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Geometric Realizations of Unitary Highest Weight Modules As Solutions of Differential Equations

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Let G be a real reductive Lie group with maximal compact subgroup K. We assume that G/K is hermitian symmetric. We consider those representations of G which are contained in a kth tensor power of the metaplectic representation. Our goal is to give a geometric construction of these representations as holomorphic sections of some vector bundles over G/K which also satisfy, in most cases, a system of homogeneous linear differential equations. For example, if $G = \mathrm{U}(2,2)$, complexifications of Maxwell's equations and the wave equation occur here. Our construction is inspired by the Radon transform from integral geometry and the Penrose transform of mathematical physics.

1. Geometric Preliminaries. The groups G for which G/K is hermitian symmetric have been classified. Our construction uses the bounded realization of G/K as one of the classical bounded symmetric domains, as follows.

	$\underline{G}$	$\underline{K}$	$\underline{G/K}$
(1.1)	$\mathrm{U}(p,q)$	$\mathrm{U}(p)\times\mathrm{U}(q)$	$\{\zeta\in\mathbb{C}^{p\times q}\mid I(\zeta):=I-{}^*\zeta\zeta\gg 0\}$
	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{U}(n)$	$\{\zeta\in\mathbb{C}^{n\times n}\mid I(\zeta)\gg 0,\ {}^t\zeta=\zeta\}$
	$\mathrm{SO}^*(2n)$	$\mathrm{U}(n)$	$\{\zeta\in\mathbb{C}^{n\times n}\mid I(\zeta)\gg 0,\ {}^t\zeta=-\zeta\}$

Here, as usual, $\mathrm{U}(p,q)=\{g\in\mathrm{GL}(p+q,\mathbb{C})\mid {}^*gI_{p,q}g=I_{p,q}\}$ where the diagonal matrix $I_{p,q}=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ determines a hermitian form $h=h_{p,q}$ of signature (p,q), I denotes the identity matrix of the appropriate size, $\mathbb{C}^{p\times q}$ denotes the space of $p\times q$ complex matrices, ${}^*\zeta={}^t\overline{\zeta}$, and $A\gg 0$ means that A is positive definite. The bounded model of G/K requires that we define

$$\mathrm{Sp}(2n,\mathbb{R})=\mathrm{SU}(n,n)\cap\mathrm{Sp}(2n,\mathbb{C})$$

and

$$\mathrm{SO}^*(2n)=\mathrm{SU}(n,n)\cap\mathrm{O}(2n,\mathbb{C}),$$

where the group $\mathrm{Sp}(2n,\mathbb{C})$ preserves the standard symplectic form b on $\mathbb{C}^{2n}$ and the group $\mathrm{O}(2n,\mathbb{C})$ preserves the symmetric form τ on $\mathbb{C}^{2n}$ with matrix $\begin{pmatrix}0&I\\I&0\end{pmatrix}$.

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The groups $\mathrm{SO}(2,q)$ and the exceptional groups E_6 and E_7 also give rise to hermitian symmetric spaces, but we omit discussion of these groups here.

One of the keys to our construction is an appropriate geometric understanding of G/K. Let $\mathbf{W} \cong \mathbb{C}^m$ denote the space on which G acts naturally by left multiplication, that is, $m = p+q$ if $G = \mathrm{U}(p,q)$ and $m = 2n$ otherwise. Now, for any $\zeta \in G/K$, we define the subspaces $V^+(\zeta)$ and $V^-(\zeta)$ of $\mathbf{W}$ to be the column spans of the matrices

$$(1.2) \qquad \begin{pmatrix} I \\ {}^*\zeta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta \\ I \end{pmatrix},$$

respectively. Then $h|V^+(\zeta) \gg 0$ and $h|V^-(\zeta) \ll 0$. Furthermore, $b|V^\pm(\zeta) = \tau|V^\pm(\zeta) \equiv 0$. Finally, the action of G on subspaces of $\mathbf{W}$ which comes from left multiplication of the matrices in (1.2) by $g \in G$ corresponds to the standard action of $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on $\zeta \in G/K$ by $g.\zeta = (A\zeta + B)(C\zeta + D)^{-1}$. Thus we obtain

LEMMA 1.3. *The sets $\{V^+(\zeta) \mid \zeta \in G/K\}$ and $\{V^-(\zeta) \mid \zeta \in G/K\}$ give two distinct embeddings of G/K as a G-orbit in a Grassmannian of complex subspaces of $\mathbf{W}$.*

