of operators on (completions of) certain representation spaces of $\mathfrak{g}$.

1. The adjoint representation and integrable highest weight representations. Let (π, V) be the adjoint representation $(\mathrm{ad}, \mathfrak{g})$, or the highest weight representation $(\pi_{\Lambda}, L(\Lambda))$ with dominant integral highest weight Λ. We denote by $P(\pi)$ the set of weights of V, and by V_{μ} the weight space of weigh μ for each element μ in the dual vector space $\mathfrak{h}^{*}$ of $\mathfrak{h}$.

Let $\underline{V}$ be the (infinite, in general) direct products of all the weight spaces V_μ over $\mu \in P(\pi)$. Then, V is regarded as a subspace of $\underline{V}$, and the action of $\mathfrak{g}$ on V is naturally extended up to $\underline{V}$.

Since we have assumed that the Cartan matrix of $\mathfrak{g}$ is symmetrizable, there exists a Hermitian form $(\cdot|\cdot)_{0,\pi}$ on V which has the following property called contravariance:

$$(\pi(x)u|v)_{0,\pi} = (u|\pi(x)^*v)_{0,\pi} \quad \text{for } \forall x \in \mathfrak{g},\ \forall u,\ v \in V.$$

In case $\pi = \pi_\Lambda$, this Hermitian form is positive definite on the whole space $L(\Lambda)$. However, in the case of adjoint representation, $(\cdot|\cdot)_{0,\mathrm{ad}}$ is guaranteed to be definite only on the subspace $\sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$, and is indefinite on $\mathfrak{h}$ if $\mathfrak{g}$ is of infinite-dimension ([Kac-Peterson, Th.1]). So, we take a positive definite Hermitian form $(\cdot|\cdot)_{\mathrm{ad}}$ on $\mathfrak{h}$, and extend it to the whole space $\mathfrak{g}$ by

$$(x|y)_{\mathrm{ad}} = (x|y)_{0,\mathrm{ad}} \quad \text{for } x,\ y \in \textstyle\sum_{\alpha\in\Delta} \mathfrak{g}^\alpha.$$

(Note that the weight space decomposition is an orthogonal decomposition with respect to $(\cdot|\cdot)_{0,\pi}$ because of the contravariance.) For $\pi = \pi_\Lambda$, put $(\cdot|\cdot)_{\pi_\Lambda} = (\cdot|\cdot)_{0,\pi_\Lambda}$.

Let $H(\pi)$ be the completion of the pre-Hilbert space $(V, (\cdot|\cdot)_\pi)$. Since the weight space decomposition is a orthogonal decomposition with respect to $(\cdot|\cdot)_\pi$, we may consider $H(\pi)$ as a subspace of $\underline{V}$. Hence, for $x \in \mathfrak{g}$ and $v \in H(\pi)$, the symbol $\pi(x)v$ has a sense as an element in $\underline{V}$.

2. The spaces of differentiable vectors. For $m = 0, 1, 2, \cdots$, we define the spaces of C^m-vectors inductively by

$$H_0(\pi) = H(\pi),$$
$$H_m(\pi) = \{v \in H_{m-1}(\pi);\ \pi(x)v \in H_{m-1}(\pi) \quad \text{for any } x \in \mathfrak{g}\}, \qquad \text{for } 0 < m < +\infty.$$

We call each element in $H_m(\pi)$ a vector of class C^m. Actually, the differentiability in this sense, is determined by one strictly dominant element in $\mathfrak{h}_{\mathbf{R}}$, as follows.

Theorem 1. *Let $m = 0, 1, 2, \cdots$, and h_0 be a strictly dominant element in $\mathfrak{h}_R$, that is, an element satisfying $\alpha(h_0) > 0$ for every positive root α. Then, it holds that*

$$H_m(\pi) = \{v \in \underline{V};\ \pi(h_0)^m v \in H(\pi)\}.$$

Fix a strictly dominant element h_0 in $\mathfrak{h}_{\mathbf{R}}$, and define an inner product $(\cdot|\cdot)_{\pi,m}$ on $H_m(\pi)$ ($m = 0, 1, 2, \cdots$) by

$$(u|v)_{\pi,m} = \Sigma_{j=0}^{m} (\pi(h_0)^j u|\pi(h_0)^j v)_{\pi} \quad \text{for } u,\ v \in H_m(\pi).$$

Clearly, $(H_m(\pi), (\cdot|\cdot)_{\pi,m})$ is a Hilbert space, and every inclusion $H_{m+1}(\pi) \hookrightarrow H_m(\pi)$ is continuous. Further, the action of $\mathfrak{g}$ on V is continuously extended to the bilinear map $\mathfrak{g}_{m+1} \times H_{m+1}(\pi) \ni (x, v) \longrightarrow \pi(x)v \in H_m(\pi)$, where $\mathfrak{g}_m = H_m(\mathrm{ad})$. In particular, for each $x \in \mathfrak{g}_{m+1}$, we may (and do) regard $\pi(x)$ as an (unbounded, in general) operator on $H_m(\pi)$ with domain $H_{m+1}(\pi)$. Thanks to the contravariance of the original Hermitian form $(\cdot|\cdot)_{0,\pi}$, this operator is closable.

