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Relative invariants of the polynomial rings over the finite and tame type quivers

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In this note we consider the following problem. Let F be one of the A_r,D_r, E_r, $\tilde{A}_r$, $\tilde{D}_r$, $\tilde{E}_r$ type quivers with r vertices and arbitrary directed arrows. For example

$$\text{(F)} \quad \begin{tikzcd} \overset{V_1}{\cdot} \arrow[r, "f_1"] & \overset{V_2}{\cdot} \arrow[r, "f_2"] & \cdot & \cdot \arrow[l] & \cdot \arrow[l] \arrow[r] & \cdots \arrow[r, "f_{r-3}"] & \overset{V_{r-2}}{\cdot} \arrow[r, "f_{r-1}"] & \overset{V_{r-1}}{\cdot} \\ & & & & & & \underset{V_r}{\cdot} \arrow[u, "f_r"'] & \end{tikzcd}$$

,where V_i is a finite dimensional vector space over some field k and if $\overset{V_i}{\cdot} \xrightarrow{f} \overset{V_j}{\cdot}$, f is a linear endomorphism from V_i to V_j.

Let $V = \bigoplus_{i \to j \ in \ F} \mathrm{Hom}(V_i, V_j)$ and let $G = GL(V_1) \times GL(V_2) \times \cdots \times GL(V_r)$. Then G acts on V naturally, i.e., for $g = (g_1, g_2, \cdots, g_r) \in G$ the action of G on V is given by $g \cdot f = g_j f g_i^{-1}$, if $\overset{V_i}{\cdot} \xrightarrow{f} \overset{V_j}{\cdot}$.

Let $S(V)$ be the polynomial ring over V and the action of G on V naturally extends to the action on $S(V)$. The first problem is as follows.

PROBLEM. *What is the relative (or absolute) invariants in $S(V)$ with respect to this action?*

Also we consider the same problems for A_r, D_r , E_r, $\tilde{A}_r$, $\tilde{D}_r$, $\tilde{E}_r$ type quivers with arbitrary directed arrows.

We give answers to the above problems for the case $k = \mathbb{C}$ (complex number) . (The same holds for any field k of characteristic 0.)

For example, let F be an A_r type quivers whose arrows are directed one way,

$$\overset{V_1}{\cdot} \xrightarrow{f_1} \overset{V_2}{\cdot} \xrightarrow{f_2} \overset{V_3}{\cdot} \xrightarrow{f_3} \cdots \xrightarrow{f_{r-1}} \overset{V_r}{\cdot}.$$

Then our theorem is given as follows:

We fix a base of each vector space V_i. Let us denote the dimension of V_i by n_i $(i = 1, 2, \cdots, r)$. Then $S(V)$ can be considered as the polynomial ring in the indeterminates $\{x_{i,j}^{(s)}\}$ where $1 \leqq i \leqq n_{s+1}$, $1 \leqq j \leqq n_s$, and $s = 1, 2, \cdots r-1$. To be precise, if we substitute some values to $x_{i,j}^{(s)}$'s, then the matrix $(x_{i,j}^{(s)})_{i,j}$ corresponds to the homomorphism f_s with respect to the above basis.

Let $M_{s+1,s}$ be the matrix $(x_{i,j}^{(s)})_{i,j}$. ($n_{s+1} \times n_s$ matrix whose (i,j)-th coefficient is the indeterminate $x_{i,j}^{(s)}$.)

DEFINITION. *For any k, ℓ with $1 \leqq k \leqq \ell \leqq r$ and $n_k = n_\ell$, we define the polynomial $P_{\ell,k}$ by*

$$P_{\ell,k} := \det(M_{\ell,\ell-1} M_{\ell-1,\ell-2} \cdots M_{k+1,k})$$

and call these polynomials by determinantal invariants.

Clearly $P_{\ell,k}$ is a relative invariant and $P_{\ell,k} \neq 0$ if and only if for any v $(k < v < \ell)$, $n_v \geqq n_k = n_\ell$. Moreover if a pair (k, ℓ) satisfies the condition that $n_v > n_k = n_\ell$ for any v ($k < v < \ell$), then we call the determinantal invariant $P_{\ell,k}$ *primitive*. Clearly any determinantal invariant can be writen as the product of the primitive ones.

THEOREM 1. *Let F be an A_r type quiver with one-way directed arrows. Then the relative invariants in $S(V)$ amount to be the monomials of the primitive determinantal invariants $P_{\ell,k}$'s. Moreover the primitive determinantal invariants are algebraic independent.*

Next let F be an $\tilde{A}_r$ type quivers whose arrows are directed one way

$$(F) \qquad \begin{tikzcd}
V_1 \arrow[r, "f_1"] & V_2 \arrow[r, "f_2"] & V_3 \arrow[r, "f_3"] & \cdots \arrow[r, "f_{i-1}"] & V_i \arrow[d, "f_i"] \\
V_r \arrow[u, "f_r"] & \cdot \arrow[l, "f_{r-1}"'] & \cdot \arrow[l] & \cdots \arrow[l] & V_{i+1} \arrow[l, "f_{i+1}"']
\end{tikzcd}$$

$S(V)$ can also be considered as the polynomial ring in the indeterminates $\{x_{i,j}^{(s)}\}$ where $1 \leqq i \leqq n_{s+1}$, $1 \leqq j \leqq n_s$, and $s = 1, 2, \cdots r$. We define the determinantal invariants and the primitive determinantal invariants just in the same way as the above. (Here we consider $V_{r+1} = V_1$ and k, ℓ run through 1 to r+1.) Since $\tilde{A}_r$ type quiver has the symmetry under the cyclic permutations, We may assume that $n_1 = \text{Minimum}\{n_1, n_2, \cdots, n_r\}$. Then we will define absolute invariants $\phi_i \in S(V)$ $(i = 1, 2, \cdots, n_1)$ as follows.

