ユニタリ表現論セミナー報告集 X, 1990
pp.41-45

Discontinuous group in a homogeneous space of reductive type

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Let H be a closed subgroup which is reductive in a real reductive linear group G. The subject of this abstract is about how large a discrete group can act properly discontinuously on a homogeneous space G/H.

If H is compact, it is a famous result due to Borel and Harish-Chandra that G/H admits a uniform lattice (resp. nonuniform lattice), i.e. there is a discrete subgroup Γ of G acting properly discontinuously and freely on G/H so that $\Gamma\backslash G/H$ is compact (resp. noncompact and of finite volume). This theorem has been a foundation of abundant theory such as Eisenstein series in harmonic analysis on $L^2(\Gamma\backslash G/H)$.

If H is not compact, however, the action of a discrete group Γ of G on G/H is *not* automatically properly discontinuous and the double coset $\Gamma\backslash G/H$ may be non-Hausdorff. In fact, it sometimes occurs that only finite subgroups of G can act properly discontinuously on G/H. This is called a *Calabi-Markus phenomenon*, since the first observation was made in **[C-M]** for $G/H = SO(n+1,1)/SO(n,1)$. On the other hand, there does exist a homogeneous space with a large discontinuous group such as a group manifold $G \times G/\operatorname{diag} G$.

In order to state our main results, let us fix notations. Let G be a real linear reductive Lie group, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of its Lie algebra. Take a maximally abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. $\mathfrak{a}$ is called a maximally split abelian subspace for G. We write $W(\mathfrak{g},\mathfrak{a})$ for the Weyl group associated to the restricted root system of $\Sigma(\mathfrak{g},\mathfrak{a})$, $\mathbf{R}\text{-rank}\, G := \dim \mathfrak{a}$ ($\leq \operatorname{rank} G \geq$) $c\text{-rank}\, G := \operatorname{rank} K$, and $\mathrm{d}(G) := \dim G/K = \dim \mathfrak{p}$. Suppose that H is a closed subgroup in G. If there exists a Cartan involution of G which stables H, then H is called *reductive in* G and G/H is called *a homogeneous space of reductive type.* Let $\mathfrak{a}_H$ be a maximally split abelian subspace for H. Then there exists an element g of G such that $\mathrm{Ad}(g)\mathfrak{a}_H \subset \mathfrak{a}$. Put $\mathfrak{a}(H) := \mathrm{Ad}(g)\mathfrak{a}_H$, which is uniquely defined up to conjugacy of $W(\mathfrak{g},\mathfrak{a})$. By *a discontinuous group* Γ *in* G/H, we mean that Γ is a discrete subgroup of G acting properly discontinuously on G/H.

Our first step is to understand how properly discontinuity breaks through the study of an analogous problem of the action of connected groups. Here is a criterion of the properness.

THEOREM A. Let G/H, G/L be homogeneous spaces of reductive type. Then the following four conditions are equivalent:

1) L acts on G/H properly.

1)′ H acts on G/L properly.

2) For any $g \in G$, $L \cap gHg^{-1}$ is compact.

2)′ $W(\mathfrak{g},\mathfrak{a}) \cdot \mathfrak{a}(L) \cap \mathfrak{a}(H) = \{0\}$.

Applying Theorem A to $\mathbf{Z} \subset \mathbf{R}$ actions, we can eventually tell precisely when Calabi-Markus phenomenon occurs for a homogeneous space of reductive type.

COROLLARY 1, [**C-M**; **Wo1**; **Wo2**; **Ku**; **Ko1**]. *Let G/H be a homogeneous space of reductive type. The followings are equivalent:*

1) *Any discontinuous group in G/H is finite.*

2) $\mathbf{R}\text{-rank}\, G = \mathbf{R}\text{-rank}\, H$.

This result should be in sharp contrast with a solvable group case:

PROPOSITION. *Let G be a solvable group and H a proper closed subgroup of G. If $\pi_1(G/H) < \infty$, then there exists a discontinuous group isomorphic to $\mathbf{Z}$ in G/H.*

A second application of Theorem A is to describe some structure of a discontinuous group in a semisimple group manifold of $\mathbf{R}$-rank 1. The following Corollary is a generalization of the result due to R.Kulkarni and F.Raymond for $G = SL(2,\mathbf{R})$ case (see Theorem 5.2 and Introduction in [**Ku-R**]).