Next we consider $\mathbf{W}^k = \mathbf{W} \otimes \mathbb{C}^k \cong \mathbb{C}^{m\times k}$. We let $\langle z, w\rangle = \mathrm{tr}({}^*wz)$ be the standard positive definite form on $\mathbf{W}^k$, and we define the G-invariant form $\langle\!\langle z, w\rangle\!\rangle = \mathrm{tr}({}^*wI_{p,q}z)$ for $G = \mathrm{U}(p,q)$, or using $I_{n,n}$ otherwise. We decompose $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{W}^k$ so that the top block z_1 has p rows if $G = \mathrm{U}(p,q)$ or n otherwise, etc. Now, for any s such that $0 \le s \le k$, we define the subspace $V_s(\zeta)$ of $\mathbf{W}^k$ to be a direct sum of $k-s$ copies of $V^+(\zeta)$ and s copies of $V^-(\zeta)$, one for each copy of $\mathbf{W}$ in $\mathbf{W}^k$. For example, $V_s := V_s(0) = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} = u \in \mathbf{W}^k \right\}$, where u_1 has $k-s$ columns and u_2 has s columns. If we define $R(\zeta) = \begin{pmatrix} I & \zeta \\ {}^*\zeta & I \end{pmatrix}$ (cf. (1.2)), then we note that

$$(1.4) \qquad V_s(\zeta) = \{ R(\zeta)u \mid u \in V_s \}.$$

2. L^2-Cohomology and the Oscillator Representation. We define an inner product on differential forms $\omega(z) = \varphi(z)\, d\bar{z}_I$ and $\xi(z) = \psi(z)\, d\bar{z}_J$ by defining

$$\langle \omega, \xi \rangle = \delta_{I,J} \int_{\mathbf{W}^k} \varphi(z)\overline{\psi(z)}\, dm(z).$$

The coboundary operator for our L^2-cohomology (see [**BR**] or [**M3**] for the definition) is the covariant differential of a line bundle over $\mathbf{W}^k$.

LEMMA 2.1 [**BR**]. *The L^2-cohomology of $\mathbf{W}^k$ is nonzero in exactly one dimension, either dimension qk for $G = \mathrm{U}(p,q)$ or dimension nk otherwise. In that* *dimension, the L^2-cohomology $\mathcal{H}(\mathbf{W}^k)$ consists of all differential forms ω such that $\omega(z) = \varphi(z)d\bar{z}_2$ where $\varphi(z)e^{\frac{1}{2}|z|^2} = f(z)$ is holomorphic in z_1 and $\bar{z}_2$ and*

$$|\omega|^2 = \int_{\mathbf{W}^k} |f(z)|^2 e^{-|z|^2}\, dm(z) < \infty.$$

The unitary representation of G on $\mathcal{H}(\mathbf{W}^k)$ is easy to describe. Let P denote the orthogonal projection from the space of all square-integrable differential forms onto $\mathcal{H}(\mathbf{W}^k)$. Then $P(\varphi\, d\bar{z}_I) = 0$ unless $z_I = z_2$, and

$$P(\varphi\, d\bar{z}_2)(z) = d\bar{z}_2 \int_{\mathbf{W}^k} \varphi(w) K(z,w)\, dm(w), \tag{2.2}$$

where $K(z,w) = e^{-\frac{1}{2}(|z|^2+|w|^2)} e^{\langle z_1, w_1\rangle + \langle w_2, z_2\rangle}$. Thus, if $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$ and $\omega = \varphi\, d\bar{z}_2$, we see that $\sigma(g) = P \circ l(g)$, that is,

$$(\sigma(g)\omega)(z) = \det D^k\, d\bar{z}_2 \int_{\mathbf{W}^k} \varphi(g^{-1}w) K(z,w)\, dm(w)\,. \tag{2.3}$$

We remark that $G \subset \mathrm{Sp}(\mathrm{Im}\langle\!\langle \cdot,\cdot \rangle\!\rangle) \cong \mathrm{Sp}(2mk, \mathbb{R})$, and it is well known that σ is the restriction to G of the oscillator representation of $\widetilde{\mathrm{Sp}}(2mk,\mathbb{R})$. The decomposition of σ uses Howe's theory of dual reductive pairs [**H**]. Thus let G' be the centralizer of G in $\mathrm{Sp}(2mk,\mathbb{R})$, given by [**H**] as follows.

