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Eigenfunctions of invariant differential operators on $U(p,q)/U(p-1,q)$

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Let p (resp. q) be a positive(resp. nonnegative) integer. Let $G = U(p,q)$ be the group of all complex $(p+q)\times(p+q)$ matrices preserving the Hermitian form $[x,y] = \overline{y}_1 x_1 + \cdots + \overline{y}_p x_p - \overline{y}_{p+1} x_{p+1} - \cdots - \overline{y}_{p+q} x_{p+q}$ and let X be a space given by $X = \{z \in \mathbf{C}^{p+q}; [z,z] = 1\}$. The action of G is transitive and the subgroup fixing $(1,0,\ldots,0)$ is $U(p-1,q)$, so we have the identification

$$X = U(p,q)/U(p-1,q).$$

We define an action of $U(1)$ on X by $zu = (z_1 u, \ldots, z_{p+q} u)$ for $x \in X, u \in U(1)$. We often identify functions on X with functions on G which are invariant under the action of $U(p-1,q)$. X is also identified with the real hyperboloid $O(2p,2q)/O(2p-1,2q)$.

Let L be the Laplace-Beltrami operator on X and let D be the differential operator on X given by $Df(x) = -i\frac{d}{d\theta} f(xe^{-i\theta})\big|_{\theta=0}$. The algebra $\mathbf{D}(X)$ of G-invariant differential operators on X is generated by L and D. Both L and$D$ can be considered as restrictions of biinvariant differential operators on G.

For $l \in \mathbf{Z}$ let $\mathcal{E}_l(X)$ be the space of C^∞-functions f on X which satisfy $Df = lf$, equivalently $f(zu) = u^{-l} f(z)$ for $z \in X, u \in U(1)$. Then $\mathcal{E}_l(X)$ can be identified with the space of C^∞-sections of the homogeneous line bundle on the semisimple symmetric space $G/H = U(p,q)/U(1)\times U(p-1,q)$ associated with a one dimentional character χ_l of H. For $s \in \mathbf{C}$ and $l \in \mathbf{Z}$ the joint eigenspace $\mathcal{E}_{s,l}(X)$ of $\mathbf{D}(X)$ is given by

$$\mathcal{E}_{s,l}(X) = \{f \in \mathcal{E}_l(X); Lf = (s^2 - \rho^2)f\}$$

where $\rho = p + q - 1$.

Let K denote the maximal compact subgroup $U(p)\times U(q)$ of G. Let $E_{i,j}$ denote a $(p+q)\times(p+q)$ matrices with (i,j)th entry 1 and all other entries being 0. Let Y be the matrix $E_{p+q,1} + E_{1,p+q}$ and let $a_t = \exp tY$ for $t \in \mathbf{R}$. Let $\mathfrak{n}$ be the sum of eigenspaces of adY in $\mathfrak{g} = Lie(G)$ with positive eigenvalues. Let $A = \{a_t; t \in \mathbf{R}\}$, $N = \exp \mathfrak{n}$, $M = Z_H(Y)$ and $M_0 = M \cap K$ be subgroups of G. Then $P = MAN$ is a parabolic subgroup of G.

We define the principal series representation associated with X by

$$\mathcal{E}_{s,l}(G/P) = \{f \in C^\infty(G); f(gma_tn) = \chi_{-l}(m)e^{(s-\rho)t}f(g) \quad \text{for all } g \in G, m \in M, t \in \mathbf{R}, n \in N\}$$

for $s \in \mathbf{C}$, $l \in \mathbf{Z}$. We define the distribution $P_{s,l}$ on G by $P_{s,l}(g) = \chi_l(h)e^{(s+\rho)t}$ if $g = na_th \in NAH$ and otherwise $P_{s,l}(g) = 0$. $P_{s,l}$ is a joint eigendistribution of $\mathbf{D}(X)$. We then define the Poisson transformation $\mathcal{P}_{s,l} : \mathcal{E}_{s,l}(G/P) \longrightarrow \mathcal{E}_{s,l}(X)$ by

$$\mathcal{P}_{s,l}\varphi(g) = \frac{1}{\Gamma(-\frac{1}{2}(s+\rho-2-|l|))}\int_K P_{s,l}(k^{-1}g)\varphi(k)dk$$

where dk is the invariant measure on K with the total measure 1.

