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Spherical Distributions on Symmetric Spaces

Consider a symmetric space $X = G/H$, where G is a non-compact connected semi-simple Lie group and $H = G^\tau$, the fixpoints of an involution τ on G. Harmonic analysis on X usually involves the following two problems:

(a) Construct intertwining operators from concrete models of representations into functions on X.

(b) Construct H-invariant distributions on X satisfying differential equations corresponding to the G-invariant differential operators on X.

It is important to have relatively explicit formulas in (a) and (b) and of course to find the invariant distributions Θ giving the constituents of $L^2(X)$. In particular for the discrete series [**6**], one would like to find the analogue of Harish-Chandra's theory of characters for the discrete series on a semi-simple Lie group.

In this lecture I describe some recent progress on the problems above obtained jointly with G. 'Olafsson, building on [**1**], [**3**], [**4**], and [**5**]. The main result is for X of Hermitian type and the representations in (a) from the holomorphic discrete series of X, where the corresponding distributions in (b) admit analytic continuations to a certain domain Ξ in the complexification $X_{\mathbf{C}}$ of X. In this special case one finds an explicit formula for Θ on Ξ, and by taking boundary values on X one gets information on the singular support of Θ. These formulas are particularly nice when the space is of Cayley type, i.e. X is also a coadjoint orbit for G. Using the compactification in [**2**] and extending it to Ξ one finds a relation between the sum of the distributions for the holomorphic discrete series of X and the Cauchy-Szegö kernel for the bounded symmetric domain for $G_1 = G \times G$.

To be a little more precise, we employ the notation of [**1**] and [**5**]:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$$

and $\mathfrak{a} \subseteq \mathfrak{k} \cap \mathfrak{q}$ maximal Abelian with positive roots Δ^+. For μ a linear functional on $\mathfrak{a}$ with $\langle \mu + \rho, \alpha \rangle < 0$ for all $\alpha \in \Delta_n^+$, we associate the representation π_μ in the holomorphic discrete series for X, when μ is the highest weight of a representation δ of K with a $K \cap H$-fixed vector v_0. The intertwining operator as in (a) is given by convolution by the function

$$\phi_\mu(x) = \delta(k_H(x^{-1})^{-1})v_0, \quad x \in P^+ K_{\mathbf{C}} H_{\mathbf{C}}.$$

Here k_H is the middle component in the decomposition $H_{\mathbf{C}} K_{\mathbf{C}} P^+$.

Theorem. *Assume X is of Cayley type and $\Xi = \Gamma(-D)X$ with $\Gamma(-D) = G\exp(-iD)$ the holomorphic semigroup corresponding to the invariant convex cone $D \subseteq \mathfrak{g}$. Then we have:*

(i) *The Hardy space $H(D)$ is G-equivalent to the direct sum of all holomorphic discrete series π_μ, with multiplicity one, and to the Hardy space H_1 for the domain $G/K \times G/K$.*

(ii) *The distributions Θ_μ for these π_μ, i.e. Θ_μ is projection onto π_μ followed by evaluation at the identity coset, admit holomorphic extensions to Ξ, where the reproducing kernel for $H(D)$ is expressed as a sum over all Θ_μ.*

(iii) *On Ξ we have the following convolution formula, for every μ as above:*

$$\Theta_\mu(x) = \int_{G/K} (\phi_\mu(y^{-1}x), \phi_\mu(y^{-1}))dy.$$

Parts of this theorem (and conjectually all) remain true in the case of X a general space of Hermitian type. Thus for a large class of examples, we can find the analogue of G_1 and a corresponding causal compactification – with several applications to the harmonic analysis of X.

From (iii) one may get some information about the singularities on X of Θ_μ, for example that they are related to the orbits in X of the boundary of the cone D. Note finally that Θ_μ can also be expressed as a Poisson integral (analogous to Harish-Chandra's formula for the usual spherical functions) for the "dual" group with Lie algebra $\mathfrak{h} + i\mathfrak{q}$.

A key role in this interplay between the geometry of X and the harmonic analysis is thus played by invariant convex cones in $\mathfrak{g}$ and in $\mathfrak{q}$ respectively. This provides the space with a causal structure, so it is perhaps not surprising that examples of physical interest arise: The Lorentz hyperboloid, De Sitter space, and the cotangent bundle of compactified Minkowsky space. The representations in question satisfy analogues of "positive energy" and may be applied to problems in geometric quantization.

One may hope that symmetric spaces of Hermitian type will be an interesting class of spaces to study further, as they allow rather explicit treatment.

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