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K-bi-finite and $Z(\mathfrak{g})$-finite Functions

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1. Introduction.

In this seminar report I would like to present some results and conjectures arisen from my joint work with David Vogan. Since I have not had a chance to show him this manuscript, I should take full resposibility for the inaccuacy occured in this report.

Let G be a linear semisimple Lie group with maximal compact subgroup K. Let $\mathfrak{g}$ be the complicification of the Lie algebra $\mathfrak{g}_0$ of G. Let $Z(\mathfrak{g})$ be the center of universal envelope algebra $U(\mathfrak{g})$. Write $\mathcal{E}(G)$ as the set of K-bi-finite and $Z(\mathfrak{g})$-finit functions on G. We attemp to study the function space $\mathcal{E}(G)$ from a representation theory point of view. We will decompose $\mathcal{E}(G)$ into direct sum of generalized joint eigenfunctions of $Z(\mathfrak{g})$ (cf. Theorem 2 in section 2) and study each summand.

2. Harish-chandra modules.

An element $X \in \mathfrak{g}$ defines a vector fields $D_L(X)$ and $D_R(X)$ on G by

$$(D_L(X)\phi)(g) = \frac{d}{dt}\phi(exp(-tX)g)|_t = 0,$$

$$(D_R(X)\phi)(g) = \frac{d}{dt}\phi(gexptX)|_t = 0$$

for any $\phi \in C^\infty(G)$. Then D_L and D_R extend to algebra homomorphisms of $U(\mathfrak{g})$ to the algebra of differential operators on G. Let $X, Y \in \mathfrak{g}$, we also define

$$D_B(X,Y)\phi = D_L(X)(D_R(Y)\phi) = D_R(Y)(D_L(X)\phi).$$

Let $f \in \mathcal{E}(G)$, we put $V_L(f) = D_L(U(\mathfrak{g}))f$, $V_R(f) = D_R(U(\mathfrak{g}))f$. Then $V_L(f)$ and $V_R(f)$ are admissible $(\mathfrak{g}, K)$-modules. We also put $V_B(f) = D_B(U(\mathfrak{g} \times \mathfrak{g}))f$, then $V_B(f)$

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is an admissible $(\mathfrak{g} \times \mathfrak{g}, K \times K)$-module. For any Harish-Chandra module V let V^* be its contragradient. V (resp. V^*) may be realized as the K-finite vectors in some admissible repersentation (π, V_π) (resp. (π^*, V_{π^*})). We define the matrix coefficient map

$$C : V \times V^* \to \mathcal{E}(G)$$

by $C_{v,v^*}(x) = \langle \pi(x)v, v^* \rangle$.

Theorem 2.1 *$f \in \mathcal{E}(G)$ is a matrix coefficient of $V_R(f)$. $V_L(f)$ is isomorphic to the contragradient of $V_R(f)$. $V_B(f)$ is isomorphic to $C(V_L(f) \otimes V_R(f))$, the image of matrix coefficient map C as $(\mathfrak{g} \times \mathfrak{g}, K \times K)$-modules. In particular, if $V_L(f)$ is irreducible, then $V_B(f) \cong V_L(f) \otimes V_R(f)$.*

Remark 2.2 The definition of the matrix coefficient map C depends on the globalization of V and V^*. But the matrix coefficients C_{v,v^*} of V are invariant of Harich-Chandra module.

3. A decomposition of $\mathcal{E}(G)$.

Let $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots such that the set of restrict roots $\Delta(\mathfrak{g}, \mathfrak{a})$ is obtained by restriction from $\mathfrak{h}$ to $\mathfrak{a}$. For each $\lambda \in \mathfrak{h}^*$, we define a character of $Z(\mathfrak{g})$

$$\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$$

to be the composition of Harish-Chandra homomorphism with evaluation at λ. Let W be the Weyl group. The definition of χ_λ depends only on the W-orbit of λ. Now we define

$$\mathcal{E}_\lambda^{gen}(G) = \{f \in \mathcal{E}(G) | (z - \chi_\lambda)^n f = 0, \text{for all } z \in Z(\mathfrak{g}) \text{ and some nonegative integer } n\};$$

$$\mathcal{E}_\lambda(G) = \{f \in \mathcal{E}(G) | (z - \chi_\lambda) f = 0, \text{for all } z \in Z(\mathfrak{g})\}.$$

Theorem 3.1 *We have the decomposition*

$$\mathcal{E}(G) = \bigoplus_{\lambda \in \mathfrak{h}^*/W} \mathcal{E}_\lambda^{gen}(G).$$

We say $\lambda \in \mathfrak{h}^*$ generic if

$$\langle \lambda, \alpha \rangle \notin \mathbf{Z}, \text{for any } \alpha \in \Delta(\mathfrak{g}, \mathfrak{a}).$$

Theorem 3.2 *For $\lambda \in \mathfrak{h}^*$ generic, $\mathcal{E}_\lambda(G)$ are function space generated by matrix coefficients of all principal series with infinitesimal character χ_λ. Assume that λ is in the positive chamber and generic, then*

$$\mathcal{E}_\lambda(G) \cong \bigoplus_{\substack{d\delta = \lambda|_{\mathfrak{t}^*} \\ \nu = \lambda|_{\mathfrak{a}^*}}} Ind(\delta \otimes \nu) \otimes Ind(\delta^* \otimes -\nu).$$

This is true because there is no essential extension of a Harish-Chandra module with a generic infinitesimal character.

