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Poisson Integrals for Homogeneous, Rank 1 Koszul domains in $\mathbf{C}^n$

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Let $\Omega \subset \mathbf{C}^n$ be a domain. We shall say that Ω is homogeneous under the real analytic Lie group G if Ω is a homogeneous G-space for which the mapping $\nu : G \times \Omega \to \Omega$ is real analytic in the G variable and holomorphic in the Ω variable. Suppose that there is a positive, G-invariant, regular, Borel measure μ on Ω. In this case, we may write $\mu = K dx$ for some non-negative function K on Ω (dx denotes Lebesgue measure on $\mathbf{C}^n$). Homogeneity implies that K is strictly positive. In [Kl], Koszul introduced the following form on $T(\Omega)$ which we refer to as the Koszul form:

(1)
$$H = \sum \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log K dz_i d\overline{z}_j.$$

The form H is invariant under any bi-holomorphic mapping which preserves the measure μ. We shall say that Ω is a Koszul domain if H is non-degenerate. This gives Ω the structure of a pseudo-Kähler manifold for which the measure preserving bi-holomorphisms are isometries. In this work, we shall make the additional assumption that Ω is rank one (see [P2]).

Any pseudo-Riemannian manifold carries a canonical real, second order operator Δ. In general, this operator will be non-elliptic and non-hypoelliptic, due to the fact that H, while non-degenerate, will usually be non-definite. The goal of this work is to study the eigenvalue problem for Δ in the spirit of works such as Helgason [He], Kashiwara, *et al.* [Ka] and Oshima [Os]. Specifically, we define an appropriate concept of 'boundary' for a rank one homogeneous Koszul domain. It turns out that relative to this boundary, $\Delta - (\lambda^2 - 1)/2$ has regular singularities in the sense of [Os] for $\lambda \notin \mathbf{Z}/2$. This allows us to define the boundary value of the solution as a hyperfunction. We then construct an operator on a subspace of the space of boundary values which inverts the boundary map. This is, by definition, the Poisson kernel. A present, we have only succeeded in proving convergence of our formulas for relatively regular boundary values, under the additional hypothesis that λ is non-real. However, these restrictions, particularly the restriction on λ, are probably inessential.

We should remark that there is very little overlap between the domains we are considering and pseudo-Riemannian symmetric spaces. The only symmetric cases covered are the unit balls in $\mathbf{C}^n$. Despite this, the class of rank one Koszul domains is quite extensive. There is an extensive structure theory of such domains, as well as an algebraic machine for producing examples. (See [P2]).

To describe an appropriate boundary, we shall require some of the structure theory described in [P2]. We shall adopt the convention that Lie groups will be denoted by upper case Roman letters and the corresponding Lie algebra will be denoted by the corresponding upper case script letter.

It was shown in [P2] that every rank one, homogeneous Koszul domain is realizable as a homogeneous nil-ball. To recall the definition, let N be a nilpotent Lie group. Let $\lambda \in \mathcal{N}^*$. A complex subalgebra $\mathcal{P}' \subset \mathcal{N}_c$ is a totally complex polarization for λ if it satisfies the following properties:

(a) $Z \in \mathcal{P}'$ if and only if $[Z, \mathcal{P}'] \subset \ker \lambda$.

(b) $\mathcal{P}' + \overline{\mathcal{P}'} = \mathcal{N}_c$.

Given a totally complex polarization, there is a canonical way of associating a Koszul domain with it. Explicitly, we let $\mathcal{P} = \mathcal{P}' \cap \ker \lambda$. By forming a quotient, we may assume that the kernel of λ contains no non-trivial ideals. In this case, the center $\mathcal{Z}$ of $\mathcal{N}$ is one dimensional and λ is non-trivial on $\mathcal{Z}$. Let Z_o be a basis for the center, normalized by the condition that $\lambda(Z_o) = 1$. Then, clearly,

$$\mathcal{N}_c = \mathcal{N} + i\mathbf{R}Z_o + \mathcal{P}.$$

Let $X = N_c/P$. From the nilpotency of N_c, this space is bi-holomorphic with $\mathbf{C}^n$ for some n. The above equality implies that the image of N in X under the quotient map is a real co-dimension one submanifold which divides X into two connected components

$$\Omega^{\pm} = N(\exp \mathbf{R}^{\pm} iZ_o)P/P.$$

We refer to $\Omega^+ = \Omega$ as the domain associated to the polarization. (Note that Ω^- is the domain associated to $(\mathcal{P}, -\lambda)$.) Such domains are what we refer to as the nil-balls.

As a C^∞ manifold,

$$\Omega^+ = \mathbf{R}^+ \times NP/P = \mathbf{R}^+ \times N/R \tag{2}$$

where $R = P \cap N$. This decomposition respects the N action. In particular, we see that N does not act transitively on Ω^+. Thus, not all nil-balls need be homogeneous.