$\underline{G}$	$\underline{G'}$
$\mathrm{U}(p,q)$	$\mathrm{U}(k)$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{O}(2k)$
$\mathrm{SO}^*(2n)$	$\mathrm{Sp}(k)$

In each case G' is compact and has rank k. In fact, the torus $\mathbb{T}$ in $\mathrm{U}(k)$ consisting of the diagonal matrices $e^{i[\theta_1,\ldots,\theta_k]} = \mathrm{diag}(e^{i\theta_1},\ldots,e^{i\theta_k})$ clearly acts on $\mathbf{W}^k$ on the right. We define eigenspaces $\mathcal{H}_\lambda(\mathbf{W}^k)$ of $\mathbb{T}$ on $\mathcal{H}(\mathbf{W}^k)$, for weights $\lambda \in \mathbb{Z}^k$, so that $\omega \in \mathcal{H}_\lambda(\mathbf{W}^k)$ if and only if $\omega(z) = f(z)e^{-\frac{1}{2}|z|^2}\, d\bar{z}_2$ where

$$f(ze^{i[\theta_1,\ldots,\theta_k]}) = e^{i(\lambda_1\theta_1+\cdots+\lambda_k\theta_k)} f(z). \tag{2.4}$$

Clearly $\mathcal{H}_\lambda(\mathbf{W}^k)$ is G-invariant. Let $\Lambda \subset \widehat{G'}$ be the set of representations of G' which occur in $\mathcal{H}(\mathbf{W}^k)$. We identify $\lambda \in \Lambda$ with its highest weight, so that λ is a nonincreasing k-tuple of integers. Howe shows that there is a one-one correspondence between the representations $\lambda \in \Lambda$ and the representations σ_λ of G which occur in $\mathcal{H}(\mathbf{W}^k)$, and furthermore that $\sigma = \bigoplus d_\lambda \sigma_\lambda$, where d_λ is the dimension of λ. Kashiwara and Vergne [**KV**] explicitly describe Λ for $G = \mathrm{U}(p,q)$ and $G = \mathrm{Sp}(2n,\mathbb{R})$ and show that there is a distinguished copy

$$\mathcal{H}(\lambda) \subset \mathcal{H}_\lambda(\mathbf{W}^k) \tag{2.5}$$

of σ_λ. In this way we obtain all unitary highest weight modules for $G = \mathrm{Sp}(2n, \mathbb{R})$ and $G = \mathrm{U}(p,q)$ **[EP]** and almost all those for $G = \mathrm{SO}^*(2n)$ **[DES]**.

Finally, we specialize our discussion of L^2-cohomology to the spaces $V_s(\zeta)$. Using (1.4), we consider $u \in V_s$ as a coordinate on $V_s(\zeta)$, and we define $\mathcal{I}(\zeta) = \begin{pmatrix} I({}^*\zeta) & 0 \\ 0 & I(\zeta) \end{pmatrix}$ (see (1.1)). Then, combining Lemma 2.1 with (2.4), we see that $\omega \in \mathcal{H}_\lambda(V_s(\zeta))$ if and only if $\omega(u) = f(u)e^{-\frac{1}{2}\langle \mathcal{I}(\zeta)u,u\rangle}\, d\bar{u}_2$ where f is holomorphic in u_1 and $\bar{u}_2$ and

$$f(ue^{i[\theta_1,\ldots,\theta_k]}) = e^{i(\lambda_1\theta_1+\cdots+\lambda_k\theta_k)} f(u). \tag{2.6}$$

The form of u then implies that f is a polynomial in u_1 and $\bar{u}_2$, hence that $\mathcal{H}_\lambda(V_s(\zeta))$ is finite-dimensional.

3. The Integral Transform. Throughout this section we fix $\lambda \in \Lambda$. We define $s = s(\lambda)$ to be the number of negative entries in λ, so that $0 \le s \le k$. We now construct a vector bundle $\mathbf{E}_\lambda$ on G/K so that the fiber $\mathbf{E}_\lambda(\zeta)$ over $\zeta \in G/K$ is

$$\mathcal{H}_\lambda(V_s(\zeta)) \otimes \mathbb{C}\, du_1. \tag{3.1}$$

By (2.6), the fiber is finite-dimensional. The bundle $\mathbf{E}_\lambda$ is a holomorphic, homogeneous vector bundle for G which is trivial topologically. We denote by $\mathcal{O}(\mathbf{E}_\lambda)$ its space of holomorphic sections.