DEFINITION. *Let $\phi_i \in S(V)$ $(i = 1, 2, \cdots, n_1)$ be the i-th elementary symmetric function of the product of matrices* $M_{1,r}M_{r,r-1}M_{r-1,r-2}\cdots M_{2,1}$, *namely*

$$\det(tI_{n_1} - M_{1,r}M_{r,r-1}\cdots M_{2,1}) = \sum_{k=0}^{n_1} \phi_i(-1)^i t^{n_1-i}.$$

It is easy to see that ϕ_i's are absolute invariants. For a relative invariant $f \in S(V)$, we call that f has *weight* $\mathfrak{k} = (k_1, k_2, \cdots, k_r) \in \mathbb{Z}^r$ if $g \cdot f = (\det g_1)^{k_1}(\det g_2)^{k_2}\cdots(\det g_r)^{k_r} f$,where $g = (g_1, g_2, \cdots, g_r) \in G = GL(n_1) \times GL(n_2) \times \cdots GL(n_r)$.

By $S(V)^{\mathfrak{k}}$, we denote the relative invariants of weight $\mathfrak{k}$ in $S(V)$. Here we can state our theorem for this case.

THEOREM 2. *Let F be an $\tilde{A}_r$ type quiver with one-way directed arrows.*
(1) The absolute invariants $S(V)^G$ is the polynomial ring of n_1 generators $\phi_1, \phi_2, \cdots, \phi_{n_1}$, namely,

$$S(V)^G = \mathbb{C}[\phi_1, \phi_2, \cdots, \phi_{n_1}].$$

(2) The relative invariants in $S(V)$ amount to be the monomials of $\phi_1, \phi_2, \cdots, \phi_{n_1-1}$ and $P_{j\,i}$'s, where $P_{j,i}$'s are the primitive determinantal invariants. $\phi_1, \phi_2, \cdots, \phi_{n_1-1}$ and $P_{j\,i}$'s are algebraic independent.
(3) As $S(V)^G$ module, $S(V)^{\mathfrak{k}}$ is a free module of rank one.

These are examples of our answers to the problem. For the other cases, we also give explicit generators of the relative invariants in $S(V)$ and prove that they are algebraic independent.

The proof of the above facts needs some combinatorics to calculate the Littlewood-Richardson coefficients explicitly for Young diagrams of the special shapes.

We also state extentions of the problem. Theorem 1 comes up naturally in the following situation.

Let P be a parabolic subgroup of GL(n) (where $n = \sum_{i=1}^r n_i$) defined by

$$P = \begin{array}{cc} \begin{array}{cccc} n_r & \cdots & n_2 & n_1 \end{array} & \\ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} & \begin{array}{c} n_r \\ \vdots \\ n_2 \\ n_1 \end{array} \end{array}$$

Let $P = LU$ be a Levi decomposition of P, where L is a reductive part of P and U is the unipotent radical of P. For example

$$L = \begin{array}{cc} \begin{array}{cccc} n_r & \cdots & n_2 & n_1 \end{array} & \\ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} & \begin{array}{c} n_r \\ \vdots \\ n_2 \\ n_1 \end{array} \end{array}$$

Let $\mathfrak{N}$ be the Lie algebra corresponding to U. Then L acts on $\mathfrak{N}$ by adjoint action, hence L acts on $\mathfrak{N}/[\mathfrak{N}\,\mathfrak{N}]$ by adjoint action This action just coincides with the acion of G on V in Theorem 1. So we can extend the problem as follows.

PROBLEM 1. *Let G be a semisimple Lie group and let P be a parabolic subgroup of G. Let $P = LU$ is a Levi decomposition of P and let $\mathfrak{N}$ be the Lie algebra corresponding to U. What is the relative invariants under the adjoint action of L on $V = \mathfrak{N}/[\mathfrak{N}\,\mathfrak{N}]$?*

It is known that the above action of L on V is prehomogenius.

PROBLEM $1'$. *Consider the problem and the problem 1 over any field k instead of the complex field (or the field of characteristic 0).*

For example, let F be an A_2 type quiver and k be a finite field

(F) $$\overset{V_1}{\bullet} \xrightarrow{f_1} \overset{V_2}{\bullet}$$

If $\dim V_1 = 1$, i.e., $V_1 = k$, then $S(V)$ is isomorphic to $S(V_2)$ and G_2 naturally acts on $S(V_2)$. It is known in this case that the absolute invariants $S(V_2)^{G_2}$ are the polynomial ring in the Dickson's invariants $I_1, I_2, \cdots, I_{n_2}$. Compared with the characteristic 0 case, (See Theorem 1) things seem to be slightly changed over a finte field,

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