ユニタリ表現論セミナー報告集 X, 1990
pp.50-52

Spherical hyperfunctions on the tangent space of symmetric spaces

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Let G be connected semisimple Lie group, σ an involutive automorphism of G and H an open subgroup of fixed points of σ. Then G/H is called a semisimple symmetric space and the tangent space at the origin of G/H is identified with $\mathfrak{q}$, where $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ is the direct decomposition of the Lie algebra $\mathfrak{g}$ corresponding to G by σ and $\mathfrak{h}$ is the Lie algebra corresponding to H.

We consider spherical hyperfunctions on $\mathfrak{q}$ that are H-invariant and simultaneously eigen hyperfunctions on $\mathfrak{q}$. There have appeared several papers dealing with spherical functions on $\mathfrak{q}$ ([1], [2], [3], [5], [9], [10]). In his paper [2], van Dijk listed up spherical distributions for the rank 1 case. On the other hand, in his paper [1], Cerezo determined the dimension of $O(p,q)$ (or $SO_0(p,q)$) invariant spherical hyperfunctions on $\mathbb{R}^{p+q}$, where $\mathbb{R}^{p+q}$ can be regarded as the tangent space of the semisimple symmetric space ; $SO_0(p+1,q)/SO_0(p,q)$. However, studying spherical hyperfunctions, the author found interesting phenomenon. That is ; if f is an H-invariant eigen hyperfunction then f is $\tilde{H}$-invariant, where $\tilde{H}$ is the connected component of the Lie group of all non-singular transformations T on $\mathfrak{q}$ such that $p(Tx) = p(x)$ for any H-invariant polynomial p and $x \in \mathfrak{q}$. In fact, $\tilde{H}$ is "large" (if $G = SL(m+1,\mathbb{R})$ and $H = GL^+(m,\mathbb{R})$, then $\dim H = m^2$ and $\dim \tilde{H} = 2m^2-m$). It seems that this phenomenon is independent of the category of functions but is dependent on H or $\tilde{H}$ orbits structure on $\mathfrak{q}$. In his paper [8], Ochiai deals with this problem as

$\mathcal{D}$-module structure generated by the Lie algebra $\mathfrak{h}$ or $\tilde{\mathfrak{h}}$ which is the Lie algebra corresponding to $\tilde{H}$.

Recently, I proved that for "generic" eigen values if f is an H-invariant eigen hyperfunction then f is $\tilde{H}$-invariant. That is ;

Theorem 1 *If* $\lambda \in \tilde{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}$, *then*

$$\mathcal{B}_{\lambda}^{H}(\mathfrak{q}) = \mathcal{B}_{\lambda}^{\tilde{H}}(\mathfrak{q}).$$

Here $\tilde{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}$ is the set of all elements $X \in \mathfrak{q}_{\mathbb{C}}$ such that $\Delta_{N-L}(X) \neq 0$, where $N = \dim \mathfrak{g}_{\mathbb{C}}$ and L is the dimension of a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and $\det(t-\mathrm{ad}X) = t^{N}+\Delta_{1}(X)t^{N-1}+\cdots+\Delta_{N}(X)$.

Thanks to Cerezo's result and Theorem 1, we can determine the dimension of spherical hyperfunctions on $\mathfrak{q}$ when rank $\mathfrak{q} = 1$ and eigen value $\mu \neq 0$. That is ;

Theorem 2. *When rank* $\mathfrak{q} = 1$, *if* $\mu \neq 0$, *then*

(1) $p = q = 1$ *case*, $\qquad \dim \mathcal{B}_{\mu}^{H}(\mathfrak{q}) = 4$,

(2) $p = 1$ *or* $q = 1$ *case*, (*except for case* (1)) $\qquad \dim \mathcal{B}_{\mu}^{H}(\mathfrak{q}) = 3$,

(3) $p > 1$ *and* $q > 1$ *case*, $\qquad \dim \mathcal{B}_{\mu}^{H}(\mathfrak{q}) = 2$,

where $p = \dim (\mathfrak{q} \cap \mathfrak{k})$ *and* $q = \dim (\mathfrak{q} \cap \mathfrak{p})$ and $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ *is a Cartan decomposition.*

Remark. In [2], Van Dijk listed up the dimension of invariant eigen distributions. Since $\mathcal{D}'_{\lambda,H}(\mathfrak{q}) \subset \mathcal{B}_{\lambda}^{H}(\mathfrak{q})$ (see [2] for the definition of $\mathcal{D}'_{\lambda,H}(\mathfrak{q})$), it is clear that $\dim \mathcal{D}'_{\lambda,H}(\mathfrak{q}) \leq \dim \mathcal{B}_{\lambda}^{H}(\mathfrak{q})$. But from Theorem 5.3 and [2], if $\lambda \neq 0$, then we have $\mathcal{D}'_{\lambda,H}(\mathfrak{q}) = \mathcal{B}_{\lambda}^{H}(\mathfrak{q})$.

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