COROLLARY 2, [**Ko2**]. *Let G be a connected noncompact reductive linear Lie group. Then the following conditions are equivalent.*

1) $\mathbf{R}$-rank $G = 1$
2) *Any torsionless discontinuous group Γ in a group manifold $G \times G/\operatorname{diag} G$ is of the following form up to switch of factor: $\Gamma = \{(\gamma, \rho(\gamma)) : \gamma \in \Phi\}$ with a subgroup $\Phi \subset G$ and with a homomorphism $\rho \colon \Phi \to G$.*

A continuous analogue of a uniform lattice is that the double coset space $L\backslash G/H$ is compact where L, H are connected subgroups. Here is a criterion for the compactness.

THEOREM B. *Under the equivalent conditions in Theorem A, the followings are equivalent:*

1) *$L\backslash G/H$ is compact in the quotient topology.*
2) $\mathrm{d}(L) + \mathrm{d}(H) = \mathrm{d}(G)$.

Applying Theorems A, B to the action of a discrete group, we have

COROLLARY 3. *If $L \subset G \supset H$ satisfies the criteria Theorem A-(2)$'$ and Theorem B-(2), then both G/H and G/L admit a (non)uniform lattice.*

Notice that if H is compact, the assumption in Corollary 3 is satisfied with $L = G$. Another stupid example is $(G, L, H) = (G' \times G', G' \times \{e\}, \operatorname{diag} G')$. In these cases, Corollary 3 is nothing but a result in [**B**], which is our starting point. The next examples are remarkable:

EXAMPLE 1. The following homogeneous spaces admit both uniform and non-uniform lattices:

1) a) $U(2,2n)/Sp(1,n)$, b) $U(2,2n)/U(1) \times U(1,n)$,

2) a) $SO(2,2n)/U(1,n)$, b) $SO(2,2n)/SO(1,2n)$,

3) a) $SO(4,4n)/Sp(1,n)$, b) $SO(4,4n)/SO(3,4n)$,

4) a) $SO(4,3)/G_{2(2)}$, b) $SO(4,3)/SO(2) \times SO(4,1)$.

Among these examples, (2-b), (3-b) were previously known in [**Ku**].

Conversely, we find two necessary conditions for the existence of a uniform lattice:

THEOREM C. *Let G/H be a homogeneous space of reductive type. If G/H admits a uniform lattice, then the following conditions hold.*

1) $\operatorname{rank} G = \operatorname{rank} H \Longrightarrow c\text{-}\operatorname{rank} G = c\text{-}\operatorname{rank} H$

2) *There does not exist a closed subgroup G' reductive in G such that*

$$\mathfrak{a}(G') \subset W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}(H) \quad \textit{and} \quad \mathrm{d}(G') > \mathrm{d}(H).$$

The first condition in Theorem C is derived from a comparison between two compact homogeneous spaces, $K/H \cap K$ (a compact subspace) and G_U/H_U (a compact real form) by the Euler characteristic ([**Ko-O**], [**Ko1**]). The second one is from a comparison among subgroups reductive in G by the cohomological dimension of a discrete subgroup.

REMARK. One of the simplest application of Theorem C (2) is a comparison of G/H with one point G/G by taking $G' := G$, yielding that if $\mathbf{R}\text{-}\operatorname{rank} G = \mathbf{R}\text{-}\operatorname{rank} H$ and if G/H is not compact then G/H does not have a uniform lattice. Alternatively this follows directly from Corollary 1.

EXAMPLE 2. Let $G/H = SO(i+j, k+l)/SO(i,k) \times SO(j,l)$, where $i \leq j, k, l$. If G/H admits a uniform lattice, then G/H is compact, or H is compact, or $0 = i < l \leq j-k$. Moreover, $jkl \equiv 0 \mod 2$. (See also (2-b), (3-b), (4-b) in Example 1.)

The following two Corollaries are applications of Theorem C to a semisimple orbit, to a semisimple symmetric space, respectively. A table of some other homogeneous spaces without uniform lattice may be seen in [**Ko1**]-II.

COROLLARY 4. *Let G be a real reductive linear Lie group, and X a semisimple element of $\mathfrak{g}$. If $G \cdot X \simeq G/Z_G(X)$ admits a uniform lattice, then the orbit $G \cdot X$ carries a G-invariant complex structure. (See also* (1-b), (2-a), (4-b) *in Example 1).*

COROLLARY 5. *If an irreducible semisimple symmetric space G/H admits a uniform lattice, then the associated symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$ is basic in ϵ-family $F((\mathfrak{g}, \mathfrak{h}^a))$ in the sense of* [**O-S**].

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