However, suppose that $t \to \delta(t)$, $t \in \mathbf{R}^+$, is a one parameter group of semi-simple, $\mathbf{R}$-split automorphisms of $\mathcal{N}$. We shall say that the pair $(\lambda, \mathcal{P}')$ is homogeneous if $\delta(t)$ preserves $\mathcal{P}'$ and $\delta(t)^*(\lambda) = t\lambda$ for all t. In this case, the action of $\delta(t)$ on N_c projects to an action on N_c/P. Furthermore, $\delta(t)(Z_o) = tZ_o$. It is follows that under these assumptions, Ω^+ is homogeneous under the group $G = N \times_s \mathbf{R}^+$ where $\mathbf{R}^+$ acts on N by means of δ.

We shall assume in the rest of this work that such a realization is given. By the boundary of Ω, we mean its topological boundary in this realization. This boundary is thus identified with N/R. In formula 2 above, we shall denote the $\mathbf{R}^+$ coordinate by 'y' and the N coordinate by 'u'.

Next, we wish to describe Δ in terms of these coordinates. Let Z_o denote the action of Z_o as a left invariant vector field in the N/R coordinate in the decomposition of formula 2.

PROPOSITION 1. *There is an N-invariant, second order differential operator Δ_b on N/R such that*

$$\Delta = y^2\left(\frac{\partial^2}{\partial y^2} + Z_o^2\right) + (n+2)y\frac{\partial}{\partial y} + y(\Delta_b + i(n+2)Z_o). \tag{3}$$

The operator Δ_b is homogeneous of degree one with respect to the one parameter group δ defined above.

An immediate corollary of this proposition is that $\Delta - q$ has regular singularities in the weak sense along N/R. This then allows us to use the theory in [Os] to define the boundary values of any hyperfunction solution to this equation.

Our main tool in the analysis of this operator is an integral operator on N referred to as the N-transform which has the remarkable property of transforming this equation into what is essentially the Casmir element of $\mathrm{Sl}(2,\mathbf{R})$ acting in the space of an induced representation.

Let $f \in L^2(N/R)$. We let the partial Fourier transformation of f in the Z_o direction at λ be denoted by f_λ. Thus

$$f_\lambda(u) = \int_{-\infty}^{\infty} f(\exp tZ_o u)e^{-i\lambda t}dt.$$

Then f_λ transforms covariantly under the group $L = (\exp \mathbf{R}Z_o)R$ with respect to the character χ defined by $\chi(\exp tZ_o u) = \exp -i\lambda t$. For a.e. λ, the function $|f_\lambda|$ is in $L^2(N/L)$ and

$$||f||^2 = \int_{-\infty}^{\infty} ||f_\lambda||^2 d\lambda. \tag{4}$$

We denote the space of all such covariant, $L^2(N/L)$, functions by $\mathcal{H}_\lambda$. We consider this space as a Hilbert space under the obvious norm.

We define, for $\lambda \neq 0$, $g_\lambda = |\lambda|^{(1-n)/2} f_\lambda \circ \delta(|\lambda|^{-1})$. Then, for $\lambda > 0$, g_λ belongs to $\mathcal{H}_1$ while for $\lambda < 0$, g_λ belongs to $\mathcal{H}_{-1}$. The mapping of $f_\lambda \to g_\lambda$ defines an isometry of $\mathcal{H}_\lambda$ onto $\mathcal{H}_{\pm 1}$. Finally, we define

$$Tf(x) = \int_0^{\infty} e^{ix\lambda}(g_\lambda, g_{-\lambda})d\lambda.$$

This $\mathcal{H}_1 \times \mathcal{H}_{-1} = \mathcal{H}$ valued integral exists for a.e. $x \in \mathbf{R}$ due to formula 4. This mapping is an isometry onto the subspace $\mathcal{L}^+$ of $L^2(\mathbf{R}, \mathcal{H})$ consisting of those elements with Fourier transformation supported in $\mathbf{R}^+$.

The point of introducing T is the effect it has on Δ_b. Observe that $\mathcal{H}_\lambda$ may be interpreted as the space of square integrable sections over N/L of the homogeneous line bundle $E = N \times_\chi \mathbf{C}$. The manifold N/L may be identified with the complex manifold N_c/P' since (as may be easily verified) $NP' = N_c$. In this complex structure, N/L is pseudo-Kahlerian and E is a holomorphic line bundle. Let Δ_b^λ be the Laplace-Beltrami operator which acts on C^∞ sections of E. Let D be the densely defined operator on $\mathcal{H}$ defined by $D = (\Delta_b^1, \Delta_b^{-1})$. Using the homogeneity of Δ_b in proposition 1, it is easily seen that T intertwines Δ_b and $\frac{\partial}{\partial x}D$. P also intertwines Z_o and $\frac{\partial}{\partial x}$.

