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PROJECTIVE HOLOMORPHIC REPRESENTATIONS OF THE RESTRICTED LINEAR GROUP

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1.1. Flagvarieties

Let H be a separable complex Hilbert space, equiped with an orthogonal decomposition

$$H = H_1 \oplus \ldots \oplus H_m ,$$

with $k_i = \dim(H_i) > 0$. Each g in B(H), the space of bounded linear operators on H, decomposes accordingly as $g = (g_{ij})$ with $g_{ij} \in B(H_j, H_i)$. The orthogonal projections of H onto H_i is denoted by p_i. To this decomposition we associate the **standard flag** F_0:

$$\{0\} \subset H_1 \ \ldots \subset \bigoplus_{j=1}^{i} H_j \ \ldots \subset H = \bigoplus_{j=1}^{m} H_j .$$

The flags in H we will consider do not differ much, in a sense to be precised below, from the standard flag. Let $\mathcal{N}(H_i, H_j)$ denote the space of nuclear operators from H_i to H_j. In this paper the **restricted linear group** $G\ell(\mathcal{N},H)$ is defined by

$$G\ell(\mathcal{N},H) = \left\{ g = (g_{ij}) \in G\ell(H),\ g_{ij} \in \mathcal{N}(H_j, H_i) \text{ for all } i \neq j \right\}$$

Clearly all g_{jj}, $1 \le j \le m$, for $g \in G\ell(\mathcal{N},H)$ are Fredholm operators with index $i(g_{jj})$. The group $G\ell(\mathcal{N},H)$ can be given an appropriate Banach Lie group structure. If $\mathbf{k} = (k_1, \ldots, k_m)$, then we define the **flagvariety** $\mathcal{F}(\mathbf{k})$ as the collection of flags F:

$$\{0\} = W_0 \subset W_1 \ \ldots \subset W_m = H ,$$

with $W_i = g(\bigoplus_{j=1}^{i} H_j)$ for some g in $G\ell(\mathcal{N},H)$. The stabilizer of F_0 in $G\ell(\mathcal{N},H)$ is given by

$$P(\mathbf{k}) = \left\{ g = (g_{ij}) \in G\ell(\mathcal{N},H) \mid g_{ij} = 0 \text{ if } i > j \right\} .$$

The space $\mathcal{F}(\mathbf{k}) = G\ell(\mathcal{N},H)/P(\mathbf{k})$ is a Banach manifold modeled on the space

$$\bigoplus_{i>j} \mathcal{N}(H_j, H_i)$$

and the action of $G\ell(\mathcal{N},H)$ on $\mathcal{F}(\mathbf{k})$ is holomorphic.
The flagvarieties corresponding to $m = 3$, $k_1 = k_3 = \infty$ en $k_2 < \infty$, turn up in a natural way in the context of the modified KP-hierarchies, see [HP].
Let $\{i_1 <\ldots< i_{r+1}\}$ be the indices for which $\dim(H_j) = \infty$ and let $\ell = (\ell_s)$ in $\mathbb{Z}^{r+1}$ be such that $\sum_{s=1}^{r+1} \ell_s = 0$. The connected components of $G\ell(\mathcal{N},H)$ are given by

$$G\ell(\ell,\mathcal{N},H) = \left\{ \begin{array}{l} g = (g_{ij}) \in G\ell(\mathcal{N},H),\ i(g_{i_s i_s}) = \ell_s \\ \text{for all } s,\ 1 \le s \le r+1 \end{array} \right\} .$$

It is not difficult to show then that $G\ell(\mathcal{N},H)$ is the semi-direct product of $G\ell(0,\mathcal{N},H)$ and a group R isomorphic to $\mathbb{Z}^r$. Since $P(\mathbf{k})$ is connected, we see that the connected components of $\mathcal{F}(\mathbf{k})$ are given by

$$\left\{ g \cdot F_0 \mid g \in G\ell(\ell,\mathcal{N},H) \right\} = \mathcal{F}(\ell,\mathbf{k}) .$$