LEMMA. *Let $s \in \mathbf{C}$ be such that $Res > 0$ and let $\varphi \in \mathcal{E}_{s,l}(G/P)$ be K finite. Then*

$$\lim_{t\to\infty} e^{(\rho-s)t}\mathcal{P}_{s,l}\varphi(a_t) = c_l(s)\varphi(e),$$

where

$$c_l(s) = \frac{(-1)^{p-1}2^{\rho-s}\Gamma(p)\Gamma(q)\Gamma(s)}{\Gamma(-\frac{1}{2}(s+\rho-2-|l|))\Gamma(\frac{1}{2}(s+\rho+l))\Gamma(\frac{1}{2}(s+\rho-l))}.$$

We define Fourier transformation $\mathcal{F}_{s,l} : \mathcal{E}_l(X) \longrightarrow \mathcal{E}_{s,l}(G/P)$ by

$$\mathcal{F}_{s,l}f(y) = \frac{1}{\Gamma(\frac{1}{2}(s-\rho+2+|l|))}\int_{G/H} f(y)P_{-s,-l}(y^{-1}g)d\dot{g}$$

where $d\dot{g}$ is invariant measure on G/Hnormalized as in [**Sc**]. Note that the integrand is a function on G/H for each $y \in G$. Both Fourier and Poisson transformations are G-equivariant.

THEOREM 1. *Let $f \in \mathcal{E}_l(X)$ be K-finite and compact supported. Then*

$$\begin{aligned} f(x) = {} & \frac{1}{2\pi}\int_0^\infty (\mathcal{P}_{i\lambda,l}\mathcal{F}_{i\lambda,l}f)(x)|c_l(i\lambda)|^{-2}d\lambda \\ & - i \sum_{\substack{0<s<\rho+|l| \\ s\equiv\rho+l \bmod 2}} (\mathcal{P}_{s,l}\mathcal{F}_{s,l}f)(x)\ Res\left[\frac{1}{c_l(\lambda)c_l(-\lambda)}; \lambda = s\right] \\ & - i \sum_{\substack{s\geq\rho+|l| \\ s\equiv\rho+l \bmod 2}} \left(\mathcal{P}_{s,l}\left.\frac{d}{d\lambda}\right|_{\lambda=s} \mathcal{F}_{\lambda,l}f\right)(x)c_{-2}\left[\frac{1}{c_l(\lambda)c_l(-\lambda)}; \lambda = s\right] \end{aligned}$$

where Res denotes the residue and $c_{-2}[\cdot; \lambda = s]$ denotes the coefficients of $(\lambda - s)^{-2}$ in the Laurent development at s.

Let $D = \{(s,l) \in \mathbf{C}\times\mathbf{Z}; s > 0, s-\rho+l \in 2\mathbf{Z}\}$ be the set of discrete spectrums appeared in the decomposition in the above theorem. For $(s,l) \in \mathbf{Z}$ let $V(s,l)$ denote the space of K-finite functions in $\mathcal{E}_{s,l}(X)$ which are square integrable with respect to the invariant measure on X.

THEOREM 2.

(1) $V(s,l)$ ($(s,l) \in \mathbf{C} \times \mathbf{Z}$) *are irreducible* $(\mathfrak{g}, K)$*-modules which are mutually inequivalent. Moreover* $V(s,l)$ *is the image of the* K*-finite functions in* $\mathcal{E}_{s,l}(G/P)$ *by the Poisson transformation* $\mathcal{P}_{s,l}$.

(2) *If* $(s,l) \in \mathbf{C} \times \mathbf{Z}$ *satisfies* $Re s \geq 0$ *and* $s \notin D$*, then* $\mathcal{P}_{s,l}$ *gives the* $(\mathfrak{g}, K)$*−isomorphism between the spaces of* K*-finite functions in* $\mathcal{E}_{s,l}(G/P)$ *and* $\mathcal{E}_{s,l}(X)$.

The proof of the theorems is done by explicit calculations on the K-finite eigenfunctions using the same method as in the papers of Faraut[**1**] and Schlichtkrull[**3**]. We also have obtained a complete description by K-types of all closed invariant subspaces of $\mathcal{E}_{s,l}(X)$.

The discrete series for the symmetric space $O(2p,2q)/O(2p-1,2q)$ is parametrized by the set $D' = \{(s,\varepsilon) \in \mathbf{C} \times \{0,1\}; s > 0, s - \rho + \varepsilon \in 2\mathbf{Z}\}$ (see Rossmann[**2**]). Let $T(s,\varepsilon)$ denote the discret series representation corresponding to the parameter $(s,\varepsilon) \in D'$. Note that the eigenvalue of Laplace-Beltrami operator L is $s^2 - \rho^2$ and $O(1)$ acts by $(-1)^\varepsilon$. Considering the isomorphism

$$U(p,q)/U(p-1,q) \simeq O(2p,2q)/O(2p-1,2q),$$

we have the following corollary of the theorems and its proof.

COROLLARY. *For* $(s,l) \in D'$ *we have the following multiplicity free decomposition*

$$T(s,\varepsilon)|_{U(p,q)} \simeq \bigoplus_{l \equiv \varepsilon \bmod 2} V(s,l).$$

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