ユニタリ表現論セミナー報告集 X, 1990
pp.37-40

A Method of Reduction to the Spherical Case in Harmonic Analysis on Real Rank 1 Semisimple Lie Groups

TAKESHI KAWAZOE

Department of Mathematics, Faculty of Science and Technology,
Keio University

Introduction. Let G be a connected semisimple Lie group with finite center. Then harmonic analysis on G is deeply related with the unitary dual $\hat{G}$ of G, in particular the Fourier transformation of $L^2(G)$ of square-integrable functions on G is defined by using the principal series and the discrete series of G. As in the case of the Eucledian space, the characterization of Fourier transforms of functions in $L^2(G)$ and $C_c^\infty(G)$ of C^∞ compactly supported functions on G is one of main problems in harmonic analysis on G, so the Plancherel formula and the Paley-Wiener theorem have been studied by various people. Let K be a maximal compact subgroup of G. Then, right K-invariant functions on G can be regarded as functions on the symmetric space $X = G/K$, so harmonic analysis on X is regarded as one for right K-invariant functions on G and the same problems have been investigated for X. In this case the Fourier transforms are consist of wave packets defined by using the principal series, so the discrete series does not contribute to harmonic analysis. This fact means that the Paley-Wiener theorem for $C_c^\infty(X)$ is simpler than the one for $C_c^\infty(G)$ and, if we apply the Paley-Wiener theorem for $C_c^\infty(G)$ to right K-invariant functions on G, we can deduce the one for $C_c^\infty(X)$. The problem we shall treat below is the reverse: is it possible to deduce the Paley-Wiener theorem for $C_c^\infty(G)$ from the one for $C_c^\infty(X)$?

Before we explain our aim more precisely we shall recall a brief history of the Paley-Wiener theorem on G. The difficulty in the proof of the theorem was in showing compactness of the support of functions whose Fourier transform are holomorphic of exponential type. Especially, if we apply the method of changing of contour of integration in the Fourier inversion formula, the main problems we had to conquare were (a) how to obtain a sharp estimate for the Harish-Chandra expansion which allows us to change the contour of integration, and (b) how to treat residues which appear during the contour change. When we treat right K-invariant functions on G, these residues don't appear, so (a) is the main problem and solved by Helgason [H] and Gangolli [G]. However, when we treat arbitrary K-finite functions on G, we encounter the residues during the contour change, so (b) is essential. This problem is solved by noting a relation between the residues and matrix coefficients of nonunitary principal series of G in which the discrete series is realized as a subrepresentation. For real rank 1 groups this was done by Campoli [C] and for arbitrary reductive Lie groups done by Arthur [A]. Another method of proving the surjectivity is completely different and is algebraic in nature. For complex semisimple Lie groups the method was used by Želobenko [Z], for any groups with one conjugacy classes of Cartan subgroups of G done by Delorme [D] and generalized for reductive Lie groups by Clozel and Delorme [CD].

Our aim is to offer a new method of proving the Paley-Wiener theorem for $C_c^\infty(G,\tau)$ of functions in $C_c^\infty(G)$ with right K-type τ, that is, we would like to reduce the problem for $C_c^\infty(G,\tau)$ to the one for $C_c^\infty(G,1) = C_c^\infty(X)$ for which the residues don't appear. As mentioned above, these residues are real obstacles in the proof of the surjectivity and come from singularities of the meromorphic functions $\Phi(\nu, a)$ and $c(\nu)^{-1}$, which appear in the Harish-Chandra expansion of generalized spherical functions on G (cf. [W], Chap. 9). Therefore, if possible, we would like to avoid using the Harish-Chandra expansion. Actually, in our reduction method the Szegö operators will play an important role and the Harish-Chandra expansion won't be used in the contour change.

Especially, the residues which appear during the contour change come from singularities of a simple modified Plancherel measure.

