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ON THE UNITARY REPRESENTATIONS OF SO(n,m).
The regular integral case.

Susana A. Salamanca-Riba
New Mexico State University
Las Cruces, NM USA

In this manuscript I want to discuss the problem of classifying the irreducible unitary representations of SO(m,n). At present I can attack this problem only when the infinitesimal character of the representation is sufficiently regular. However, in order to avoid some unpleasant technicalities I shall assume that the group in question is the identity component of SO(m,n) and the infinitesimal character of the representation is regular and integral.

To be more precise, if G is any reductive real Lie group I denote the Lie algebra of G by $\mathfrak{g}_0$ and the complexification of the latter by $\mathfrak{g}$ and use likewise notation for other groups. Let K be the maximal Compact subgroup of G, $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$, the Cartan decomposition of $\mathfrak{g}_0$ and $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. If q is a subalgebra of $\mathfrak{g}$ which is invariant under the Cartan involution θ, L its Levi factor, that is, its normalizer in G and (λ,V), a representation of L, Zuckerman [1977], has constructed a class of ($\mathfrak{g}$, K) modules $\mathfrak{R}_q(\lambda)$ of K-finite q-intertwining homomorphisms of $\mathcal{U}(\mathfrak{g})$ into V. If λ is unitary and under certain conditions, Vogan ([1984]) showed that $\mathfrak{R}_q(\lambda)$ is also unitary. We denote this representation by $A_q(\lambda)$ The result that I want to consider is the following

Theorem 1. Let $G=SO_0(n,m)$ and (π, $\mathcal{H}$) be an irreducible unitary represetation of G with regular integral infinitesimal character, then its Harish-Chandra (H-C) module X is isomorphic to a Zuckerman module $A_q(\lambda)$, where λ is one dimensional.

This is a particular case of a conjecture of Vogan [1986], which states that if X is an irreducible unitary H-C module of a real reductive Lie group with integral infinitesimal character, then X is isomorphic to a Zuckerman module $A_q(\lambda)$ (where λ is not necessarily one dimensional). In the sufficiently regular case the conjecture says that λ is one dimensional.

I have already proved this conjecture in the regular integral case, when G is SL(n, R), SU(p,q) and Sp(n, R) [1988] and again, the generalization to the regular case is not too difficult to do and will be part of a future paper. Other people, Enright [1979], Speh [1981], Baldoni Silva-Barbasch [1983], Barbasch [1987], Barbasch-Vogan[1985] and Vogan[1986] among others, have contributed to partial results of this conjecture.

The basic tools for the proof of this theorem are some necessary and sufficient conditions (D. A. VOGAN-G. ZUCKERMAN [1984]) for the lowest K-type (LKT) of the modules $A_q(\lambda)$ -this is a representation of K ocurring in X, which minimizes a certain norm defined on K^- and a necessary condition for any K-type of a unitary representation namely, Parthasarathy's Dirac operator inequality (see A. Borel-N. Wallach [1980] and S.A. Salamanca-Riba [1988])

The proof is by contradiction. We first assume that X is an irreducible H-C module of G equiped with a non-zero Hermitian form and that the LKT of X fails to satisfy the necessary and sufficient conditions previously mentioned. Using this information one tries to show that the representation fails to satisfy Parthasarathy's inequality on this K-type, and therefore conclude that X is not unitary. For a generic K-type this is technically very messy and unpractical, and one is forced to try to find another K-type of X such that the signature of the Hermitian form restricted to these two K-types is indefinite. Here is where the hard technical problem of the paper lies.

To do this we need to reduce the problem to H-C modules of proper reductive subgroups whose LKT's have a very simple form and Parthasarathy's inequality is very easy to test. Studying the failure of this inequality a little closer one can find a list of candidates of K-types where the Hermitian form will be indefinite.

Here is how the reduction step is achieved: Vogan [1979] (see also [1981]) attaches to a LKT of an H-C module a representation (π^M, X^M) of a proper reductive subgroup M, such that X is isomorphic to a Zuckerman module $\mathfrak{R}_s(\pi^M)$, where M is the normalizer in G of s. Using this result and Vogan [1984], I prove the following

Theorem 2. Let $G=SO_0(n,m)$ and X, an irreducible H-C module which is not a Zuckerman module $A_q(\lambda)$ and assume X is endowed with a non-zero Hermitian form. Then the following hold

(a) There are a θ-stable parabolic subalgebra s' and a representation $(\pi^{M'}, V^{M'})$ of $M'=N_G(s')$ such that $X \cong \mathfrak{R}_{s'}(\pi^{M'})$.

(b) $V^{M'}$ is endowed with a Hermitian form $< , >^{M'}$

(c) There are lowest $(M \cap K)$-types $(LK_{M'}T)$ δ, δ' of $\pi^{M'}$ such that the restriction of $< , >^{M'}$ to $V(\delta) \oplus V(\delta')$ is indefinite.

(d) Set $\mu=\mathfrak{R}_{s' \cap k}(\delta)$ and $\mu'=\mathfrak{R}_{s' \cap k}(\delta')$, then μ and μ' are K-types of X

Having a representation satisfying (a)-(d) Vogan shows (see S. A. Salamanca-Riba [1988], Theorem 5.8) that the signature of < ,> on $V(\mu)\oplus V(\mu')$ is equal to the signature of $< , >^{M'}$ on $V(\delta)\oplus V(\delta')$, which proves Theorem 1.

In most cases M'=M and M' is always a product of the form

$M'=U(r_1,s_1)xU(r_2,s_2)x...xU(r_t,s_t)xSO_o(n',m')$, where if n=2p or 2p+1, and m=2q or 2q+1, then $n'=p-2\sum r_i$ and $m'=q-2\sum s_t$. This reduces the problem to the case when G is U(p,q) and a proper subgroup $SO_o(n',m')$ where n' and n and m'and m have the same pairity.

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