At the construction of the line bundles over $\mathcal{F}(\mathbf{k})$ we need a convenient description of the flags as a certain space of embeddings modulo a group of automorphisms. We give this description for the component $\mathcal{F}(0,\mathbf{k})$ containing the standard flag. Similar descriptions can be given for the other components. Let $\mathcal{P}$ be the collection of continuous embeddings $\omega: \bigoplus_{i=1}^{m-1} H_i \to H$ that decompose as (ω_{ij}) with $1 \le i \le m-1$, $1 \le j \le m$ and $\omega_{ij} \in B(H_j, H_i)$ satisfying $\omega_{ij} \in \mathcal{N}(H_j, H_i)$ if $i \ne j$ and $\omega_{ii} - Id \in \mathcal{N}(H_i, H_i)$. To each ω in $\mathcal{P}$ we associate a flag F_ω:

$$\{0\} \subset \omega(H_1) \subset \omega(H_1 \oplus H_2) \ \dots \subset \omega\Big(\bigoplus_{i=1}^{m-1} H_i\Big) \subset H$$

in $\mathcal{F}(0,\mathbf{k})$. The subgroup $\mathcal{T}$ of $\mathrm{Aut}\Big(\bigoplus_{i=1}^{m-1} H_i\Big)$ is defined by

$$\left\{ t \in \mathrm{Aut}\Big(\bigoplus_{i=1}^{m-1} H_i\Big),\ t = (t_{ij}) \text{ with } \begin{cases} t_{ij} = 0 \text{ if } i < j, \\ t_{ij} \in \mathcal{N}(H_j, H_i) \text{ if } j < i \\ t_{ii} - \mathrm{Id} \in \mathcal{N}(H_i, H_i) \end{cases} \right\}$$

and acts on $\mathcal{P}$ naturally by composition on the right. We have then

1.2. Lemma. The map $\omega \mapsto F_\omega$ induces a bijection between $\mathcal{P}/\mathcal{T}$ and $\mathcal{F}(0,\mathbf{k})$.

2. Line bundles over $\mathcal{F}(\mathbf{k})$

Note that the group $\mathcal{T}$ is chosen such that for each $\alpha = (\alpha_1, \dots, \alpha_{m-1})$ in $\mathbb{Z}^{m-1}$ we can define a holomorphic character χ_α of $\mathcal{T}$ by

$$\chi_\alpha(t) = \chi_\alpha((t_{ij})) = \prod_{i=1}^{m-1} \det(t_{ii})^{\alpha_i} .$$

Thanks to Lemma 1.2 we can associate to each χ_α a line bundle L_α over $\mathcal{F}(0,\mathbf{k})$ by taking the space $\mathcal{P} \times \mathbb{C}$ and dividing out the following action of $\mathcal{T}$:

$$(\omega, \lambda) \mapsto (\omega \circ t^{-1},\ \chi_\alpha(t)\lambda) .$$

The quotientspace is the holomorphic line bundle L_α over $\mathcal{F}(0,\mathbf{k})$. The class of (ω,λ) is denoted by $[\omega,\lambda]$. By using similar descriptions for the other components one can define a line bundle L_α over $\mathcal{F}(\mathbf{k})$. If one tries to lift the natural action of $G\ell(\mathcal{N},H)$ on the base space to one on L_α, one is led to the introduction of a central extension of $G\ell(\mathcal{N},H)$. Consider first the action of $G\ell(0,\mathcal{N},H)$ on $\mathcal{F}(0,\mathbf{k})$. Since for $\omega \in \mathcal{P}$ and $g \in G\ell(0,\mathcal{N},H)$ the embedding $g \circ \omega$ does not have to belong to $\mathcal{P}$, we will have "push" it back in the required shape. For each g in $G\ell(0,\mathcal{N},H)$ the operator $g_+ : \bigoplus_{i=1}^{m-1} H_i \to \bigoplus_{i=1}^{m-1} H_i$ decomposes as (g_{ij}) with

$1 \le i \le m-1$ and $1 \le j \le m-1$. Let Q be the subgroup of $G\ell(\overset{m-1}{\underset{i=1}{\oplus}} H_i)$ consisting of all q that decompose as (q_{ij}), $1 \le i \le m-1$, $1 \le j \le m-1$, with $q_{ij} = 0$ if $i > j$ and $q_{ij} \in \mathcal{N}(H_j, H_i)$ if $j > i$. Consider now

$$\mathcal{E} = \left\{ (g,q) \in G\ell(0,\mathcal{N},H) \times Q \;\middle|\; g_+ q^{-1} - Id \in \mathcal{N}(\overset{m-1}{\underset{i=1}{\oplus}} H_i, \overset{m-1}{\underset{i=1}{\oplus}} H_i) \right\}.$$