Remark 3.3 If λ is not generic, $\mathcal{E}_\lambda(G)$ can be much larger than the matrix coefficients of all principal series with infinitesimal character χ_λ. For example, let $G = \mathrm{SL}(2, \mathbf{R})$ and $\lambda = \rho$ the half sum of all positive roots. We can construct a Harish-Chandra module V with infinitesimal character χ_λ and with composition series

$$0 = V_0 \subset V_1 \subset V_2 \subset V_3 = V$$

so that $V_3/V_2 \cong D^+$, $V_2/V_1 \cong \mathbf{C}$ and $V_1/V_0 \cong D^-$, where $\mathbf{C}$ is the 1-dimensional representation and D^+ and D^- are two discrete series. The matrix coefficients of V is contained in $\mathcal{E}_\lambda(G)$ but not is the space generated by the matrix coefficients of all principal series.

4. Some conjectures.

Let τ_1 and τ_2 be finit-dimensional representations of K on spaces U_1 and U_2 respectively, and let $\tau = (\tau_1, \tau_2)$. A function $F \in C^\infty(G, \mathrm{Hom}(U_1, U_2))$ is call τ-spherical if

$$F(k_1 x k_2) = \tau_1(k_1) F(x) \tau_2(k_2), \text{for all } k_1, k_2 \in K, x \in G.$$

By theorem 3.1, the dimension of all τ-spherical joint eigenfuctions of $Z(\mathfrak{g})$ with eigencharacter χ_λ is independent of λ if λ is generic.

Conjecture 4.1 If G is split, then the dimension of all τ-spherical eigenfunctions of $U(\mathfrak{g})$ with eigencharacter χ_λ is a constant indepent of λ.

Remark 4.2 It is not very hard to verify that Conjecture 4.1 is true for $G = SL(2, \mathbf{R})$. K-bi-finite joint eigenfuctions of $Z(\mathfrak{g})$ can be viewed as joint eigenfunctions of $G \times G$-invariant differential operators on symmetric spacs $G \times G/d(G)$. By use of Flensted-Jensen's duality principle and Helgason's conjecture, we know that $\mathcal{E}_\lambda(G)$ is isomorphic as $(\mathfrak{g} \times \mathfrak{g}, K \times K)$ to $K_{\mathbf{C}}$-finite distributions sections $\mathcal{D}_{\lambda, K_{\mathbf{C}}}(G_{\mathbf{C}}/B)$ on a line bundle over flag manifold $G_{\mathbf{C}}/B$, where $K_{\mathbf{C}}$ and $G_{\mathbf{C}}$ denote the complixifications of K and G respectively and B denotes a Borel subgroup of $G_{\mathbf{C}}$. For $G = \mathrm{SL}(2, \mathbf{R})$ the module $\mathcal{D}_{\lambda, K_{\mathbf{C}}}(G_{\mathbf{C}}/B)$ is computable.

For generic $\lambda \in \mathfrak{h}^*$, we see that $\mathcal{E}_\lambda(G)$ are linear combinations of matrix coefficients of all principal series with infinitesimal character χ_λ. It is well-known that matrix coefficients of principal series are closely relates to Eisenstein integrals (cf. [K] or [T]). All matrix coefficients of principal series can be recovered from Eisenstein integrals. Therefore $\mathcal{E}_\lambda(G)$ can be recovered from Eisenstein integrals for λ generic. However this conclusion is not true for non-generic λ (cf. Remark 3.3). On the other hand if G is split an Eisenstein integral is a vector-valued holomorphic function with respect to λ. It may vanish at a non-generic λ. If an Eisenstein integral has a zero of order k at a non-generic point λ_0, we define its k-th derivatives with respect to λ to be generalized Eisenstain integrals. We conjecture that for λ non-generic the generalized Eisenstein integrals play the same roles as Eisenstein integrals for λ generic, i.e.

Conjecure 4.3 If G is split and λ is not generic, then τ-spherical joint eigenfunctions of $Z(\mathfrak{g})$ of eigencharacter χ_λ are generalized Eisenstein integrals.

Remark 4.4 It is easy to verify that this conjecture is true for $G = \mathrm{SL}(2, \mathbf{R})$. The argument is same as we used in Remark 4.2.

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