

The Pompeiu Problem for Groups

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The Pompeiu problem has its origins in classical analysis in $\mathbb{R}^n$ (see [2,3,4,5] for a discussion and some history). In this context it may be stated as follows. Let $D \subset \mathbb{R}^n$ be a bounded measurable set of positive Lebesgue measure and f a locally integrable function on $\mathbb{R}^n$. Then, if $\int_{\sigma(D)} f = 0$ for all rigid motions σ of $\mathbb{R}^n$, is $f = 0$?

In this generality the answer, even for $\mathbb{R}^2$, is no: D a disc and f the solution to a certain Neumann problem [2] provide a counterexample. The question of which regions in $\mathbb{R}^n$ or in symmetric spaces have this Pompeiu property (i.e. $\int_{\sigma(D)} f = 0$ implies $f = 0$) has thus received considerable attention not least because of its connections with other interesting problems (see the preceding references). Apart from the articles mentioned above useful insight into why this question is of importance is given in the survey [10].

This talk deals with the following variant of the problem which has links with harmonic analysis on locally compact groups and with their representation theory. For which locally compact groups G is it true that for every relatively compact Borel set $E \subset G$ of positive Haar measure and $f \in L^1(G)$,

$$\int_{xEy} f = 0 \text{ for all } x, y \in G \text{ implies } f = 0? \quad (*)$$

If this holds for G then we call G a *Pompeiu group*. Notice that we have restricted our class of allowed functions f to $L^1(G)$ so that it follows that in this sense $\mathbb{R}^n$ is a Pompeiu group [1]. If we were to allow other classes of functions than L^1 then the results are strikingly different.

This question has been first studied in some detail in [8] where partial results are obtained: principally, semisimple Lie groups are not Pompeiu in general while the Heisenberg group and the motion group of $\mathbb{R}^2$ where shown to be Pompeiu. Certain extensions have recently been obtained in [9]: semidirect products of vector groups with vector groups are Pompeiu, and the motion group of $\mathbb{R}^n$, $n \geq 4$, is not. It turns out that there are far reaching generalizations of these results (joint work with A. Carey and W. Moran [6]).

To start with, it is worthwhile to mention that the question of whether there always exist Pompeiu sets, that is, relatively compact Borel subsets E of G with property (*), is much easier. In fact, generalizing arguments of Rana

[7] for abelian groups. it can be shown that given any first countable locally compact group G , every compact subset of G of positive Haar measure contains a Pompeiu set [5].

Establishing the preceding fact as well as the results below requires a reformulation of the problem in terms of representation theory. Let G be an arbitrary locally compact group, and denote by $\hat{G}_r$ the reduced dual of G . Suppose E is a relatively compact Borel subset of G of positive Haar measure and χ_E the characteristic function of E . Then E is a Pompeiu set if and only if there exists a dense subset D of $\hat{G}_r$ such that $\pi(\chi_E) \neq 0$ for all $\pi \in D$.

Next, it is easy to see that G cannot be Pompeiu provided that it has a non-trivial compact normal subgroup. This fact leads to various assumptions on the groups which preclude the existence of compact normal subgroups. The main results proved in [5] are as follows:

- (i) A discrete group G is a Pompeiu group if and only if its finite conjugacy class subgroup is torsion free.
- (ii) Suppose G is a semidirect product $G = K \ltimes N$, where K is a compact Lie group and N is an abelian normal subgroup. Then G is a Pompeiu group if and only if N contains no non-trivial compact element and for all λ in some dense subset of $\hat{N}$, the stability subgroup of λ in K is trivial.
- (iii) Let N be a closed normal subgroup of G such that G/N is abelian and has no non-trivial compact elements. Then if N is Pompeiu, G is also Pompeiu.

It can be deduced from (iii) that every connected and simply connected solvable Lie group is a Pompeiu group, and also that an arbitrary nilpotent locally compact group is Pompeiu if and only if it contains no non-trivial compact elements.

The proofs exploit fairly extensive knowledge of the representation theory and structure of the groups involved.

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