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THE PENNEY–FUJIWARA PLANCHEREL FORMULA FOR HOMOGENEOUS SPACES WITH MONOMIAL SPECTRUM

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The Penney-Fujiwara Plancherel Formula [PFPF] is a distribution-theoretic version of the direct integral decomposition of the quasi-regular representation $\tau = Ind_H^G 1$. Here G is a connected Lie group and H is a closed subgroup. Aside from its intrinsic interest, such a formula has the advantage over an abstract representation-theoretic decomposition in that it yields the explicit intertwining operator that effects the decomposition. The main feature of [**3**] and this short note is the presentation of such a formula in several general situations in which the generic representations that occur in the decomposition are monomial.

The main results of [**3**] can be summarized as follows. We fix right Haar measures dg, dh on G, H and a smooth function $q = q_{H,G}$ on G satisfying $q_{H,G}(hg) = \frac{\Delta_H(h)}{\Delta_G(h)} q_{H,G}(g)$, where Δ denotes modular functions. These choices uniquely determine a quasi-invariant measure $d\dot{g}$ on $H\backslash G$ satisfying

$$\int_G f(g)q(g)dg = \int_{H\backslash G}\int_H f(hg)dhd\dot{g}$$

Suppose it is true generically that, for each irreducible representation π that occurs in τ, there is a closed subgroup B of G and a character χ of B for which $\pi = \pi_\chi = Ind_B^G\chi$. We realize the representation π_χ in the space $\mathcal{H}_\chi = \{f : G \to \mathbb{C}, f(bg) = \chi(b)f(g), \int_{B\backslash G} |f|^2 d\dot{g} < \infty\}$, where $d\dot{g}$ is the quasi-invariant measure on $B\backslash G$ now, and G acts by $\pi_\chi(g)f(x) = f(xg)[q_{B,G}(xg)/q_{B,G}(x)]^{1/2}$. We further assume that BH is closed and $q_{H\cap B,H}q_{H\cap B,B} \equiv 1$ on $H \cap B$. Under these conditions we can define a (conjugate-linear)distribution

$$\beta_\chi : f \to \int_{H\cap B\backslash H} \bar{f} q_{B,G}^{1/2} q_{H\cap B,H}^{-1} q_{H,G}^{-1/2} d\dot{h}, f \in C_c^\infty(G,B,\chi).$$

Then the following facts are true.

(i) β_χ is relatively invariant under the action of H with modulus $q_{H,G}^{-1/2}$.

(ii) $\pi_\chi(\omega)\beta_\chi \in C^\infty(G,B,\chi)$ for any test function ω on G; in fact

$$\pi_\chi(\omega)\beta_\chi(g) = \int_{H\cap B\backslash B} \omega_H(bg)\bar{\chi}(b) q_{B,G}^{-1/2}(bg) q_{H,G}^{1/2}(bg) q_{H\cap B,B}^{-1}(b) d\dot{b},$$

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where

$$\omega_H(g) = \Delta_G(g)^{-1} q_{H,G}^{-1/2}(g) \int_H \omega(g^{-1}h^{-1})\Delta_G(h)^{-1} q_{H,G}^{-1/2}(h)dh.$$

(iii)The matrix coefficient $< \pi_\chi(\omega)\beta_\chi, \beta_\chi >$of β_χ is well-defined and equals

$$(1)\quad \int_{H\cap B\backslash H} \int_{H\cap B\backslash B} \omega_H(bh)\bar{\chi}(b) q_{B,G}^{-1/2}(b) q_{H,G}^{1/2}(h^{-1}bh) q_{H\cap B,B}^{-1}(b) q_{H\cap B,H}^{-1}(h) d\dot{b} d\dot{h}.$$

I wish to emphasize that these results require NO structural assumptions on G other than those already stipulated. Now suppose we have a nice parameter space for the characters χ, say Λ. The PFPF amounts to the elucidation of a measure $d\chi$ on Λ for which one can prove the integral formula

$$(2)\qquad \int_\Lambda < \pi_\chi(\omega)\beta_\chi, \beta_\chi > d\chi = < \tau(\omega)\alpha_\tau, \alpha_\tau > .$$

Here α_τ is the canonical (relatively invariant) distribution on $\mathcal{H}_\tau^\infty$, and the right side of (2) equals $\omega_H(e)$ (see [**3**]). It is possible to deduce from formula (2) (see [**3**]) that the intertwining operator for the direct integral decomposition

$$\tau = \int_\Lambda^\oplus \pi_\chi d\chi$$

is

$$\Omega(g) \to \int_{H\cap B\backslash B} \Omega(bg)\bar{\chi}(b) q_{B,G}^{-1/2}(bg) q_{H,G}^{1/2}(bg) q_{H\cap B,B}^{-1}(b) d\dot{b}$$

where Ω is any H-invariant test function on G.

When can we actually carry out this program? That is, for which homogeneous spaces G/H whose spectrum is monomial can we actually progress from the computation of the matrix coefficients (1) to the PFPF (2)? [**3**] contains two general situations in which this is accomplished: (a) G simply connnected completely solvable, H connected; and (b) G/H an abelian symmetric space, i.e. G is a semidirect product $G = HV$ where V is a closed normal vector subgroup and H is arbitrary (other than $\hat{V}/H$ is countably separated). In this note I would like to announce several other general situations where we can derive the PFPF from the matrix coefficients. I will list three.

(1) G exponential solvable. The work here generalizes (a) above precisely as Fujiwara's theory in [**1**] subsumes that of [**2**].

(2) $G = RV$, where R is connected reductive, V is a normal vector subgroup and $H = SU$ is of the same form with $S \subset R, U \subset V$. Furthermore, suppose the generic stabilizer in R of elements from $U^\perp$ is trivial. In this case the parameter space $\Lambda = U^\perp/S$, and there is a common polarization for all $\chi \in \Lambda$, namely $B = V$. Moreover, there is a natural realization of the Plancherel measure as a pseudo-image of Lebesgue measure on $U^\perp$, depending on the choices of the q functions.

(3) Start with $G = SL(2, \mathbb{R}).\mathbb{R}^2$. G may be realized as the group of real matrices of the form

$$G = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, ad - bc = 1.$$

Conjugation of G by the diagonal matrix with entries (1,-1,1) is an involution. The stability subgroup is

$$H = \begin{pmatrix} a & 0 & x \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

G/H is a symmetric space and $\tau = \int^{\oplus} \pi_t dt$ where $\pi_t = Ind_B^G \chi_t$,

$$B = \begin{pmatrix} 1 & b & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and $\chi_t(b, x, y) = e^{ibt}e^{iy}$. In this case Lebesgue measure dt is the Plancherel measure and the derivation of the PFPF involves a Fourier inversion argument.

REMARKS. (i) Number (3) is representative of a class of semidirect products (of semisimple by vector groups) for which I can derive the PFPF. The key feature of those (as well as the spaces in number 2) is not only that the generic representations are monomial, but that there is a common polarization which is normalized by H.

(ii) A very interesting situation in which one has a common real polarization not normalized by H is that of a Riemannian symmetric space G/K (G a semisimple Lie group, K a maximal compact subgroup). The common polarization is a minimal parabolic subgroup. It is possible that one might be able to simplify the usual presentation of the spherical Plancherel measure by the techniques described here.

(iii) I have also considered situations in which there exist common positive (non-real) totally complex polarizations—especially various oscillator groups. I will discuss these at the conference as well as in another manuscript.

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