3. Negative spaces. Let $m = 0, 1, 2, \cdots$. For each $v \in H(\pi)$, the map $H_m(\pi) \ni u \longrightarrow (u|v)_{\pi} \in \mathbf{C}$, defines a continuous linear form on $H_m(\pi)$. We denote by $\|v\|_{\pi,-m}$ the norm of this linear form. Let $H_{-m}(\pi)$ be the completion of the normed linear space $(H(\pi), \|\cdot\|_{\pi,-m})$. Actually, $H_{-m}(\pi)$ is a Hilbert space, and we have a decreasing sequence of Hilbert spaces:

$$\begin{aligned}&\underline{V} \supset \cdots \supset H_{-m-1}(\pi) \supset H_{-m}(\pi) \supset \cdots \\ &\quad \supset H(\pi) = H_0(\pi) \supset \cdots \supset H_m(\pi) \supset H_{m+1}(\pi) \supset \cdots \supset V.\end{aligned}$$

By definition, $(\cdot|\cdot)_{0,\pi}$ gives a non-degenerate continuous pairing of $H_{-m}(\pi)$ and $H_m(\pi)$. Thanks to the contravariance of

$(\cdot|\cdot)_{0,\pi}$, the action of each $x \in \mathfrak{g}_{m+1}$ on $H_{m+1}(\pi)$ is extended to a continuous linear map of $H_{-m}(\pi)$ into $H_{-m-1}(\pi)$ by

$$(\pi(x)u|v)_{0,\pi} = (u|\pi(x)^{*}v)_{0,\pi} \qquad \text{for } u \in H_{-m}(\pi),\ v \in H_{m+1}(\pi)$$

4. Groups corresponding to completions $\mathfrak{k}_m$ of $\mathfrak{k}$. Let $\mathfrak{k}_m$ be the closure of $\mathfrak{k}$ in $\mathfrak{g}_m$, and x an arbitrary element in $\mathfrak{k}_{m+2}$ (m = 0, 1, 2, $\cdots$). As noted above $\pi(x)$ is regarded as an closable operator on $H_m(\pi)$ and on $H_{-m-1}(\pi)$, with domains $H_{m+1}(\pi)$ and $H_{-m}(\pi)$, respectively. By definiton of $\mathfrak{k}$, contravariance means $\mathfrak{k}$-invariance, and so $(\cdot|\cdot)_{0,\pi}$ is $\mathfrak{k}$-invariant. Making use of this fact, we obtain the following evaluation as for the action of the operator $\pi(x)$.

Theorem 2. *i) For any $v \in H_{m+1}(\pi)$, there hold the inequality*

$$|(1-\pi(x))u|_{\pi,m} \geqq (1-C|x|_{ad,m+1})|u|_{\pi,m}.$$

ii) For any $v \in H_{-m}(\pi)$, there holds the inequality

$$\|(1+\pi(x))v\|_{\pi,-m-1} \geqq c(1-c'\|x\|_{ad,m+1})\|v\|_{\pi,-m-1}.$$

Here, C, c, and c' are positive real numbers independent of x, u, and v, $\|\cdot\|_{\pi,m}$ is the norm on $H_m(\pi)$ induced from the inner product $(\cdot|\cdot)_{\pi,m}$, and $|\cdot|_{\pi,m}$ is also the norm on $H_m(\pi)$ defined by

$$|v|_{\pi,m} = \Sigma_{j=0}^{m} (\pi(h_0)^j v|\pi(h_0)^j v)^{1/2} \quad \text{for } v \in H_m(\pi).$$

Clearly, the norms $\|\cdot\|_{\pi,m}$ and $|\cdot|_{\pi,m}$ are equivalent to each other.

Thanks to the evaluation ii) of this theorem, we see that, for any sufficiently small $\varepsilon > 0$, the images of the operators $1\pm\varepsilon\pi(x)$: $H_{m+1}(\pi) \longrightarrow H_m(\pi)$ are dense in $H_m(\pi)$. This fact, together with the evaluation i) in Theorem 2, implies that the criterion in [Yosida] as for the exponentiability of a closed operator on a Banach space, applies to the closure of the operator $\pi(x)$: $H_{m+1}(\pi) \longrightarrow H_m(\pi)$. Thus, we have

Theorem 3. *Let $m = 0, 1, 2, \cdots$, and $x \in \mathfrak{k}_{m+2}$. Then, there exists a unique strongly continuous 1-parameter group $e^{t\pi(x)} = \exp(t\pi(x))$ $(t \in \boldsymbol{R})$ of bound operators on $H_m(\pi)$ whose infinitesimal generator is the closure of the operator $\pi(x)$: $H_{m+1}(\pi) \longrightarrow H_m(\pi)$.*

Let K_m^π be the group of bounded operators on $H(\pi)$ generated by the 1-parameter groups $e^{\pi(x)}$ with $x \in \mathfrak{k}_m$ $(m = 2, 3, 4, \cdots)$. We get a decreasing sequence of groups:

$$K_2^\pi \supset K_3^\pi \supset \cdots \supset K_m^\pi \supset K_{m+1}^\pi \supset \cdots .$$

The above theorem says that each subgroup K_m^π preseves the $(m-2)$-times differentiablity.

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