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Note on Weyl-Harish-Chandra character formula for tamely ramified supercuspidal representations of GL_n

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Abstract. 1. Let F be a p-adic field and E be a tamely ramified extension of degree $n \geq 2$ over F. Let s and r denote the ramification index and the residue degree of E/F respectively. We denote by $\mathfrak{O}_E$ and $\mathfrak{P}_E$ the maximal order of E and the maximal ideal of $\mathfrak{O}_E$ respectively. Let GL_n be defined over F and set $G = GL_n(F)$. Let T be an elliptic torus of GL_n such that $T(F) \simeq E^\times$, where $E^\times$ denotes the multiplicative group of E. We may identify $T(F)$ with $E^\times$. We get the parahoric subalgebra A_s of $M_n(F)$ and the parahoric subgroup K_s of G associated with E/F. Moreover we have the filtrations A_s^m ($m \geq 0$) of $M_n(F)$ and $K_s^m = 1 + A_s^m$ ($m \geq 1$) of G (cf. [H], [M], [C] and [T]).

2. Let L be a Galois closure of E over F with Galois group Γ. Then Γ acts on T naturally and acts on the root system Φ of SL_n with respect to $T \cap SL_n$. For each $\alpha \in \Phi$, we denote by $\Gamma\alpha$ the Γ-orbit of α in Φ. Let $\Phi/\pm\Gamma$ denote the set of Ω

$= \Gamma\alpha \cup -\Gamma\alpha$, $\alpha \in \Phi$. We call Ω symmetric if $\Omega = \Gamma\alpha = -\Gamma\alpha$ and non-symmetric otherwise.

3. Let $N_{GL_n}(T)$ be the normalizer of T in GL_n and set $W(T) = N_{GL_n}(T)(F)/T(F)$ (called the small Weyl group). A quasi-character θ of $T(F) = E^\times$ is said to be regular if ${}^w\theta \neq \theta$ for any $w \neq 1$ in $W(T)$. We can show the following.

LEMMA. If a quasi-character θ is regular and its conductoral exponent f is ≥ 2, there is an unique element u_θ in $E^\times$ such that

$$\theta(1+x) = \psi(tr(u_\theta x)), \quad x \in \mathfrak{P}_E^{1-f}$$

and $E = F[u_\theta]$, where ψ denotes an additive character of F of conductoral exponent 1.

4. **PROPOSITION.** Let θ be a regular quasi-character of $T(F) = E^\times$ of conductoral exponent $N \geq 2$. Then, from θ and u_θ in Lemma, we can construct an irreducible representation of the subgroup $H = E^\times K_s^{m-1}$ (resp. $E^\times K_s^m$) of G with $N = 2m-1$ (resp. $2m$). Moreover the induced representation of it from H to

G is an irreducible supercuspidal representation of G.

See [H], [M], [C] or [T].

We will take the representations constructed by T. Takahashi [T]. We denote by κ_θ and π_θ the representations of H and G respectively.

T.Takahashi determined the character formula for κ_θ ([T, Proposition 2.6.7]).

5. **THEOREM** (Weyl-Harish-Chandra character formula). Notations being as above. Let θ be a regular unitary character of $E^\times$ of conductoral exponent $N \geq 2$ and moreover with certain additional condition(*) on the "semi-simple" conductors (see [G, Theorem 5]). If $t \in \mathfrak{O}_E^\times$ is regular i.e., $E = F[t]$, and satisfies $\mathrm{val}_L(t^\alpha - 1) \leq (N+2)/3$, $\alpha \in \Phi$, then

$$\mathrm{Tr}\ \pi_\theta(t) = (-1)^{(r-1)N} \sum_{w \in W(T)} \lambda(w,t)\theta({}^w t)/|D(t)|^{1/2}$$

where val_L denotes the extended valuation of L from the normalized valuation of F,

$$\lambda(w,t) = \prod_{\Phi/\pm\Gamma} \varepsilon_\Omega^{\ v_t(w,\Omega)},$$

$\varepsilon_\Omega = 1$ (resp. -1) if Ω is non-symmetric (resp. symmetric), $v_t(w,\Omega) = \mathrm{val}_L(({}^w t)^\alpha - 1)$, for some $\alpha \in \Phi$,

$$D(t) = \prod_{\alpha\in\Phi} (t^{\alpha/2} - t^{-\alpha/2}),$$

and $|x|$ denotes the module of x in L.

This theorem can be verified by the above result [T, Proposition 2.6.7] and by the method of the proofs of [G, Theorem 4 and Theorem 5].

REMARK. $\lambda(w,t)$ in the theorem may be simplified.

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