Notation. Let G be a connected real rank 1 semisimple Lie group with finite center and suppose that G has a compact Cartan subgroup of G. Let $G=ANK$ be an Iwasawa decomposition of G and for $g \in G$ let $g=e^{H(g)}n(g)\kappa(g) \in ANK$. Let M be the centralizer of A in K, $N_K(A)$ the normalizer of A in K and $W=N_K(A)/M$ the Weyl group for (G,A). Let $\mathfrak{g}=\mathfrak{a}+\mathfrak{n}+\mathfrak{k}$ be the corresponding decomposition of the Lie algebra $\mathfrak{g}$ of G and $\mathfrak{m}$ the Lie algebra of M. We denote the complexification of the Lie algebra $\mathfrak{b}$ by $\mathfrak{b}_c$ and the dual of $\mathfrak{b}$ by $\mathfrak{b}^*$. For simplicity, we put $\mathcal{F}=\mathfrak{a}^*$ and $\mathcal{F}_c=\mathfrak{a}_c^*$. Let $\mathfrak{t}_c$ and $\mathfrak{h}_c$ be the compact and noncompact Cartan subalgebras of $\mathfrak{g}_c$ respectively and $\mathfrak{h}_c^-$ the Cartan subalgebra of $\mathfrak{m}_c$ contained in $\mathfrak{h}_c$. We fix compartible orderings in the root spaces for $(\mathfrak{g}_c,\mathfrak{t}_c)$, $(\mathfrak{g}_c,\mathfrak{h}_c)$ and $(\mathfrak{g}_c,\mathfrak{a}_c)$. Let α be the positive noncompact simple root for $(\mathfrak{g}_c,\mathfrak{h}_c)$ and H_α the element in $\mathfrak{a}$ such that $\alpha(H_\alpha)=2$.

Let $(\tau,V_\tau) \in \hat{K}$ and $(\sigma,H_\sigma) \in \hat{M}$. We suppose that $\tau \in \hat{K}_\sigma = \{\zeta \in \hat{K}; [\zeta,\sigma] \neq 0\}$ and we regard H_σ as a subspace of V_τ. Then for $\nu \in \mathcal{F}_c$ the Szegö operator $S^\tau(\sigma,\nu) : C^\infty(K,\sigma) \to C^\infty(G,\tau)$ is defined by

$$S^\tau(\sigma,\nu)f(g) = \int_K e^{(\rho+i\nu)H(kx^{-1})}\tau(\kappa(kg^{-1}))^{-1}f(k)dk. \tag{1}$$

On the other hand, for $f \in C_c^\infty(G)$ the Fourier transform associated with the principal series $(\pi(\sigma,\nu),C^\infty(K,\sigma))$ is defined by

$$\hat{f}(\varphi;\sigma,\nu)(k) = \int_G f(g)\pi(\sigma,\nu;g^{-1})\varphi(k)dg \quad (k \in K), \tag{2}$$

where $\varphi \in C^\infty(K,\sigma)$.

Reduction formula. In order to carry out a reduction method we assume that σ and τ are embedded suitably in a finite dimensional representation π of G and satisfy the following conditions: Let $\lambda_\tau \in \mathfrak{t}_c^*$ be the highest weight of τ and $\lambda_\sigma \in \mathfrak{t}_c^*$ a weight of π that is highest in H_σ. For each weight γ we denote by φ_γ a corresponding normalized weight vector. Then, there exists a weight $-\mu \in \mathfrak{h}_c^*$ of π satisfying

(i) $-\mu$ is lowest with respect to $(\mathfrak{g}_c,\mathfrak{a}_c)$,

(ii) $\lambda_\sigma\,|_{\mathfrak{h}_c^-} = -\mu\,|_{\mathfrak{h}_c^-}$,

(iii) $P_\tau(\varphi_{-\mu}) = c\varphi_{\lambda_\sigma} \quad (c \neq 0)$,

where $P_\tau : U_\pi \to V_\tau$ is the orthogonal projection of U_π to V_τ. Under the assumption that

(A) (σ,H_σ) is the restriction of τ to the M-cyclic subspace generated by φ_{λ_τ}

we can find π and $-\mu$ satisfying the above conditions and deduce the following reduction formula, which reduces the harmonic analysis for $\hat{f}(\sigma,\nu)$ to the one for $\hat{f}(1,\nu)$.

REDUCTION FORMULA. *Let us suppose that σ and τ satisfy (A) and f is in $C_c^\infty(G)$. Then for each $i \in I_\tau$*

$$\begin{aligned}< S^\tau(\sigma,\nu+i\mu_\alpha)\hat{f}(\varphi_{\tau,i,1};\sigma,\nu+i\mu_\alpha)(g), v_{\tau,i} >&= \sum_{j,s\in I_\tau}\Pi_{ij}(g) \sum_{p\in I_\eta,\eta\in\hat{K}_{1,,\tau}} C_{\tau,i,s;\eta,p}P_\eta(\nu)^{-1}\\ &\times S^1(1,\nu)((-H_\alpha)^{n(\eta)}E^\eta_{p1}(\overline{\pi_{sj}}f))^\wedge(\psi_{1,1};1,\nu)(g) \quad (\nu \in \mathcal{F}_c, g \in G).\end{aligned}$$

where Π_{ij} and π_{sj} are matrix coefficients of π, $\mu_\alpha =< \mu, \alpha > \alpha / < \alpha, \alpha >$, $n(\eta)$ is an integer and $H_\alpha \in \mathfrak{a}$ is regarded as a right invariant differential operator on G.

