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Generalized Weil representations for SL(n)

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0. Introduction.

In a previous paper [P-SA], generalized Weil representations were constructed for the group G = SL(n,k), for even n , and k a finite field, by extending Lion-Vergne's version [L-V] of A. Weil's original method [W], applied to SL(2,k), to suitable higher order analogues of the Heisenberg group in one degree of freedom (which we call "Grassmann-Heisenberg groups"). The construction was restricted to the case where n is even, because only in this case there exist reasonable analogues of the Schrödinger representation for our Grassmann-Heisenberg groups, whose isomorphy type will be fixed by the natural action of the whole group G (in the case where n is odd one obtains only Schrödinger-like representations whose isomorphy type is fixed by very small subgroups of G)

In the joint work with J. Pantoja presented here a more geometric approach to this construction is given, using the method of contraction of a complex G - vector bundle over a base point with the help of a suitable G - equivariant connection [SA]. A first specialization of this method allows us to recover the representations constructed in [P-SA]. A second specialization allows us to obtain generalized Weil representations also for SL(n,k) , n odd.

It should be noticed that our method applies as well to a local field

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instead of a finite base field, case to which we have reduced ourselves here only for technical simplicity.

1. Construction of representations by contraction.

We recall here briefly the results in [SA]. Let G be a finite group and $\xi = (E,B,p,\eta)$ a complex G-vector bundle, where as usual E denotes the total space, B the base space, p the projection from E to B, and η the action of G on E and B. We have then $p(\eta_g(v)) = \eta_g(p(v))$ ($g \in G$, $v \in E$). We denote by E_b the fiber $p^{-1}(b)$ over $b \in B$.

To obtain a linear representation of G out of ξ, by contraction over a base point $b_0 \in B$, one needs an equivariant connection on ξ, in the following sense.

DEFINITION 1. An equivariant connection on the complex G-vector bundle $\xi = (E,B,P,\eta)$ is a family of linear isomorphisms $\Gamma = \{\gamma_{b,a}\}_{a,b \in B}$ such that

(i) each $\gamma_{b,a}$ is a linear isomorphism from the fiber E_a onto the fiber E_b and every $\gamma_{a,a}$ is the identity;

(ii) we have

$$\gamma_{\eta_g(b),\eta_g(a)} \circ \eta_g = \eta_g \circ \gamma_{b,a}$$

for all $a,b \in B$, $g \in G$;

(iii) we have $\gamma_{c,b} \circ \gamma_{b,a} = \mu_\Gamma(c,b,a)\gamma_{c,a}$, for all $a,b,c \in B$, for a suitable function $\mu_\Gamma : B \times B \times B \longrightarrow \mathbb{C}^\times$ called the multiplier of Γ.

We say that the connection Γ is flat iff its multiplier μ_Γ is the constant function 1.

PROPOSITION 1. Let us fix a base point $b \in B$ and suppose that an equivariant connection G is given on the complex G-vector bundle ξ with base B. A projective representation (V,σ) of G, with multiplier $\mu_\sigma(g,h) = \mu_\Gamma(b,\eta_g(b),\eta_{gh}(b))$ for $g,h \in G$, may be constructed as follows. Let $V =$

E_b and define σ by

$$\sigma_g(v) = \gamma_{b,h_g(b)}(\eta_g(v)) \qquad (g\in G.\ v\in V).$$

We call (V,σ) the representation of G obtained by contraction of ξ over b according to Γ

$\square$

2. A geometric approach to the construction of generalized Weil representations for $G = SL(n,k)$, n even and k a finite field.

PROPOSITION 2. The generalized Weil representation $(E(\ell,c),\rho)$ of $G = SL(V) \cong SL(n,k)$, for even n , constructed in [P-SA] , may be obtained by the contraction procedure recalled above according to the following data:

(i) the complex G-vector bundle $\xi = (E,B,p,\eta)$ is defined as follows. Let

$$W = \Lambda^1 V\oplus\Lambda^3 V\oplus\ldots\oplus\Lambda^{n-1}V,$$

$$\langle\omega,\zeta\rangle = (\omega\wedge\zeta)_n \qquad (\omega,\zeta\in W),$$