LEMMA 3.2 **[M3]**. *Let $\omega = \varphi\, d\bar{z}_2 \in \mathcal{H}(\mathbf{W}^k)$, let $\zeta \in G/K$, and let $0 \le s \le k$. Then, the restriction $\varphi|V_s(\zeta)$ is square-integrable.*

We may now construct our integral transform. As in (2.5), let the representation σ_λ of G corresponding to λ act on the space $\mathcal{H}(\lambda) \subset \mathcal{H}_\lambda(\mathbf{W}^k)$. We define the integral transform $\Phi_\lambda : \mathcal{H}(\lambda) \to \mathcal{O}(\mathbf{E}_\lambda)$ so that if $\omega = \varphi\, d\bar{z}_2 \in \mathcal{H}(\lambda)$, then

$$(\Phi_\lambda(\omega)(\zeta))(u) = P_{s,\zeta}(\varphi|V_s(\zeta)\, d\bar{u}_2) \wedge du_1,$$

where as in (2.2), $P_{s,\zeta}$ is the projection onto $\mathcal{H}(V_s(\zeta))$. This integral converges, by Lemma 3.2. By combining (2.6) and (2.2) with (1.4), we compute that

$$(\Phi_\lambda(\omega)(\zeta))(u) = d\bar{u}_2 \wedge du_1 \int_{V_s} \varphi(R(\zeta)v)\, K_{s,\zeta}(u,v)\, dm(v), \tag{3.3}$$

where $K_{s,\zeta}(u,v) = \det I(\zeta)^k\, e^{-\frac{1}{2}(\langle \mathcal{I}(\zeta)u,u\rangle + \langle \mathcal{I}(\zeta)v,v\rangle)}\, e^{\langle I({}^*\zeta)u_1,v_1\rangle + \langle I(\zeta)v_2,u_2\rangle}$.

THEOREM 3.4 **[M2, M3]**. *The transform Φ_λ is well-defined, injective, and intertwines the actions of G on $\mathcal{H}(\lambda)$ and on $\mathcal{O}(\mathbf{E}_\lambda)$.*

4. The Differential Equations. In this section we assume that $G = \mathrm{U}(p,q)$, and we fix $\lambda \in \Lambda \subset \mathbb{Z}^k$. Let $k = r + s$, where s was defined in (3.1). We discuss the differential equations in the trivialization of $\mathbf{E}_\lambda$. Let $\omega \in \mathcal{H}(\lambda)$ be such that $\omega(z) = f(z)\, e^{-\frac{1}{2}|z|^2}\, d\bar{z}_2$. Then the section $\Phi_\lambda(\omega)$ of $\mathbf{E}_\lambda$ gives rise to the vector-valued function

$$(\Psi\omega)(\zeta, u) = \int_{V_s} f(R(\zeta)v)\, e^{-\frac{1}{2}\langle v,v\rangle}\, e^{\langle u_1,v_1\rangle + \langle v_2,u_2\rangle}\, dm(v), \tag{4.1}$$

where $\Psi\omega$ is holomorphic in ζ and is a polynomial in the variables u_1 and $\bar{u}_2$. Consider the $(p+s)\times(q+r)$ matrix

$$\nabla = \begin{pmatrix} \partial/\partial\zeta & \partial/\partial u_1 \\ {}^t\partial/\partial\bar{u}_2 & 0 \end{pmatrix},$$

where $\partial/\partial\zeta$ is the matrix with entries $\partial/\partial\zeta_{ij}$, etc. We define the differential operator $\nabla_{I,J}$ to be the $(k+1)\times(k+1)$ minor of ∇ which includes the $r+1$ rows $\{i_1,\dots,i_{r+1}\} \subset \{1,\dots,p\}$ of $\partial/\partial\zeta$ and all s rows of ${}^t\partial/\partial\bar{u}_2$, and includes the $s+1$ columns $\{j_1,\dots,j_{s+1}\} \subset \{1,\dots,q\}$ of $\partial/\partial\zeta$ and all r columns of $\partial/\partial u_1$. We see that $\nabla_{I,J}$ is linear in $\partial/\partial\zeta$.

THEOREM 4.2 **[M1]**. *Let $\lambda \in \Lambda \subset \mathbb{Z}^k$ be given, with $k = r+s$, and let our other notations be as above. Assume that $G = \mathrm{U}(p,q)$ satisfies $p \geq r+1$ and $q \geq s+1$. Then*

$$\nabla_{I,J}(\Psi\omega) \equiv 0$$

for any I, J and any $\omega \in \mathcal{H}(\lambda)$.

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