Remark. When $G = SU(n,1)$, $K = S(U(n) \times U(1))$ is not semisimple and thus, it has one dimensional representations $\tau_\ell (\ell \in \mathbf{Z})$ defined by the ℓ-th power of the $(n+1, n+1)$ entry. Then, for τ_ℓ and $\sigma_\ell = \tau_\ell \mid_M$ we can find π and μ satisfying the desired properties stated above. This means that all arguments are also applicable to σ_ℓ and τ_ℓ.

Let $C^\infty_{c,\tau,\sigma}(G)$ denote the set of functions f in $C^\infty_c(G)$ whose principal part f^P are of the form

$$f^P(g) = \sum_{i \in I_\tau} \int_{\mathcal{F}} < S^\tau(\sigma,\nu)(\hat{f}(\varphi_{\tau,i};\sigma,\nu))(g), v_{\tau,i} > \mu(\sigma,\nu) d\nu. \tag{3}$$

Then, it is easy to see that

$$C^\infty_c(G) = \oplus_{\sigma \in \hat{M}} \oplus_{\tau \in \hat{K}_\sigma} C^\infty_{c,\tau,\sigma}(G).$$

When $\eta = 1$, the Paley-Wiener theorem for $C^\infty_{c,1,1}(G) = C^\infty_c(X)$ is given as follows (see [H] and [G]).

For $f \in C^\infty_{c,1,1}(G)$ we define the Fourier transform $\hat{f}_{1,1}$ as

$$\hat{f}_{1,1} = \hat{f}(\psi_{1,1}; 1, \nu)(k) \quad (\nu \in \mathcal{F}_c, k \in K). \tag{4}$$

On the other hand, we define $PW_{1,1}(\mathcal{F} \times K)$ as the set of all $\omega(\nu,k)$ in $L^2(\mathcal{F} \times K, \mu(1,\nu) d\nu dk)$ satisfying the following conditions:

(1) $\omega(\nu,k)$ is a holomorphic function of uniform exponential type.
(2) As a function of $\nu \in \mathcal{F}_c$

$$S^1(1,\nu)\omega(\nu,\cdot)(g) \quad (g \in G)$$

is W-invariant.

Then the Paley-Wiener theorem for $C^\infty_{c,1,1}(G)$ can be stated as follows.

THEOREM. *The map $f \mapsto \hat{f}_{1,1}$ is a bijection between $C^\infty_{c,1,1}(G) = C^\infty_c(X)$ and $PW_{\eta,1}(\mathcal{F} \times K)$.*

Main theorem. We now combine the reduction formula with the Paley-Wiener theorem for $C^\infty_{c,1,1}(G)$. Then, we can deduce the Paley-Wiener theorem for $C^\infty_{c,\tau,\sigma}(G)$ without using the Harish-Chandra expansion of the τ spherical functions on G. Our main theorem can be stated as follows. Let $\Omega = (\omega_i(\nu,k); i \in I_\tau)$ be a d_τ-tuple of H_σ-valued functions on $\mathcal{F}_c \times K$. Then for each $\tau' \in \hat{K}$ we define $PW_{\tau,\sigma}(\mathcal{F} \times K)_{\tau'}$ as the set of $\Omega = (\omega_i(\nu,k))$ satisfying

(1) Each $\omega_i(\nu,k)$ is a holomorphic function of uniform exponential type.
(2)

$$\sum_{i \in I_\tau} < S^\tau(\sigma,\nu)\omega_i(\nu,\cdot)(g), v_{\tau,i} >$$

is W-invariant.
(3) There exist functions $\omega_{i;i'}(\nu)$ $(i \in I_\tau, i' \in I_{\tau'})$ and $\omega_{s,j;\eta,p;\eta',p'}(\nu)$ $(s \in I_\pi, \eta \in \hat{K}_{1,\tau}, p \in I_\eta$, $\eta' \in \hat{K}_{1,\tau'}, p' \in I_{\eta'})$ satisfying the following conditions