For every $g \in G\ell(0,\mathcal{N},H)$ there is a q in Q such that $(g,q) \in \mathcal{E}$. The group $P(\mathbf{k})$ embeds naturally into $\mathcal{E}$ by $p \mapsto (p,p_+)$. The group $\mathcal{T}$ embeds into $\mathcal{E}$ as a normal subgroup via $t \mapsto (Id,t)$, since it is the kernel of the natural projection $\mathcal{E} \to G\ell(0,\mathcal{N},H)$. In $\mathcal{T}$ we have the normal subgroup $\mathcal{T}_1$ of $\mathcal{E}$ given by $\{t \in \mathcal{T},\ \chi_\alpha(t) = 1\}$. The group $\mathcal{T}/\mathcal{T}_1$ is central in $\mathcal{E}/\mathcal{T}_1$ and we have obtained in this way a central extension $G\ell(0,\mathcal{N},H)^\sim$ of $G\ell(0,\mathcal{N},H)$ by $\mathcal{T}/\mathcal{T}_1 \cong \mathbb{C}^*$. The space $\mathcal{E}$ can be equiped with an appropriate manifold structure to ensure that $\mathcal{E}/\mathcal{T}_1$ is a Banach Lie group extension of $G\ell(0,\mathcal{N},H)$. The Lie group $\mathcal{E}/\mathcal{T}_1$ acts as follows on the line bundle L_α over $\mathcal{F}(0,\mathbf{k})$

$$[\omega,\lambda] \mapsto [g\circ\omega\circ q^{-1},\lambda].$$

This action is holomorphic and covers the one on $\mathcal{F}(0,\mathbf{k})$. Further we have

2.1. Lemma

(a) The natural inner action of the subgroup R of $G\ell(\mathcal{N},H)$ on $G\ell(0,\mathcal{N},H)$ can be lifted to $G\ell(0,\mathcal{N},H)^\sim$. This defines a central extension $G\ell(\mathcal{N},H)^\sim$ of $G\ell(\mathcal{N},H)$, that is the semi-direct product of R and $G\ell(0,\mathcal{N},H)^\sim$.

(b) The holomorphic action of $G\ell(0,\mathcal{N},H)^\sim$ on $(L_\alpha, \mathcal{F}(0,\mathbf{k}))$ can be extended to a holomorphic action of $G\ell(\mathcal{N},H)^\sim$ on $(L_\alpha, \mathcal{F}(\mathbf{k}))$.

A more fancy way to introduce the line bundle L_α over $\mathcal{F}(\mathbf{k})$ would be to consider the holomorphic character $\tilde{\chi}_\alpha$ of the subgroup $P(\mathbf{k})\cdot\mathcal{T}/\mathcal{T}_1$ of $G\ell(\mathcal{N},H)^\sim$ defined by

$$\tilde{\chi}_\alpha(pt) = \chi_\alpha(t) \text{ with } p \in P(\mathbf{k}) \text{ and } t \in \mathcal{T}$$

and to form the associated holomorphic line bundle over the base space $G\ell(\mathcal{N},H)^\sim$modulo $P(\mathbf{k})\mathcal{T}/\mathcal{T}_1 = \mathcal{F}(\mathbf{k})$.

Let $\Gamma(L_\alpha)$ be the space of holomorphic sections of L_α, equiped with the topology of uniform convergence on compact subsets of $\mathcal{F}(\mathbf{k})$. We can now state the main result.

2.2. Theorem

If $\alpha_1 < \alpha_2 <\ldots \alpha_{m-1} < 0$, then $\Gamma(L_\alpha)$ is non-zero and contains a dense Hilbert subspace H, that is invariant under the action of $G\ell(\mathcal{N},H)^{\sim}$ and that is an irreducible $G\ell(\mathcal{N},H)^{\sim}$-module.

The proof of the theorem runs roughly along the following lines. First one consider the case m equal to 2 that can be partly found in [PS]. In that case one restricts the elements of $\Gamma(L_\alpha)$ to a dense nested sequence of finite dimensional subflagvarieties. By using the well-known results in the finite dimensional case and analyzing the connecting maps, one finds a dense stable Hilbert subspace. Its irreducibility is a consequence of cyclicity and positivity, see [CR]. Finally one composes the results found for the case m = 2 to build a Hilbert subspace of sections in the general case. For detailed proofs of the results we refer to [H].

[CR] Carey, A.L. and Ruysenaars, S.N.M., On Fermion Gauge Groups, Current Algebras and Kac-Moody Algebras, Acta Appl. Math. 10 (1987), pp. 1-86.

[H] Helminck, G.F., Holomorphic line bundles over infinite dimensional flagvarieties, to appear.

[HP] Helminck, G.F. and Post, G.F., Geometrical Interpretation of the Bilinear Equations for the KP-Hierarchy, Letters in Mathematical Physics 16 (1988), pp. 359-364.

[PS] Pressley, A. and Segal, G., Loop groups, Clarendon Press, Oxford, 1986.