$B = \{\ell < V \mid \dim \ell = 1\}$. We put for all $\ell\in B$,

$$W_\ell = \ell\oplus\ell\wedge\Lambda^2 V\oplus\ldots\oplus\ell\wedge\Lambda^{n-2}V,$$

and

$$E_\ell = \{f:W\longrightarrow\mathbb{C} \mid f(\omega+\zeta) = \psi(c\langle\omega,\zeta\rangle)f(\omega) \quad \forall\omega\in W,\ \forall\zeta\in W_\ell\},$$

$$E = \bigcup_{\ell\in B} E_\ell,$$

p is the obvious projection and η is given by

$$(\eta_g f)(\omega) = f(\Lambda g^{-1}\omega)(g\in G,\ \omega\in W),$$

$$\eta_g(\ell) = g(\ell)(g\in G,\ \ell\in B);$$

(ii) the connection $\Gamma = \{\gamma_{\ell',\ell}\}_{\ell,\ell'\in B}$ is defined by

$\gamma_{\ell,\ell} = \mathrm{Id}_{E_\ell}$ for all $\ell\in B$ and

$$(2)(\gamma_{\ell',\ell} f)(\omega) = q^{-d(n)/2} \sum_{\zeta\in W_{\ell'}/W_\ell\cap W_{\ell'}} \psi(-c\langle\omega,\zeta\rangle)f(\omega+\zeta) \text{for all } f\in W_\ell,\ \omega\in W,$$

$\ell',\ell\in B,\ \ell\neq\ell'$. $\square$

3. Generalized Weil representations for $G = SL(n,k)$, n odd, k a finite field.

Let V be a k-vector space of odd dimension $n = 2m+1$. Put $W = \Lambda^1 V \oplus \Lambda^2 V \oplus \ldots \oplus \Lambda^{n-1} V$ and define an alternating form $\langle\ ,\ \rangle$ on W by

(3) $$\langle \omega, \zeta \rangle = (\omega^{\sim} \wedge \zeta)_n$$

for $\omega, \zeta \in W$, where $\omega^{\sim} = \sum_i (-1)^i \omega_i$. Then $(W, \langle\ ,\ \rangle)$ is a $2(2^{2m}-1)$-dimensional non degenerate symplectic space.

Let us denote by B the set of all one dimensional subspaces ℓ of V. For each $\ell \in B$ let

(4) $$W_\ell = \ell \oplus (\ell \wedge \Lambda^1 V) \oplus \ldots \oplus (\ell \wedge \Lambda^{n-2} V).$$

Then W_ℓ is a Lagrangian subspace of $(W, \langle\ ,\ \rangle)$, of dimension $2^{2m}-1$. For $\ell, \ell' \in B$, put

(5) $$W_{\ell,\ell'} = W_\ell / W_\ell \cap W_{\ell'}$$

Let us define the complex G-vector bundle $\eta = (E,B,p,\tau)$ as follows:

B stands as above for the set of all one-dimensional subspaces ℓ of W ;

$E_\ell = \{f : W \longrightarrow \mathbb{C} \mid f(\omega + \zeta) = \psi(\langle \omega,\zeta \rangle) f(\omega) \quad \forall \omega \in W,\ \forall \zeta \in W_\ell \}$;

$E = \bigcup_{\ell \in B} E_\ell$;

and p is the canonical projection from E to B.

The group $G = SL(V)$ acts in a natural way on η by

$(\tau_g f)(\omega) = f(\Lambda g^{-1} \omega) \qquad (g \in G,\ \omega \in W,\ f \in E),$

$\tau_g(\ell) = g(\ell) \qquad (g \in G,\ \ell \in B).$

We define the family of linear transformations

$\Gamma = \{\gamma_{\ell',\ell}\}_{\ell,\ell' \in B}$ on the fibers of η by

(6) $$(\gamma_{\ell',\ell} f)(\omega) = q^{-2^{n-3}} \sum_{\zeta \in W_{\ell',\ell}} \psi(-\langle \omega,\zeta \rangle) f(\omega + \zeta)$$

for $f \in E_\ell$, $\omega \in W$, $\ell, \ell' \in B$, $\ell \neq \ell'$ and $\gamma_{\ell,\ell} = Id_{E_\ell}$ for all $\ell \in B$.

Theorem. The family $\Gamma = \{\gamma_{\ell',\ell}\}_{\ell,\ell' \in B}$ of linear transformations defined by (6) is a G-equivariant flat connection on the complex G-vector bundle η , and so gives raise by contraction over any line $\ell < V$ to a $q^{2^{n-1}-1}$ - dimensional representation (E_ℓ, ρ) of $G = SL(V) \cong SL(n,k)$, which we call generalized Weil representation of G .

□

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