(a)

$$\omega_i(\nu,k)=\sum_{i'\in I_{\tau'}}\omega_{i;i'}(\nu)\varphi_{\tau',i'}(k)\quad (k\in K).$$

(b) If we put

$$\omega_{s,j;\eta,p}(\nu,k)=\sum_{p'\in I_{\eta'},\eta'\in\hat{K}_{1,\tau'}}\omega_{s,j;\eta,p;\eta',p'}(\nu)\psi_{\eta',p'}(k)\quad (k\in K),$$

then,

$$\tilde{\omega}_i(\nu+i\mu_\alpha,k)_j=\sum_{s\in I_\pi,p\in I_\eta,\eta\in\hat{K}_{1,\tau}}C_{\tau,i,s;\eta,p}\omega_{s,j;\eta,p}(k)\quad (k\in K),$$

where for $f\in C^\infty(K,\sigma)$ $\tilde{f}(k)=\sum_{i\in I_\sigma}<f(k),v_{\tau,i}>u_{\pi,i}$ and $\tilde{f}_j(k)=<\pi(k)^{-1}\tilde{f}(k),u_{\tau,j}>$.
(c) For $s,j\in I_\pi, p\in I_\eta$ and $\eta\in\hat{K}_1$

$$P_\eta(\nu)\omega_{s,j;\eta,p}(\nu,k)\in PW_{1,1}(\mathcal{F}\times K).$$

(d) For $\lambda'\in\Lambda_2$

$$\begin{aligned}&D_\nu^{m(\lambda')}\omega_{s(\lambda'),j(\lambda');\eta(\lambda'),q(\lambda');\eta'(\lambda'),q'(\lambda')}(\nu)|_{\nu=\xi(\lambda')}\\&\quad=\sum_{\lambda\in\Lambda_{01}}A_{\lambda',\lambda}\omega_{i(\lambda);i'(\lambda)}(\zeta(\lambda))+\sum_{\lambda\in\Lambda_{02}}A_{\lambda',\lambda}D_\nu^{m(\lambda)}\omega_{s(\lambda),j(\lambda);\eta(\lambda),q(\lambda);\eta'(\lambda),q'(\lambda)}(\nu)|_{\nu=\xi(\lambda)}.\end{aligned}$$

On the other hand, for $f\in C^\infty_{c,\tau,\sigma}(G)_{\tau'}$ we define the Fourier transform $\hat{f}_{\tau,\sigma}$ as

$$\hat{f}_{\tau,\sigma}=(\hat{f}(\varphi_{\tau,i};\sigma,\nu)(k);i\in I_\tau)\quad(\nu\in\mathcal{F}_c,k\in K).\tag{5}$$

Then we can obtain

THE PALEY-WIENER THEOREM FOR $C^\infty_{c,\tau,\sigma}(G)_{\tau'}$. *Let $\sigma\in\hat{M}$ and $\tau\in\hat{K}_\sigma$ satisfy (A). Then for each $\tau'\in\hat{K}_\sigma$ the map $f\mapsto\hat{f}_{\tau,\sigma}$ is a bijection between $C^\infty_{c,\tau,\sigma}(G)_{\tau'}$ and $PW_{\tau,\sigma}(\mathcal{F}\times K)_{\tau'}$.*

REFERENCES

[A] Arthur, J.G., *A Paley-Wiener theorem for reductive groups*, Acta Math. **150** (1983), 1-89.

[C] Campoli, O.A., *Paley-Wiener type theorems for rank-1 semisimple Lie groups*, Unión Matemática Argentina **29** (1980), 197-221.

[CD] Clozel, L. and P. Delorme, *Le théorème de Paley-Wiener invarinat pour les groupes de Lie réductifs*, Invent. Math. **77** (1984), 247-453.

[D] Delorme, P, *Théorème de type Paley-Wiener pour les groupes de Lie semisimples réels avec une seule classe de conjugasion de sons groupes des Cartan*, J. Func. Anal. **47** (1982), 26-63.

[G] Gangolli, R., *On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups*, Ann. of Math. **93** (1971), 150-165.

[H] Helgason, S., *An analogue of the Paley-Wiener theorem for the Fourier transform on certain symmetric spaces*, Math. Ann. **165** (1966), 297-308.

[W] Warner, G., "Harmonic Analysis on Semi-Simple Lie Groups II," Springer-Verlag, New York, 1972.

[Z] Želobenko, D.P., "Harmonic Analysis on Complex Semisimple Lie Groups," (in Russian), WAUKA, Moscow, 1974.