Using T, we next define an isometry S of $L^2(\Omega)$ onto a subspace $\mathcal{M}^+$ of $L^2(\mathbf{R}^+ \times \mathbf{R}, y^{-2}dydx, \mathcal{H})$. Explicitly, we define $Sf(x,y) = Tf_y(x)$ where $f_y(u) = y^{-(n-1)/2}f(y,u)$. We obtain the following fundamental result.

PROPOSITION 2. *The operator S intertwine Δ with the operator*

$$y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{1-n^2}{4} - iy\frac{\partial}{\partial x}(D+n-1).$$

This may be put into a more familiar form if (as we shall assume) D has a self adjoint closure. Then there is a unitary representation σ of $\mathbf{R}$ in $\mathcal{H}$ whose infinitesimal generator is $i(D+n-1)$. Letting $g(x,y,\mu) = \sigma(-\mu)Sf(x,y)$, we see that $-i(D+n-1)$ in the above formula becomes simply $\frac{\partial}{\partial \mu}$. The operator of proposition 2 then becomes

$$y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{1-n^2}{4} + y\frac{\partial^2}{\partial x \partial \mu}.$$

This is just the Laplace-Beltrami operator of the simply connected covering group Q of $\mathrm{Sl}(2,\mathbf{R})$ expressed in $NA\tilde{K}$ coordinates. More precisely, let a basis for the Lie algebra be defined by

$$H_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, N_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We identify Q with $\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}$ by means of the mapping

$$(y,x,\mu) \to (\exp xN_0)(\exp(\frac{1}{2}\ln y)A_0)(\exp \mu H_0).$$

We may consider σ as a representation of the subgroup $\tilde{K} = 0 \times 0 \times \mathbf{R}$. Let π be the unitary representation of Q induced from σ. Then, our transformation S intertwines Δ with $\pi(C) + (1-n^2)/2$ where C is the Casmir element of Q.

Following ideas in [Ka], a boundary theory for the eigenvectors of $\pi(C)$ may be constructed. Furthermore, it turns out that the operators S and T intertwine the corresponding boundary operators for Δ and $\pi(C)$. This allows us reduce the problem of computing the Poisson kernel for Δ to (a) computing the Poisson kernels for $\pi(C)$ and (b) to explicitly computing the representation σ. Problem (b) is a problem which we have considered at length is a previous work [P1]. This computation may be carried out in a broad (but far from complete) set of cases. Problem (a) also is solvable, at least with appropriate restrictions on λ and the space of allowed boundary values. We shall discuss this theory briefly.

Let $\mathcal{P}$ be the parabolic subalgebra which is the span of N_0 and A_0 and let $\mathcal{P}^t$ be the opposite parabolic. Let $R_o = P \times P^t$ be the connected subgroup of $M = Q \times Q$ which corresponds to $\mathcal{P} \times \mathcal{P}^t$. Let $Z = \exp(\pi \mathbf{Z}) H_0$ (the center of Q) and define ΔZ to be the subgroup of $Q \times Q$ consisting of elements (z, z^{-1}) where $z \in Z$. Let $R = R_o \Delta Z$. There is a unique character τ_λ of R, trivial on ΔZ, whose differential equals equals $\lambda + 1$ at $(A_0, 0)$ and $\lambda - 1$ at $(0, A_0)$. Let E be the homogeneous line bundle over $R \setminus M$ defined by τ_λ and let $\sigma_\lambda(g)$ denote right translation of sections of E by $g \in Q \times Q$. Then, any $\mathcal{H}$ valued eigenfunction to $\pi(C)$ corresponding to $q = (\lambda^2 - 1)/2$ has a boundary value b_λ which is a generalized section of $E \otimes \mathcal{H}$. This boundary value will satisfy

$$\sigma_\lambda(e, k) b = \sigma(k^{-1}) b$$

for all $k \in \tilde{K}$, where σ_λ denotes right translation of sections.

Conversely, given sufficiently regular generalized sections of this bundle satisfying this transformation property, it is possible to obtain an eigenfunction for $\pi(C)$. For $\operatorname{im} \lambda > 0$, the formula takes the form

$$Pb(g) = C \sum_n (-1)^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sigma_\lambda(g, e) b)(\exp((\mu + n\pi) H_o, \nu H_o)) e^{i|n|\pi\lambda} \cos^{\lambda - 1}(\mu - \nu) d\mu d\nu$$

where C is a constant independent of b. The formula for $\operatorname{im} \lambda < 0$ is similar.

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